UNIVERSITY OF CALIFORNIA
SANTA CRUZ

# Generalized Monopoles of Gauge Groups

**David M. Reiman**

**University of California Santa Cruz**

# Outline

1. Gauge Theory Formalism

2. Generalized Monopoles
    2.1 The Higgs Vacuum
    2.2 Homotopy Groups
    2.3 The Little Group

3. Applications

4. Conclusion

5. Bibliography

# Gauge Theory Formalism

Consider a general gauge group G with a faithful representation given by D(g) and a scalar field which transforms under the action of D(g)

$$\phi \rightarrow D(g)\phi \tag{1}$$

A local gauge transformation is one for which $g = g(x)$, which spoils the previously covariant transformation of the kinetic term

$$\partial^{\mu}\phi \rightarrow D(g)\partial^{\mu}\phi + \partial^{\mu}D(g)\phi \tag{2}$$

# Gauge Theory Formalism (cont.)

To restore the covariance we introduce gauge fields and associate them with matrices of the Lie algebra L(G)

$$\mathbf{W}^{\mu} = W_{a}^{\mu} T^{a} \in L(G) \tag{3}$$

Our newly covariant derivative is then given by

$$D^{\mu} = \partial^{\mu} + ie\mathbf{W}^{\mu} \tag{4}$$

# Gauge Theory Formalism (cont.)

Where the gauge fields now transform in the following manner

$$\mathbf{W}^{\mu} \rightarrow g\mathbf{W}^{\mu}g^{-1} + \frac{i}{e}(\partial^{\mu}g)g^{-1} \tag{5}$$

Then gauge field tensor is then given by

$$\mathbf{G}^{\mu\nu} = \partial^{\mu}\mathbf{W}^{\nu} - \partial^{\nu}\mathbf{W}^{\mu} + ie[\mathbf{W}^{\mu}, \mathbf{W}^{\nu}] \tag{6}$$

# Gauge Theory Formalism (cont.)

By noting that

$$[D^\mu, D^\nu]\phi = ieD(g)G^{\mu\nu} \tag{7}$$

The transformation of the gauge field tensor can be gleaned

$$\mathbf{G}^{\mu\nu} \to g\mathbf{G}^{\mu\nu}g^{-1} \tag{8}$$

## Gauge Theory Formalism (cont.)

We then have

$$\mathbf{G}_a^{\mu\nu}\mathbf{G}_{\mu\nu}^a = \frac{1}{k}Tr[G^{\mu\nu}G_{\mu\nu}] \tag{9}$$

Which is clearly invariant under the action of the gauge group

$$\frac{1}{k}Tr[G^{\mu\nu}G_{\mu\nu}] \rightarrow \frac{1}{k}Tr[gG^{\mu\nu}g^{-1}gG_{\mu\nu}g^{-1}] \rightarrow \frac{1}{k}Tr[G^{\mu\nu}G_{\mu\nu}] \tag{10}$$

# Outline


1. Gauge Theory Formalism
2. Generalized Monopoles
    2.1 The Higgs Vacuum
    2.2 Homotopy Groups
    2.3 The Little Group
3. Applications
4. Conclusion
5. Bibliography

# The Higgs Vacuum

We now study a scalar field governed by the following Lagrangian

$$\mathscr{L} = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} + (D^\mu \phi)^\dagger D_\mu \phi - V(\phi) \tag{11}$$

Where we assume $V(\phi)$ is symmetric under G. Additionally, we take $\phi$ to satisfy the following equations at all points in space besides a finite number of regions we call monopoles

$$V(\phi) = 0 \tag{12}$$

$$D^\mu \phi = 0 \tag{13}$$

# The Higgs Vacuum (cont.)

We define the vacuum manifold as the set of all field configurations which lie at the minimum of the potential

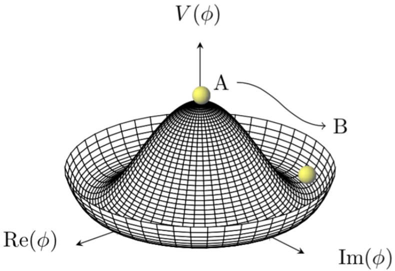
$$M_0 = \{\phi \mid V(\phi) = 0\} \tag{14}$$

Note that since $V(\phi)$ is invariant under the action of G...

$$V(D(g)\phi) = 0 \rightarrow D(g)\phi \in M_0 \tag{15}$$

# The Higgs Vacuum (cont.)

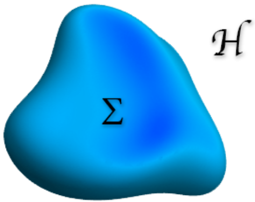
We consider a theory with a non-trivial vacuum manifold and whose corresponding field configurations are related via a single orbit




This is equivalent to considering a scalar field with a non-vanishing vacuum expectation value which we will call a *Higgs field*.

## Homotopy Groups

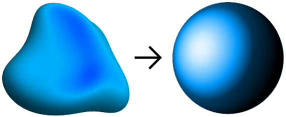
We consider a large region of space $\mathscr{H}$ in which the equations governing the previously defined Higgs vacuum hold to a good approximation.




A compact monopole region $\mathscr{M}$ surrounded by $\mathscr{H}$ is enclosed exactly once within a surface $\Sigma$. Then, the field provides a continuous mapping $\phi : \Sigma \to M_0$ (from physical space to the vacuum manifold).

## Homotopy Groups (cont.)

To allow for the movement and evolution of $\mathscr{M}$, $\Sigma$ must vary continuously in time.

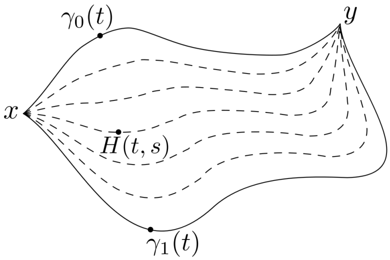


Thus, the mapping $\phi$ of the surface $\Sigma$ at any given time must be *homotopic* to itself at all other times.

$$\phi(\mathbf{r}, t_1) \sim \phi(\mathbf{r}, t_2) \tag{16}$$

# Homotopy Groups (cont.)

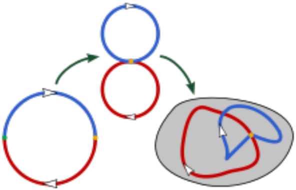
A homotopy is a continuous deformation between two mappings in the same space. Let's first consider a map between the real line interval $I = [0, 1]$ and a topological space $\mathscr{X}$ .




The paths $\gamma_0(t)$ and $\gamma_1(t)$ are said to be homotopic because there exists a continuous deformation between them.

# Homotopy Groups (cont.)

This defines an equivalence class to which all paths that are homotopic to a given path $\gamma(t)$ belong. Paths can be combined under the operation of *path concatenation*.



This provides a group operation for paths. Path concatenation (or composition) combines two paths for which the terminal point of the first path is the initial point of the second path.

# Homotopy Groups (cont.)

For example, if the path $\gamma_0(t)$ takes us from $x_0 \rightarrow y_0$ and path $\gamma_1(t)$ takes us from $x_1 \rightarrow y_1$ then the concatenated path first takes us along $\gamma_0$ then along $\gamma_1$. This defines our group operation as follows.

$$\gamma_0\gamma_1(t) = \begin{cases} \gamma_0(2t) & \text{for } 0 \leq t \leq 0.5 \\ \gamma_1(2t-1) & \text{for } 0.5 \leq t \leq 1 \end{cases} \tag{17}$$

Which is only defined for $y_0 = x_1$. The set of equivalence classes of *closed* paths (or maps) about a basepoint $x_0$ equipped with the concatenation operation forms a topological group.

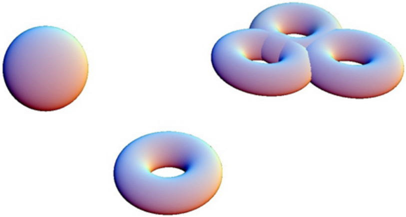
# Homotopy Groups (cont.)

We verify by checking the group axioms.

- The identity element sends the entire interval $I$ to $x_0$. It's concatenation with any other element yields the element itself.
- The inverse element for a given map $\gamma(t)$ is simply $\gamma(1 - t)$. This path sends us directly backwards along the original path.
- The group is closed since the concatenation of two loops with the same basepoint $x_0$ is itself a loop with basepoint $x_0$.
- Concatenation is associative **up to a path-homotopy** due to the previously defined parameterization.

# The Fundamental Group

The fundamental group with basepoint $x_0$ in a topological space $X$ is denoted $\Pi_1(X, x_0)$. This group is a *topological invariant* which records information about the shape and number of holes in a given topological space.



The topological invariance of the fundamental group allows us to classify topological spaces and their homeomorphisms.

# Higher Order Groups

In general, the *n*-th homotopy group, $\Pi_n(X)$, is the group of maps from the n-sphere, $S^n$, to the topological space X.

We will also use the zeroth class $\Pi_0(X)$ which is the map between two points and the space X. If the class is trivial, all points are path-connected in X and we call the space *connected*.

However, to employ homotopy theory in the description of magnetic monopoles, we also must be familiar with the properties of the little group H of a vector $\phi_0 \in M_0$.

## The Little Group

The Little group of a point $\phi$ is defined as follows

$$H_\phi = \{h \in G \mid D(h)\phi = \phi\} \tag{18}$$

If we associate a point $\phi$ with an element $g \in G$ via $\phi = D(g)\phi_0$, two elements of $G$ can only be associated with the same point if they belong to the same right coset space of H in G.

$$D(g_1)\phi_0 = D(g_2)\phi_0 \rightarrow D(g_1^{-1}g_2)\phi_0 = \phi_0 \tag{19}$$

# The Little Group (cont.)

Which requires

$$g_1^{-1} g_2 \in H_{\phi_0} \tag{20}$$

Points on the vacuum manifold $M_0$ are identified with an element of the right coset space $G/H$. This means that given a gauge group $G$, once $H$ has been determined we can uncover the structure of $M_0$.

$$M_0 \simeq G/H \tag{21}$$

## The Little Group (cont.)

An important result in homotopy theory provides us the following

$$\Pi_1(G/H) \simeq \Pi_0(H) \tag{22}$$
$$\Pi_2(G/H) \simeq \Pi_1(H) \tag{23}$$

Since the map $\phi : \Sigma \to M_0$ defines an element of the homotopy group $\Pi_2(G/H)$, the second isomorphism provides a description of the topological quantum numbers in terms of the fundamental group of the little group, H.

# Outline

1. Gauge Theory Formalism

2. Generalized Monopoles
   2.1 The Higgs Vacuum
   2.2 Homotopy Groups
   2.3 The Little Group

3. Applications

4. Conclusion

5. Bibliography

# Applications

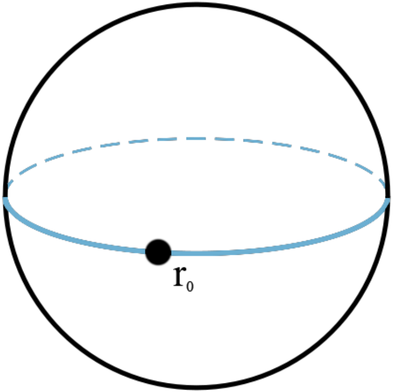
To relate the gauge fields of H to the topological quantum numbers, we employ the second defining equation of the Higgs vacuum.

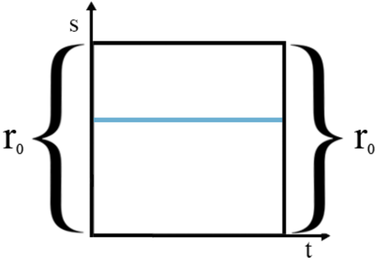
$$D^{\mu}\phi = 0 \tag{24}$$

It is useful to parameterize the surface $\Sigma$ as the unit square with its perimeter all mapped to the same point $\phi(\mathbf{r}_0) = \phi_0$

$$\Sigma = \{\mathbf{r}(s,t) \mid 0 \leq s \leq 1, 0 \leq t \leq 1\} \tag{25}$$

# Applications (cont.)



# Applications (cont.)

Writing $D_t = \frac{dr_i}{dt} D_i$, the map $\phi$ is defined by

$$D_t \phi = 0 \tag{26}$$

With the associated boundary condition $\phi(s, 0) = \phi_0$. Expanding out the covariant derivative yields Schrodinger's differential equation

$$\frac{\partial \phi}{\partial t} = ieD(\mathbf{W}^i)\phi \frac{\partial r^i}{\partial t} \tag{27}$$

## Applications (cont.)

This equation can be solved via a time evolution operator which is written in terms of a time-ordered exponential

$$U(s,t) = \mathscr{T} \exp\left[ie \int_0^t \mathbf{W}^i \frac{\partial r^i}{\partial t'} dt'\right] \tag{28}$$

Then the solution to the differential equation in $\phi$ is given by

$$\phi(s,t) = U(s,t)\phi_0 \tag{29}$$

$$\phi(s,t) = \mathscr{T} \exp\left[ie \int_0^t \mathbf{W}^i \frac{\partial r^i}{\partial t'} dt'\right]\phi_0 \tag{30}$$

# Applications (cont.)

The quantity $U(s, 1)$ corresponding to a loop with $s$ fixed is the path-dependent phase factor for a closed loop

$$h(s) = U(s, 1) = \mathscr{T} \exp\left[ie \int_0^1 \mathbf{W}^i \frac{\partial r^i}{\partial t'} dt'\right] \to \langle \mathbf{r}_0 | U(t, 0) | \mathbf{r}_0 \rangle \quad (31)$$

It corresponds to a closed loop in the group $H_{\phi_0}$ since the boundaries of the unit square are mapped to the same point $r_0$ and $\phi(r_0) = \phi_0$.

$$h(0) = h(1) = 1 \quad (32)$$

# Applications (cont.)

This closed loop can be further related to the field strength tensor $\mathbf{G}^{\mu\nu}$ via a non-Abelian Stokes' theorem generalization [3]

$$h^{-1}\frac{dh}{ds} = ie\int_0^1 g^{-1}\mathbf{G}_{ij}g\frac{\partial r^i}{\partial t}\frac{\partial r^j}{\partial s}dt \tag{33}$$

Where the left-hand side of the above equation is an element of L(H)

$$h^{-1}\frac{dh}{ds} = c_a t^a \text{ for } t^a \in L(H) \tag{34}$$

# Applications (cont.)

For a $U(1)$ gauge group the equation simplifies

$$h(s) = \exp\left[ie \int_{\Sigma} \mathbf{B} \cdot \mathbf{dS}\right] \tag{35}$$

And we recognize the equation $h(1) = 1$ as Dirac's famous electromagnetic charge quantization condition

$$eg = 2n\pi \tag{36}$$

# Applications (cont.)

Perhaps a more interesting application is the quantization of electromagnetic charges in the presence of a color gauge group, $K$, under which the electric charge remains a color singlet state.

Since $h^{-1}\frac{dh}{ds}$ is in the Lie algebra of H, we may write the following.

$$h^{-1}\frac{dh}{ds} = \frac{ie}{a}\alpha(s)\phi + i\beta_a(s)K^a \tag{37}$$

Where 1 and $K^a$ are the generators of $U(1)$ and $K$ respectively and the constant $a$ is the length of $\phi$ in the vacuum manifold $M_0$.

# Applications (cont.)

The solution to the differential equation is given by

$$h(s) = k(s) \exp\left[i\frac{e}{a}\Phi(s)\right] \tag{38}$$

Where $\Phi(s)$ is the magnetic flux through a surface spanning the loop $\Gamma_s$ for $s =$ constant and $k(s) \in K$, the color gauge group. Setting $h(1) = 1$ and $\Phi(1) = g$, the total magnetic charge enclosed within $\Sigma$

$$\exp\left[igQ\right] = k \in K \tag{39}$$

# Applications (cont.)

Since $Q$ is a color singlet, $k$ must commute with all of $K$ and therefore lives in the center of $K$, $Z(K)$. Taking $k \in SU(N)$, Schur's lemma requires that $k$ take the following form

$$k = \lambda I_N \text{ for } \lambda = \exp[2\pi im/N] \text{ and } m \in Z \tag{40}$$

We now take $|c\rangle$ to be the particle representations of the color group $K$ which may be electromagnetically charged under the $U(1)$ group

$$k\,|c\rangle = \exp\left[ig_0 Q\right]|c\rangle = \exp\left[2\pi im/N\right]|c\rangle \tag{41}$$

# Applications (cont.)

Which implies that the arguments of the exponentials must be equal

$$q = \frac{2\pi m}{N g_0} = \frac{m q_0}{N} \text{ for } m \in Z \tag{42}$$

In the relevant case of QCD with $K = SU(3)$ and hence $N = 3$

$$q = \frac{m}{3} e = 0, \pm\frac{1}{3} e, \pm\frac{2}{3} e, \pm e, \ldots \tag{43}$$

## Applications (cont.)

This is the generalization of the fractional charge of quarks to an electromagnetic gauge group in the presence of a more general color group $SU(N)$. We've seen that for $N = 3$ we have the gauge group of the strong interaction with the fractional quark charges in thirds of e.

The above result implies some connection between colored magnetic monopoles and the fractional charges of quarks which has been investigated in part by 't Hooft [4] and Corrigan & Olive [5]

# Outline

1. Gauge Theory Formalism

2. Generalized Monopoles
   2.1 The Higgs Vacuum
   2.2 Homotopy Groups
   2.3 The Little Group

3. Applications

4. Conclusion

5. Bibliography

# Conclusion

- Group theory in combination with homotopy theory provide a general topological treatment of monopoles in quantum theory in terms of topological invariants
- The topological features can be encapsulated in the gauge fields of H which carry the long-range characteristics of the monopole
- Quantization of electric charge in the presence of larger gauge groups can be approached via topological features of magnetic monopoles
- The general treatment allows for easy access to larger gauge groups which may be of interest to grand unified theories

# Outline

1. Gauge Theory Formalism

2. Generalized Monopoles
   2.1 The Higgs Vacuum
   2.2 Homotopy Groups
   2.3 The Little Group

3. Applications

4. Conclusion

5. Bibliography

# Bibliography

[1] P. Goddard and D. I. Olive | *Magnetic Monopoles in Gauge Field Theories* | Reports on Progress in Physics, Volume 41, Number 9

[2] Luis J. Boya et al. | *Homotopy and Solitons* | Fortschr. Phys., 26: 175–214

[3] J. Goldstone | *Unpublished Lectures Given at Cambridge University*

[4] G. 't Hooft | *Monopoles and Dyons in the SU(5) Model* | 1976 Nucl. Phys. B 105 538

[5] E. Corrigan et al. | *Magnetic monopoles in SU(3) gauge theories* | 1976 Nucl. Phys. B 110 237-47