Graded Lie Algebras and Representations of Supersymmetry Algebras

Physics 251 Group Theory and Modern Physics

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Overview

Graded Lie Algebras

- What is a graded algebra?

Graded Lie Algebras

- What is a graded algebra?
- How do we construct a graded algebra?

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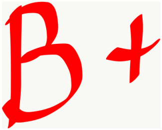
- What is a graded algebra?
- How do we construct a graded algebra?
- It's actually much easier than you think

Graded Lie Algebras

$\underline{\mathfrak{su}\left(2\right)}$

Graded Lie Algebras

$\underline{\mathfrak{su}\,(2)}$



Graded Lie Algebras

$\underline{\mathfrak{so}(3)}$

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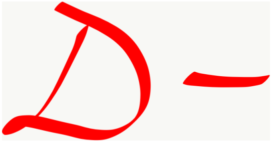
$\underline{\mathfrak{so}(3)}$



Graded Lie Algebras

$\underline{E_6}$

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Linear Vector Spaces

- $\vec{v}_0, \vec{v}_1, \vec{v}_2, \cdots \in V$

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- Vector addition (+), Abelian operation such that $\vec{v}_i + \vec{v}_j = \vec{v}_j + \vec{v}_i \in V$

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- Vector addition $(+)$, Abelian operation such that $\vec{v}_i + \vec{v}_j = \vec{v}_j + \vec{v}_i \in V$
- Scalar multiplication $(\cdot)$, $c \in F, \vec{v} \in V \rightarrow c \cdot \vec{v} \in V$

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- $\vec{v}_k \times (\vec{v}_i + \vec{v}_j) = \vec{v}_k \times \vec{v}_i + \vec{v}_k \times \vec{v}_j$ Distributive Property

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Other potential properties of the vector product

- $\vec{v}_i \times \vec{v}_j = \vec{v}_j \times \vec{v}_i$ — Commutativity
- $\vec{v}_i \times \vec{v}_j = -\vec{v}_j \times \vec{v}_i$ — Anti-commutativity

Lie Algebra

A linear vector space $\mathfrak{g}$ over a field F where we define the vector product as a non-associative, alternating bilinear map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ denoted by the Lie Bracket $[.,.]$ which obeys the Jacobi identity

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- $[x, [y, z]] \neq [[x, y], z] \qquad x, y, z \in \mathfrak{g} \qquad$ Non-Associative

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- $[ax + by, z] = a[x, z] + b[y, z]$ $\quad a, b \in F$ $\quad x, y, z \in \mathfrak{g}$ $\quad$ Bi-linearity

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- $[ax+by,z] = a[x,z] + b[y,z] \quad a,b \in F \quad x,y,z \in \mathfrak{g}$ Bi-linearity
- $[x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0 \quad x,y,z \in \mathfrak{g}$ Jacobi Identity

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- Lie's 3$^{\text{rd}}$ Theorem shows that any finite dimensional Lie algebra over $\mathbb{R}$ or $\mathbb{C}$ corresponds uniquely to a connected Lie group up to covering
- The Lie algebra is often easier to deal with than the Lie Group, so we can study the group through the associated algebra

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$$[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j} \tag{3}$$

Graded Lie Algebra Example: $\mathfrak{sl}(2, \mathbb{C})$

$$\begin{aligned} a &\in \mathfrak{sl}(2) \\ \operatorname{tr}(a) &= 0 \end{aligned} \tag{4}$$

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$$\begin{aligned} X &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ Y &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ H &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \tag{5}$$

Graded Lie Algebra Example: $\mathfrak{sl}(2, \mathbb{C})$

$$[X, Y] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{6}$$

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&= \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix} = -2Y
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Let $\mathfrak{g}_{-1} = span\,(X)$, $\mathfrak{g}_0 = span\,(H)$, $\mathfrak{g}_1 = span\,(Y)$

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Let $\mathfrak{g}_{-1} = span(X)$, $\mathfrak{g}_0 = span(H)$, $\mathfrak{g}_1 = span(Y)$

We can then write $\mathfrak{sl}(2) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$

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Graded Lie Algebra Example: $\mathfrak{su}(3)$

The Gell-Mann matrices are traceless Hermitian generators of $\mathfrak{su}(3)$

[1] The f_{abc} are antisymmetric under the permutation of any pair of indices.

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$$[\lambda_a, \lambda_b] = 2if_{abc}\lambda_c, \quad \text{where } a, b, c = 1, 2, 3, \ldots, 8 \tag{8}$$

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abc	f_{abc}	abc	f_{abc}	abc	f_{abc}
123	1	345	$\frac{1}{2}$	147	$\frac{1}{2}$
367	$-\frac{1}{2}$	156	$-\frac{1}{2}$	458	$\frac{\sqrt{3}}{2}$
246	$\frac{1}{2}$	678	$\frac{\sqrt{3}}{2}$	257	$\frac{1}{2}$

Table: Non-zero structure constants[1] f_{abc} of $\mathfrak{su}(3)$

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(9)

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$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{9}$$

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$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{9}$$

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\lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, &\quad \lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},
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\lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, & \lambda_8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},
\end{aligned}
\tag{9}
$$

Graded Lie Algebra Example: $\mathfrak{su}(3)$

Defining

$$T_a \equiv \frac{1}{2}\lambda_a$$
$$\left(F_l^k\right)_{ij} = \delta_{li}\delta_{kj} - \frac{1}{3}\delta_{kl}\delta_{ij}$$

the Gell-Mann matrices can be written in the Cartan-Weyl basis[2]

[2]The requires us to complexify the $\mathfrak{su}(3)$ algebra to $\mathfrak{sl}(3,\mathbb{C})$

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$$\begin{aligned} \lambda_1 &= F_1^2 + F_2^1, & \lambda_2 &= -i\left(F_1^2 - F_2^1\right), \\ \lambda_4 &= F_1^3 + F_3^1, \end{aligned} \tag{10}$$

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the Gell-Mann matrices can be written in the Cartan-Weyl basis[2]

$$\begin{aligned} \lambda_1 &= F^2_1 + F^1_2, & \lambda_2 &= -i\left(F^2_1 - F^1_2\right), \\ \lambda_4 &= F^3_1 + F^1_3, & \lambda_5 &= -i\left(F^3_1 - F^1_3\right), \end{aligned} \tag{10}$$

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$$[T_i, F_\alpha] = \alpha_i F_\alpha \tag{12}$$

Where $i = 3, 8$ and $F_\alpha = \left\{F_1^2, F_2^1, F_1^3, F_3^1, F_2^3, F_3^2\right\}$.

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The root vectors α_i are (excluding the 0 vectors)

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We can now see how to decompose $\mathfrak{sl}(3,\mathbb{C})$ into a graded Lie algebra. We identify the subalgebras by the root vector (including the 0 vector) associated with the subvector space.

Cartan Subalgebra

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The Lie algebra is defined by the Lie bracket of its generators (vector product), which is anti-commutative. In Supersymmetry (SUSY) we introduce fermionic generators, which are defined using an extension of the Lie bracket $\{.,.\}$ which is commutative.

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In order to define a closed SUSY algebra we need to relax the Alternativity of the Lie bracket, but we still want a subalgebra to be defined with the regular Lie bracket $\Rightarrow$ Graded Lie Algebra

Lie Superalgebras

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Introduce a Lie Superalgebra: A $\mathbb{Z}_2$ graded algebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ that satisfies generalized Lie algebra axioms

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- $[x, y] = -(-1)^{|x||y|}[y, x]$ Superskew Symmetry

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- $[x, y] = -(-1)^{|x||y|}[y, x]$ Superskew Symmetry
- $(-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||x|}[y, [z, x]] + (-1)^{|z||y|}[z, [x, y]] = 0$
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And $|x|, |y|, |z| \in \{0, 1\}$ is called the *degree* of x (even or odd)

Lie Superalgebras

The Super Jacobi Identity on the previous slide can be written in a more compact form

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If $x \in \mathfrak{g}_1$, $|x| = 1$. $\mathfrak{g}_1$ is a linear representation of $\mathfrak{g}_0$ and there exists a symmetric $\mathfrak{g}_0$-equivariant linear map $\{.,.\} : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_0$

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$$f(\theta) = a + \theta b \tag{21}$$

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$$\sigma^0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Superspace

Spinors transform under Lorentz transformations as

$$\psi'_\alpha = M_\alpha{}^\beta \psi_\beta \qquad \bar{\psi}'_{\dot{\alpha}} = {M^*}_{\dot{\alpha}}{}^{\dot{\beta}} \bar{\psi}_{\dot{\beta}}$$

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And the familiar Dirac γ-matrices in the Weyl-basis

$$\gamma^\mu = \begin{pmatrix} \mathbb{0} & \sigma^\mu \\ \bar{\sigma}^\mu & \mathbb{0} \end{pmatrix}$$

Supersymmetry Algebras

$$
\begin{aligned}
[P^\mu, P^\nu] &= 0 \\
[M_{\mu\nu}, P_\rho] &= i\eta_{\nu\rho}P_\mu - i\eta_{\mu\rho}P_\nu \\
[M_{\mu\nu}, M_{\rho\sigma}] &= i\eta_{\mu\sigma}M_{\nu\rho} - i\eta_{\mu\rho}M_{\nu\sigma} - i\eta_{\nu\sigma}M_{\mu\rho} + i\eta_{\nu\rho}M_{\mu\sigma} \\
\left[P_\mu, {Q_\alpha}^L\right] &= \left[P_\mu, \bar{Q}_{\dot{\alpha}L}\right] = 0 \\
\left[M_{\mu\nu}, {Q_\alpha}^L\right] &= \frac{1}{2}{\sigma_{\mu\nu\alpha}}^\beta {Q_\beta}^L \\
\left[M_{\mu\nu}, \bar{Q}_{\dot{\alpha}L}\right] &= \frac{1}{2}{\bar{\sigma}_{\mu\nu\dot{\alpha}}}^{\dot{\beta}} \bar{Q}_{\dot{\beta}L} \\
\left\{{Q_\alpha}^L, \bar{Q}_{\dot{\alpha}M}\right\} &= 2{\sigma_{\alpha\dot{\alpha}}}^\mu P_\mu \delta^L_M \\
\left\{{Q_\alpha}^L, {Q_\beta}^M\right\} &= \epsilon_{\alpha\beta}X^{LM} \\
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\tag{23}
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SUSY Representations

Of particular interest to us are the anticommutation relations

$$\begin{aligned} \left\{Q_\alpha{}^L, \bar{Q}_{\dot{\alpha}M}\right\} &= 2\sigma_{\alpha\dot{\alpha}}{}^{\mu} P_\mu \delta^L_M \\ \left\{Q_\alpha{}^L, Q_\beta{}^M\right\} &= 0 \\ \left\{\bar{Q}_{\dot{\alpha}L}, \bar{Q}_{\dot{\beta}M}\right\} &= 0 \end{aligned} \tag{24}$$

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From the definition of the supersymmetry generators

$$(-1)^{N_F} Q_\alpha = -Q_\alpha (-1)^{N_F} \tag{25}$$

SUSY Representations

For a finite dimensional representation[3] we can take the trace of the operator $(-1)^{N_F}\left\{{Q_\alpha}^A, \bar{Q}_{\dot{\beta}B}\right\}$

$$\begin{aligned}
\mathrm{Tr}\left[(-1)^{N_F}\left\{{Q_\alpha}^A, \bar{Q}_{\dot{\beta}B}\right\}\right] &= \mathrm{Tr}\left[(-1)^{N_F}\left({Q_\alpha}^A\bar{Q}_{\dot{\beta}B} + \bar{Q}_{\dot{\beta}B}{Q_\alpha}^A\right)\right] \\
&= \mathrm{Tr}\left[(-1)^{N_F}{Q_\alpha}^A\bar{Q}_{\dot{\beta}B}\right] + \mathrm{Tr}\left[(-1)^{N_F}\bar{Q}_{\dot{\beta}B}{Q_\alpha}^A\right] \\
&= -\mathrm{Tr}\left[{Q_\alpha}^A(-1)^{N_F}\bar{Q}_{\dot{\beta}B}\right] + \mathrm{Tr}\left[{Q_\alpha}^A(-1)^{N_F}\bar{Q}_{\dot{\beta}B}\right] \\
&= 0
\end{aligned} \tag{26}$$

[3] The trace is undefined for infinite dimensional representations

SUSY Representations

Using the anticommutation relations of the $Q, \bar{Q}$ we can also show

$$\begin{aligned} \mathrm{Tr}\left[(-1)^{N_F}\left\{Q_\alpha{}^A, \bar{Q}_{\dot{\beta}B}\right\}\right] &= 2\sigma_{\alpha\dot{\beta}}{}^\mu \delta^A{}_B \mathrm{Tr}\left[(-1)^{N_F} P_\mu\right] \\ &= 0 \end{aligned} \tag{27}$$

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For fixed non-zero momentum this requires

$$\mathrm{Tr}\left[(-1)^{N_F}\right] = 0 \tag{28}$$

$\Rightarrow$ Representations of supersymmetry must contain an equal number of bosonic and fermionic states.

Raising and Lowering Operators

Boost to the rest frame such that for a massive, one-particle state $P_\mu = (-M, 0, 0, 0)$

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The SUSY algebra in this frame takes the form

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The indices A and B run from 1 to N, where N is the number of supersymmetries.

Constructing Massive Representations

We can rescale the SUSY generators

$$\begin{aligned} {a_\alpha}^A &\equiv \frac{1}{\sqrt{2M}} {Q_\alpha}^A \\ \left({a_\alpha}^A\right)^\dagger &\equiv \frac{1}{\sqrt{2M}} \bar{Q}_{\dot{\alpha} A} \end{aligned} \tag{30}$$

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These operators obey the following anticommutation relations in the rest frame

$$\begin{aligned} \left\{{a_\alpha}^A, \left({a_\alpha}^B\right)^\dagger\right\} &= {\delta_\alpha}^\beta {\delta^A}_B \\ \left\{{a_\alpha}^A, {a_\beta}^B\right\} &= \left\{\left({a_\alpha}^A\right)^\dagger, \left({a_\beta}^B\right)^\dagger\right\} = 0 \end{aligned} \tag{31}$$

Constructing Massive Representations

We now recognize the algebra of ${Q_\alpha}^A$ and $\bar{Q}_{\dot{\beta}B}$ as being isomorphic to the algebra of $2N$ fermionic creation and annihilation operators ${a_\alpha}^A$ and $\left({a_\alpha}^B\right)^\dagger$

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Define the vacuum Ω through the condition ${a_\alpha}^A\Omega = 0$
And build up states through successive application of the creation operator $\left({a_\alpha}^A\right)^\dagger$

$$\Omega^{(n)\alpha_1\alpha_2\ldots\alpha_n}{}_{A_1A_2\ldots A_n} = \frac{1}{\sqrt{n!}}\left({a_{\alpha_1}}^{A_1}\right)^\dagger\left({a_{\alpha_2}}^{A_2}\right)^\dagger\ldots\left({a_{\alpha_n}}^{A_n}\right)^\dagger\Omega \qquad (32)$$

Constructing Massive Representations

Because the $\left(a_\alpha{}^A\right)^\dagger$ anticommute, $\Omega^{(n)\alpha_1\alpha_2\ldots\alpha_n}{}_{A_1A_2\ldots A_n}$ must be antisymmetric under the exchange of a pair of indices $\alpha_iA_i \leftrightarrow \alpha_jA_j$.

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The α's take values from 1 to 2, and the A's take values from 1 to N, there are therefore $2N$ unique values for each pair of indices αA. We cannot construct a totally antisymmetric tensor if $n > 2N$. As an example let $N = 1$, the state

$$\Omega^{121}{}_{111} = -\Omega^{121}{}_{111} \tag{33}$$

Having switched the first and third pair of indices, implies $\Omega^{121}{}_{111} = 0$

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Therefore fundamental irreducible massive multiplet $\Omega^{\alpha_1 \alpha_2 \ldots \alpha_{2N}}_{A_1 A_2 \ldots A_{2N}}$ has dimension 2^{2N} and contains 2^{2N-1} fermionic states and 2^{2N-1} bosonic states.

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The highest spin state comes from symmetrizing as many spin indices as possible while antisymmetrizing in the other index. This leads to a maximum spin of $\frac{N}{2}$

$N = 1$ Massive Supersymmetry Representations

For $N = 1$ the fundamental representation is of dimension 4, and consists of the massive, one-particle states

$$\begin{gathered} \Omega \\ (a_\alpha)^\dagger \Omega \\ \frac{1}{\sqrt{2}} (a_\alpha)^\dagger (a_\beta)^\dagger \Omega = -\frac{1}{\sqrt{2}} \epsilon^{\alpha\beta} (a^\gamma)^\dagger (a_\gamma)^\dagger \Omega \end{gathered} \tag{35}$$

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There are two states with spin 0 and one state of spin-$\frac{1}{2}$.

Massive Supersymmetry Representations

Spin	Ω_0	$\Omega_{\frac{1}{2}}$	Ω_1	$\Omega_{\frac{3}{2}}$
0	2	1		
$\frac{1}{2}$	1	2	1	
1		1	2	1
$\frac{3}{2}$			1	2
2				1

Table: $N = 1$ Massive Supersymmetry representations

Spin	Ω_0	$\Omega_{\frac{1}{2}}$	Ω_1
0	5	4	1
$\frac{1}{2}$	4	6	4
1	1	4	6
$\frac{3}{2}$		1	4
2			1

Table: $N = 2$ Massive Supersymmetry representations

Massive Supersymmetry Representations

Spin	Ω_0	$\Omega_{\frac{1}{2}}$
0	14	14
$\frac{1}{2}$	14	20
1	6	15
$\frac{3}{2}$	1	6
2		1

Table: $N = 3$ Massive Supersymmetry representations

Spin	Ω_0
0	42
$\frac{1}{2}$	48
1	27
$\frac{3}{2}$	8
2	1

Table: $N = 4$ Massive Supersymmetry representations

Massless Supersymmetry Representations

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$$\begin{aligned} \left\{ Q_\alpha{}^A, \bar{Q}_{\dot{\beta}B} \right\} &= \begin{pmatrix} 4E & 0 \\ 0 & 0 \end{pmatrix} \delta^A{}_B \\ \left\{ Q_\alpha{}^A, Q_\beta{}^B \right\} &= \left\{ \bar{Q}_{\dot{\alpha}A}, \bar{Q}_{\dot{\beta}B} \right\} = 0 \end{aligned} \tag{36}$$

Massless Supersymmetry Representations

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Once again, rescaling the Q and $\bar{Q}$

$$\begin{aligned} a^A &\equiv \frac{1}{2\sqrt{E}} Q_1{}^A \\ a^\dagger{}_A &\equiv \frac{1}{2\sqrt{E}} \bar{Q}_{\dot{1}}{}^A = \left(a^A\right)^\dagger \end{aligned} \tag{37}$$

Massless Supersymmetry Representations

The algebra in this frame consists of N creation and annihilation operators, ${a^\dagger}_A$ and a^A which obey the anticommutation relations

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Because ${Q_2}^A$ and $\bar{Q}_{\dot{2}A}$ totally anticommute they must be identically 0. We therefore lose the spinor index, and the representations are antisymmetric in only the A and B indices. The a^A operator annihilates the state of lowest helicity $\underline{\lambda}$

$$a^A \Omega_{\underline{\lambda}} = 0 \tag{39}$$

Massless Supersymmetry Representations

The states are built in the same way as in the massive case, by successive application of the creation operator ${a^\dagger}_A$ on the Clifford vacuum $\Omega_{\underline{\lambda}}$

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Because we now lack a spinor index to differentiate between states we have 2^N states with a $\frac{N!}{n!(N-n)!}$ degeneracy.

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The antisymmetry in the A index requires the highest helicity to be $\overline{\lambda} = \underline{\lambda} + \frac{N}{2}$

Massless Supersymmetry Representations

hel.	λ							
	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$
2								1
$\frac{3}{2}$							1	1
1						1	1	
$\frac{1}{2}$					1	1		
0				1	1			
$-\frac{1}{2}$			1	1				
-1		1	1					
$-\frac{3}{2}$	1	1						
-2	1							

Table: $N = 1$ Massless Supersymmetry representations

Massless Supersymmetry Representations

hel.	λ -2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
2							1
$\frac{3}{2}$						1	2
1					1	2	1
$\frac{1}{2}$				1	2	1	
0			1	2	1		
$-\frac{1}{2}$		1	2	1			
-1	1	2	1				
$-\frac{3}{2}$	2	1					
-2	1						

Table: $N = 2$ Massless Supersymmetry representations

Massless Supersymmetry Representations

	λ					
hel.	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$
2						1
$\frac{3}{2}$					1	3
1				1	3	3
$\frac{1}{2}$			1	3	3	1
0		1	3	3	1	
$-\frac{1}{2}$	1	3	3	1		
-1	3	3	1			
$-\frac{3}{2}$	3	1				
-2	1					

Table: $N = 3$ Massless Supersymmetry representations

Massless Supersymmetry Representations

hel.	λ -2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0
2					1
$\frac{3}{2}$				1	4
1			1	4	6
$\frac{1}{2}$		1	4	6	4
0	1	4	6	4	1
$-\frac{1}{2}$	4	6	4	1	
-1	6	4	1		
$-\frac{3}{2}$	4	1			
-2	1				

Table: $N = 4$ Massless Supersymmetry representations

References and Further Reading

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