$\mathcal{G}$	Conditions on $\mathbf{A} \in \mathcal{G}$	$\mathcal{L}$	Conditions on $\mathbf{a} \in \mathcal{L}$	n
$GL(N, \mathbb{C})$	-	$gl(N, \mathbb{C})$	-	$2N^2$
$GL(N, \mathbb{R})$	$\mathbf{A}$ real	$gl(N, \mathbb{R})$	$\mathbf{a}$ real	N^2
$SL(N, \mathbb{C})$	$\det \mathbf{A} = 1$	$sl(N, \mathbb{C})$	$\mathrm{tr}\, \mathbf{a} = 0$	$2N^2 - 2$
$SL(N, \mathbb{R})$	$\mathbf{A}$ real, $\det \mathbf{A} = 1$	$sl(N, \mathbb{R})$	$\mathbf{a}$ real, $\mathrm{tr}\, \mathbf{a} = 0$	$N^2 - 1$
$U(N)$	$\mathbf{A}^\dagger = \mathbf{A}^{-1}$	$u(N)$	$\mathbf{a}^\dagger = -\mathbf{a}$	N^2
$SU(N)$	$\mathbf{A}^\dagger = \mathbf{A}^{-1}$, $\det \mathbf{A} = 1$	$su(N)$	$\mathbf{a}^\dagger = -\mathbf{a}$, $\mathrm{tr}\, \mathbf{a} = 0$	$N^2 - 1$
$U(p, q)$	$\mathbf{A}^\dagger \mathbf{g} = \mathbf{g}\mathbf{A}^{-1}$	$u(p, q)$	$\mathbf{a}^\dagger \mathbf{g} = -\mathbf{g}\mathbf{a}$	N^2
$SU(p, q)$	$\mathbf{A}^\dagger \mathbf{g} = \mathbf{g}\mathbf{A}^{-1}$, $\det \mathbf{A} = 1$	$su(p, q)$	$\mathbf{a}^\dagger \mathbf{g} = -\mathbf{g}\mathbf{a}$, $\mathrm{tr}\, \mathbf{a} = 0$	$N^2 - 1$
$O(N, \mathbb{C})$	$\tilde{\mathbf{A}} = \mathbf{A}^{-1}$	$so(N, \mathbb{C})$	$\tilde{\mathbf{a}} = -\mathbf{a}$	$N^2 - N$
$SO(N, \mathbb{C})$	$\tilde{\mathbf{A}} = \mathbf{A}^{-1}$, $\det \mathbf{A} = 1$	$so(N, \mathbb{C})$	$\tilde{\mathbf{a}} = -\mathbf{a}$	$N^2 - N$
$O(N)$	$\tilde{\mathbf{A}} = \mathbf{A}^{-1}$, $\mathbf{A}$ real	$so(N)$	$\tilde{\mathbf{a}} = -\mathbf{a}$, $\mathbf{a}$ real	$\frac{1}{2}(N^2 - N)$
$SO(N)$	$\tilde{\mathbf{A}} = \mathbf{A}^{-1}$, $\mathbf{A}$ real, $\det \mathbf{A} = 1$	$so(N)$	$\tilde{\mathbf{a}} = -\mathbf{a}$, $\mathbf{a}$ real	$\frac{1}{2}(N^2 - N)$
$O(p, q)$	$\tilde{\mathbf{A}}\mathbf{g} = \mathbf{g}\mathbf{A}^{-1}$, $\mathbf{A}$ real	$so(p, q)$	$\tilde{\mathbf{a}}\mathbf{g} = -\mathbf{g}\mathbf{a}$, $\mathbf{a}$ real	$\frac{1}{2}(N^2 - N)$
$SO(p, q)$	$\tilde{\mathbf{A}}\mathbf{g} = \mathbf{g}\mathbf{A}$, $\mathbf{A}^{-1}$ real, $\det \mathbf{A} = 1$	$so(p, q)$	$\tilde{\mathbf{a}}\mathbf{g} = -\mathbf{g}\mathbf{a}$, $\mathbf{a}$ real	$\frac{1}{2}(N^2 - N)$
$SO^*(N)$	$\tilde{\mathbf{A}} = \mathbf{A}^{-1}$, $\mathbf{A}^\dagger \mathbf{J} \mathbf{A} = \mathbf{J}$	$so^*(N)$	$\tilde{\mathbf{a}} = -\mathbf{a}$, $\mathbf{a}^\dagger \mathbf{J} = -\mathbf{J}\mathbf{a}$	$\frac{1}{2}(N^2 - N)$
$Sp(\frac{N}{2}, \mathbb{C})$	$\tilde{\mathbf{A}}\mathbf{J}\mathbf{A} = \mathbf{J}$	$sp(\frac{N}{2}, \mathbb{C})$	$\tilde{\mathbf{a}}\mathbf{J} = -\mathbf{J}\mathbf{a}$	$N^2 + N$
$Sp(\frac{N}{2}, \mathbb{R})$	$\tilde{\mathbf{A}}\mathbf{J}\mathbf{A} = \mathbf{J}$, $\mathbf{A}$ real	$sp(\frac{N}{2}, \mathbb{R})$	$\tilde{\mathbf{a}}\mathbf{J} = -\mathbf{J}\mathbf{a}$, $\mathbf{a}$ real	$\frac{1}{2}(N^2 + N)$
$Sp(\frac{N}{2})$	$\tilde{\mathbf{A}}\mathbf{J}\mathbf{A} = \mathbf{J}$, $\mathbf{A}^\dagger = \mathbf{A}^{-1}$	$sp(\frac{N}{2})$	$\tilde{\mathbf{a}}\mathbf{J} = -\mathbf{J}\mathbf{a}$, $\mathbf{a}^\dagger = -\mathbf{a}$	$\frac{1}{2}(N^2 + N)$
$Sp(r, s)$	$\tilde{\mathbf{A}}\mathbf{J}\mathbf{A} = \mathbf{J}$, $\mathbf{A}^\dagger \mathbf{G} \mathbf{A} = \mathbf{G}$	$sp(r, s)$	$\tilde{\mathbf{a}}\mathbf{J} = -\mathbf{J}\mathbf{a}$, $\mathbf{a}^\dagger \mathbf{G} = -\mathbf{G}\mathbf{a}$	$\frac{1}{2}(N^2 + N)$
$SU^*(N)$	$\mathbf{J}\mathbf{A}^* = \mathbf{A}\mathbf{J}$, $\det \mathbf{A} = 1$	$su^*(N)$	$\mathbf{J}\mathbf{a}^* = \mathbf{a}\mathbf{J}$, $\mathrm{tr}\, \mathbf{a} = 0$	$N^2 - 1$

Table 8.1: The real Lie algebras $\mathcal{L}$ of some important linear Lie groups $\mathcal{G}$. $\mathbf{A}$ and $\mathbf{a}$ are $N \times N$ matrices, which are complex unless otherwise stated; $\mathbf{g}$ is an $N \times N$ diagonal matrix with p diagonal elements $+1$ and q $(= N - p)$ diagonal elements -1, $p \geq q \geq 1$. In the last six entries N is even, and $\mathbf{J}$ and $\mathbf{G}$ are the $N \times N$ matrices defined in Equations (8.35) and (8.36).

In the above table, $\mathbf{A}$ is an invertible matrix, whereas no such condition is imposed on the matrix $\mathbf{a}$. In addition, the transpose of a matrix is denoted by placing a tilde above the corresponding symbol, the matrix adjoint is denoted by a dagger, and the complex conjugate of the matrix is denoted by a star.

Table 8.1 lists the details of the real Lie algebras belonging to a number of important linear Lie groups that can be obtained this way. In Table 8.1 $\mathbf{J}$ and $\mathbf{G}$ are the $N \times N$ matrices defined by

$$\mathbf{J} = \begin{bmatrix} \mathbf{0} & \mathbf{1}_{N/2} \\ -\mathbf{1}_{N/2} & \mathbf{0} \end{bmatrix}, \tag{8.35}$$

and

$$\mathbf{G} = \begin{bmatrix} -\mathbf{1}_r & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_s & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{1}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_s \end{bmatrix}, \tag{8.36}$$

where $1 \leq r \leq \frac{1}{2}N$ and $s = \frac{1}{2}N - r$.

That the exponential mapping remains invaluable even for *non-compact* linear Lie groups is demonstrated by the following theorem.

Theorem VIII *Every* element of the connected subgroup of *any* linear Lie group $\mathcal{G}$ can be expressed as a *finite* product of exponentials of its real Lie algebra $\mathcal{L}$.

Proof See, for example, Appendix E, Section 2, of Cornwell (1984).

These results may be *summarized* by the statement that the matrix exponential function *always* provides a mapping of $\mathcal{L}$ *into* $\mathcal{G}$. This is *onto* if $\mathcal{G}$ connected and *compact*, and even when $\mathcal{G}$ is connected but *non-compact* every element of $\mathcal{G}$ is expressible as a *finite* product of exponentials of members of $\mathcal{L}$.

These two pages are taken from J.F. Cornwell, "Group Theory in Physics: An Introduction," (Academic Press, San Diego, CA, 1997).