Clifford Algebras

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Outline

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Motivation: Finding a Nice Vector Algebra

- Many ways of representing vectors:
  - 2 dimensions: complex numbers, Gibbs vectors
  - 3 dimensions: quaternions, Gibbs vectors
  - 4 dimensions: Minkowski 4-vectors, gamma matrices
  - Spinors
- All have their strengths and weaknesses
- It would be nice to unify these!

Motivation: Shortcomings of Gibbs Vectors

- Need two multiplication operations
- Cannot divide vectors
- Can’t exponentiate vectors
- Doesn’t distinguish between polar and axial (pseudo-) vectors
    - Rotations (axial vectors) don’t “add” intuitively
- Transformation requires large ugly NxN matrices
- Have to be careful of gimbal lock (degeneracy in Euler rotation matrices)

Motivation: Quaternions

- Extension of complex numbers
- Has four components: $q = a + b \cdot i + c \cdot j + d \cdot k$
- $i \cdot i = j \cdot j = k \cdot k = -1$
- $i \cdot j = k$, $j \cdot k = i$, $k \cdot i = j$
- $j \cdot i = -k$, $k \cdot j = -i$, $i \cdot k = -j$

Motivation: Quaternion Example

- Any two rotations multiplied together gives a third rotation
- Vectors represented as $v = v_x i + v_y j + v_z k$
- Rotations can be represented as quaternions with

$$\begin{aligned} q &= e^{\frac{a}{2}} = e^{\frac{\theta}{2}(u_x i + u_y j + u_z k)} \\ &= \cos\frac{\theta}{2} + (u_x i + u_y j + u_z k)\sin\frac{\theta}{2} \end{aligned}$$

- Perform rotations with $v' = qvq^{-1}$
- Simple example:
  - Rotate quarter turn about x-axis
  - Rotate quarter turn about z-axis
  - What is combined rotation?

Motivation: Quaternion Example

$$
\begin{aligned}
q &= \cos\frac{\theta}{2} + (u_x i + u_y j + u_z k)\sin\frac{\theta}{2} \\
q_x &= \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}(1+i) \\
q_z &= \cos\frac{\pi}{4} + k\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}(1+k) \\
q_x \cdot q_z &= \frac{1}{\sqrt{2}}(1+i) \cdot \frac{1}{\sqrt{2}}(1+k) \\
&= \frac{1}{2}(1+k+ik+i) \\
&= \frac{1}{2}(1+i-j+k) \\
&= \cos\frac{\pi}{3} + \frac{1}{\sqrt{3}}(i-j+k)\sin\frac{\pi}{3}
\end{aligned}
$$

Motivation: Quaternion Discussion

- Pros vs. Gibbs vectors:
  - Rotation very elegant
  - Avoids gimbal lock
  - Less elements to each component
  - Division algebra; thus easy inverses
- Cons vs. Gibbs vectors:
  - Negative norm / dot product, so need to be careful to conjugate
    - This breaks symmetry
  - Difficult to extend to Minkowski space

Introducing Clifford Algebras

- $C\ell(\mathbb{R}^n)$ describes associative algebra in n-space
- Introduce $\{e_1, e_2, \ldots, e_n\}$
- $e_i^2 = +1$
- $e_i e_j = -e_j e_i$
    - $(e_i e_j)^2 = e_i e_j e_i e_j = -e_i e_j e_j e_i = -1$
    - This is known as a bivector

Recovering Complex Numbers from $C\ell(\mathbb{R}^2)$

- Consider even components:
    - $(a + xe_1 + ye_2 + be_1e_2) \rightarrow (a + be_1e_2)$
- Multiply two even-vectored elements:

$$(a_1 + b_1e_1e_2)(a_2 + b_2e_1e_2) = (a_1a_2 - b_1b_2) + (a_1b_2 + a_2b_1)e_1e_2$$

- This forms subalgebra
- Isomorphic to complex numbers, with $e_1e_2 \rightarrow i$

Application: Representing vectors in $C\ell(\mathbb{R}^2)$

- Numbers represented as $a + xe_1 + ye_2 + be_1e_2$
  - Scalar, vector, and bivector components
- Multiply two vectors:

$$\begin{aligned}(x_1e_1 + y_1e_2)(x_2e_1 + y_2e_2) &= (x_1x_2 + y_1y_2) + (x_1y_2 - x_2y_1)e_1e_2 \\ &= v_1 \cdot v_2 + v_1 \wedge v_2\end{aligned}$$

- Intuition: view e_i as basis i
- Intuition: view e_ie_j as cross product of basis i and basis j

Application: Rotation by θ in $\mathbb{R}^2$

$$
\begin{aligned}
v &= xe_1 + ye_2 \\
v' &= v\cos\theta + vi\sin\theta = ve^{i\theta} \\
v' &= v\cos\theta - iv\sin\theta = e^{-i\theta}v \\
cos\theta + i\sin\theta &= (cos\frac{\theta}{2} + i\sin\frac{\theta}{2})^2 \\
v' &= e^{-i\frac{\theta}{2}}ve^{i\frac{\theta}{2}}
\end{aligned}
$$

Matrix reps of Cℓ($\mathbb{R}^2$)

$$1 \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad e_1 \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$e_2 \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad e_1 e_2 \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$a + xe_1 + ye_2 + be_1e_2 \to \begin{pmatrix} a+x & y+b \\ y-b & a-x \end{pmatrix}$$

$C\ell(\mathbb{R}^3)$

$$a + v_1e_1 + v_2e_2 + v_3e_3 + w_1e_2e_3 + w_2e_3e_1 + w_3e_1e_2 + be_1e_2e_3$$

- Define $j = e_1e_2e_3$
- Represent vector as $v_xe_1 + v_ye_2 + v_ze_3$

$$\begin{aligned}
vw &= (v_xe_1 + v_ye_2 + v_ze_3)(w_xe_1 + w_ye_2 + w_ze_3) \\
&= v_xw_x + v_yw_y + v_zw_z \\
&\quad v_yw_ze_2e_3 + v_zw_xe_3e_1 + v_xw_ye_1e_2 \\
&\quad -v_xw_ze_3e_1 - v_zw_ye_2e_3 - v_yw_xe_1e_2 \\
&= v \cdot w + jv \times w
\end{aligned}$$

Application: Vectors and Rotations via $C\ell(\mathbb{R}^3)$

- Rearranging:

$$\begin{aligned} v \cdot w &= \frac{1}{2}(vw + wv) \\ jv \times w &= \frac{1}{2}(vw - wv) \end{aligned}$$

- Dot / cross product are just symmetric / anti-symmetric parts of product
- Represent rotations as $q = e^{\frac{a}{2}} = e^{\frac{1}{2}(a_x e_1 + a_y e_2 + a_z e_3)}$
- Like quaternions / $C\ell(\mathbb{R}^2)$: $v' = qvq^{-1}$

Last notes on $C\ell(\mathbb{R}^3)$

$$a + v_1e_1 + v_2e_2 + v_3e_3 + w_1e_2e_3 + w_2e_3e_1 + w_3e_1e_2 + be_1e_2e_3$$

- Center is $a + be_1e_2e_3$, isomorphic to compex numbers
- Even elements isomorphic to quaternions
- Center and even elements each form subalgebras

Application: Spacetime

- Represent events as $S = t + xe_1 + ye_2 + ze_3$
- $$\begin{aligned} S\bar{S} &= (t + xe_1 + ye_2 + ze_3)(t - xe_1 - ye_2 - ze_3) \\ &= t^2 - x^2 - y^2 - z^2 \end{aligned}$$

- Lorentz boosts: $\ell = e^{\frac{1}{2}(v+jw)}$
- $s' = \ell s \ell^{-1}$
  - Can't get this with quaternions because you need a positive square
  - Can't get this with Gibbs vectors because you can't exponentiate them
  - Instead, with Gibbs 4-vectors, need 4x4 boost matrices

Application: Electromagnetism

$$
\begin{aligned}
F &= E + jB \\
\nabla &= e_1\partial_x + e_2\partial_y + e_3\partial_z \\
(\partial_t + \nabla)F &= \frac{\rho}{\epsilon} - \mu J \\
F' &= \ell F \ell^{-1}
\end{aligned}
$$

Application: Spinors

- $C\ell(\mathbb{R}^3)$ isomorphic to 2x2 complex matrices
- Represent Pauli spin matrices as e_1, e_2, e_3
- Split spinors as follows:

$$\begin{aligned}\psi &\equiv \begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix} \to \begin{pmatrix}\psi_1 & 0 \\ \psi_2 & 0\end{pmatrix} \\ &\to \begin{pmatrix}1 & 0 \\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 \\ i & 0\end{pmatrix}, \begin{pmatrix}0 & 0 \\ -1 & 0\end{pmatrix}, \begin{pmatrix}i & 0 \\ 0 & 0\end{pmatrix} \\ &\to \frac{1}{2}(1 + e_3), \frac{1}{2}(e_2 e_3 + e_2), \frac{1}{2}(e_3 e_1 - e_1), \frac{1}{2}(e_1 e_2 + e_1 e_2 e_3)\end{aligned}$$

- $\langle s_z \rangle = \psi e_3 \psi$, etc. Can calculate all 3 at once

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