Coleman-Callan-Wess-Zumino Construction

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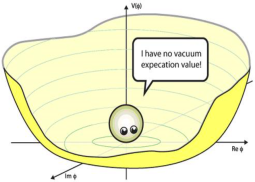

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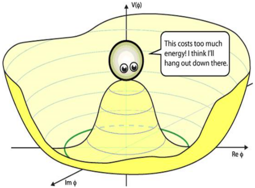

Overview

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   - Construction of NGB and Transformation Properties
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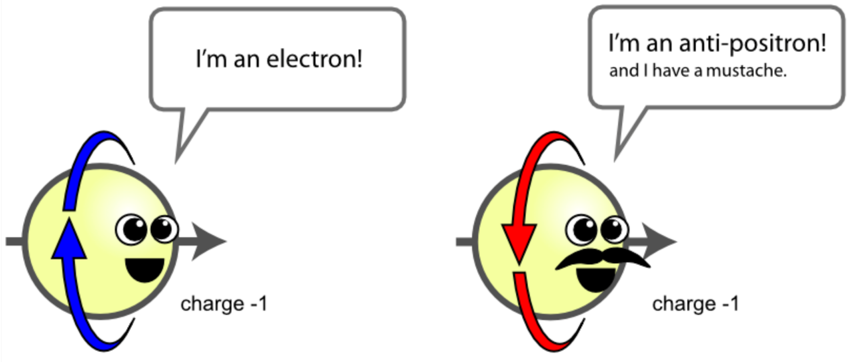
Part I: Introduction

Motivation

- Many quantum field theories exhibit symmetry breaking patterns from a group $\mathcal{G}$ to a subgroup $\mathcal{H}$.
- When a symmetry group is broken down to subgroup, the observable degrees of freedom (DOF) will change.
- By Goldstone's theorem, we while find $N_{\mathcal{G}} - N_{\mathcal{H}}$ Goldstone boson after symmetry breaking.
- In order to describe the observable DOF, a general method for constructing Lagrangians made out of Goldstone bosons is needed.
- The Coleman-Callan-Wess-Zumino (CCWZ) Construction provides a systematic way to describe low-energy DOF.

Chiral Symmetry Breaking



Chiral Symmetry Breaking

- In the Standard Model (SM) the QCD Lagrangian for light quarks is

$$\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{4}\,\mathrm{Tr}(G_{\mu\nu}G^{\mu\nu}) + i\left(q_R^\dagger \overline{\sigma}_\mu D^\mu q_R + q_L^\dagger \overline{\sigma}_\mu D^\mu q_L\right) + \text{mass terms}$$

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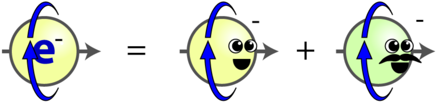
$$q_R = \begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix} \qquad q_L = \begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} \qquad \text{where} \qquad u_R = \begin{pmatrix} u_{R,r} \\ u_{R,g} \\ u_{R,b} \end{pmatrix}$$

R, L refer to right and left-handed particles



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R, L refer to right and left-handed particles

u, d, s stand for the up, down and strange quark

$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\mu$

A^a_μ gauge fields (gluons)

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$$D_\mu q_R = \partial_\mu \begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix} - igA_\mu^a \tau_a \begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix} \qquad \begin{pmatrix} u_{R,r} \\ u_{R,g} \\ u_{R,b} \end{pmatrix}$$

D_μ is the covariant derivative,

τ_a are the generators of SU(3) which act on triplets u_R, etc.,

and A_μ^a are the gauge-fields (gluons)

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$$\sigma_\mu = (\mathbb{1}_{2\times 2}, \boldsymbol{\sigma}) \qquad \overline{\sigma}_\mu = (\mathbb{1}_{2\times 2}, -\boldsymbol{\sigma})$$

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- QCD Lagrangian exhibits a global *chiral* symmetry: $\mathcal{G} = \mathrm{SU}(3)_L \otimes \mathrm{SU}(3)_R$ in the chiral (massless) limit:

$$q_L \to \exp\left(i\theta_L^a \tau_a\right) q_L \qquad q_R = \begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix} \to \exp\left(i\theta_R^a \tau_a\right) \begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix}$$

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$$\begin{aligned} q_R^\dagger\overline{\sigma}_\mu D^\mu q_R \to \left(e^{i\theta_R^a\tau_a}q_R\right)^\dagger\overline{\sigma}_\mu D^\mu\left(e^{i\theta_R^b\tau_b}q_R\right) &= q_R^\dagger e^{-i\theta_R^a\tau_a}\overline{\sigma}_\mu D^\mu e^{i\theta_R^b\tau_b}q_R \\ &= q_R^\dagger\overline{\sigma}_\mu D^\mu e^{-i\theta_R^a\tau_a}e^{i\theta_R^b\tau_b}q_R \\ &= q_R^\dagger\overline{\sigma}_\mu D^\mu q_R \end{aligned}$$

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- $\mathcal{G}$ is broken down to the subgroup $\mathcal{H} = \text{SU}(3)_V$ $(\theta_a = \theta_b)$ due to quark condensate: $\langle\Omega|\,\bar{q}q\,|\Omega\rangle \neq 0$ below confinement scale Λ_{QCD}.

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$$0 \neq \langle\Omega|\,\bar{q}q\,|\Omega\rangle = \langle\Omega|\,q_R^\dagger q_L + q_L^\dagger q_R\,|\Omega\rangle$$

Since $\langle\Omega|\,\bar{q}q\,|\Omega\rangle$ is invariant under $\mathrm{SU}(3)_V$ $(\theta_a = \theta_b)$ but not under $\mathrm{SU}(3)_A$ $(\theta_a = -\theta_b)$

$$\mathrm{SU}(3)_L \otimes \mathrm{SU}(3)_R \cong \mathrm{SU}(3)_V \otimes \mathrm{SU}(3)_A \to \mathrm{SU}(3)_V$$

Goldstone's Theorem

When a continuous symmetry group $\mathcal{G}$ is broken down to a subgroup $\mathcal{H} \subset \mathcal{G}$ in which the broken generators do not leave the vacuum invariant, then there will be a massless scalar for every broken generator called a Nambu-Goldstone Boson.

High and Low-Energy DOF

- Bellow the confinement scale, quarks are no longer the observable DOF. The new DOF are Nambu-Goldstone bosons (NGB): pions, kaons etc.

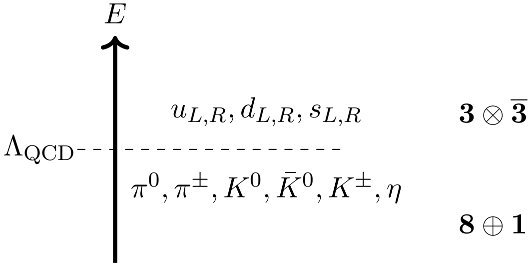

Figure: Schematic diagram showing the relevant DOF as a function of energy in QCD.

NOTE

- I have lied a bit. The actually symmetry group of the classical Lagrangian is $U(3)_R \otimes U(3)_L \cong SU(3)_R \otimes SU(3)_L \otimes U(1)_V \otimes U(1)_A$
- The $U(1)_A$ is not good quantum symmetry, it is anomalous
- The symmetry breaking pattern is actually $SU(3)_R \otimes SU(3)_L \otimes U(1)_V \to SU(3)_V \otimes U(1)_V$
- Due to the non-zero mass terms in the QCD Lagrangian:

$$\mathcal{L}_M = -\left(q_R^\dagger M q_L + q_L^\dagger M q_R\right), \qquad M = \mathrm{diag}(m_u, m_d, m_s)$$

  the $SU(3)_V \otimes SU(3)_A$ symmetry is explicitly broken, but approximately still present since $m_u, m_d, m_s \ll \Lambda_{\mathrm{QCD}}$.
- The pions, Kaons, etc. are then called psuedo-Nambu-Golstone bosons.

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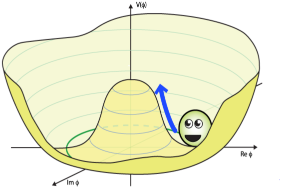
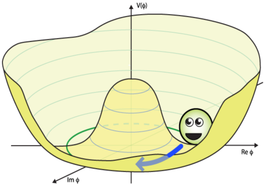
- We will find that the correct way to parameterize the NGB is

$$\Sigma = \exp\left(\frac{i\sqrt{2}}{f_\pi}\Pi^a\lambda_a\right)$$





Π^a modes

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Π^a are the NBG

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λ_a Gell-Mann matrices

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f_π is a constant, called the pion decay constant. It is determined, empirically, to be $f_\pi \approx 130.4$ MeV.

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$$\Sigma = \exp\left(\frac{i\sqrt{2}}{f_\pi}\Pi^a\lambda_a\right)$$

Σ will under $\mathrm{SU}(3)_R \otimes \mathrm{SU}(3)_L$ transform as

$$\Sigma \to R\Sigma L^\dagger$$

for $(R, L) \in \mathrm{SU}(3)_R \otimes \mathrm{SU}(3)_L$.

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$$\Sigma = \exp\left(\frac{i\sqrt{2}}{f_\pi}\Pi^a\lambda_a\right)$$

- The Lagrangian describing the light mesons will be given by

$$\mathcal{L} = \frac{f_\pi^2}{4}\,\mathrm{Tr}\left(\partial_\mu\Sigma^\dagger\partial^\mu\Sigma\right) + \cdots$$

Part II
CCWZ Construction

Construction of States from Vacuum

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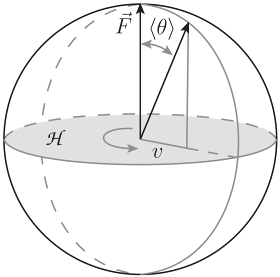
- Suppose these field acquire a non-zero expectation value $\langle\Omega|\,\mathbf{\Phi}\,|\Omega\rangle = \boldsymbol{F}$ which is invariant under a subgroup $\mathcal{H} \subset \mathcal{G}$
- $\mathcal{H}$ is the little group



e.g. $\mathcal{G} = \mathrm{SO}(3) \to \mathcal{H} = \mathrm{SO}(2)$

Construction of States from Vacuum

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- Suppose these field acquire a non-zero expectation value $\langle\Omega|\,\mathbf{\Phi}\,|\Omega\rangle = \boldsymbol{F}$ which is invariant under a subgroup $\mathcal{H}\subset\mathcal{G}$
- We want to identify the NGB, one for each broken generator. One candidate is:

$$\mathbf{\Phi}(x) = \exp\left(\frac{i\sqrt{2}}{F_0}\Theta_A(x)T^A\right)\boldsymbol{F}$$

T^A generators of the Lie algebra of $\mathcal{G}$

$\Theta_A(x)$ potentially massless, scalar fields (have no potential since a constant Θ_a yields an equivalent vacuum)

F_0 constant with mass dimension $[F_0] = m^1$

Identification of NGB

- Define T^a to be the unbroken generators (generators that leave vacuum invariant) and $\hat{T}^{\hat{a}}$ to be the broken generators

$$T^a \boldsymbol{F} = 0 \qquad \text{and} \qquad \hat{T}^{\hat{a}} \boldsymbol{F} \neq 0$$

Little a index for unbroken generators

Little $\hat{a}$ index for broken generators

Identification of NGB

- Define T^a to be the unbroken generators (generators that leave vacuum invariant) and $\hat{T}^a$ to be the broken generators
- A generic group element of $g \in \mathcal{G}$ can be written as

Fundamental formula of CCWZ

$$g = \exp\left(i\alpha_A T^A\right) = \exp\left(if_{\hat{a}}[\alpha]\hat{T}^{\hat{a}}\right)\exp(if_a[\alpha]T^a)$$

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⋆ *Infinitesimal proof*

$$\exp\left(i\alpha_A T^A\right) = I + i\alpha_{\hat{a}}\hat{T}^{\hat{a}} + i\alpha_a T^a + \mathcal{O}(\alpha^2)$$

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Thus,

$$f_{\hat{a}}[\alpha] = \alpha_{\hat{a}} + \mathcal{O}(\alpha^2)$$

$$f_a[\alpha] = \alpha_a + \mathcal{O}(\alpha^2)$$

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- Since T^a leaves the vacuum invariant, we can write $\mathbf{\Phi}$ as

$$\begin{aligned}\Phi(x) = \exp\left(\frac{i\sqrt{2}}{F_0}\Theta_A T^A\right)\boldsymbol{F} &= \exp\left(\frac{i\sqrt{2}}{F_0}\Pi_{\hat{a}}\hat{T}^{\hat{a}}\right)\exp(i\xi(x)T^a)\boldsymbol{F}\\ &= \exp\left(\frac{i\sqrt{2}}{F_0}\Pi_{\hat{a}}\hat{T}^{\hat{a}}\right)\boldsymbol{F}\end{aligned}$$

Since $\exp(i\xi(x)T^a)\boldsymbol{F} = \exp(0)\boldsymbol{F} = \boldsymbol{F}$

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- Since T^a leaves the vacuum invariant, we can write $\mathbf{\Phi}$ in terms of the Goldstone boson matrix

Goldstone Boson Matrix

$$\mathbf{\Phi}(x) = U[\Pi]\boldsymbol{F} \qquad \text{where} \qquad U[\Pi] \equiv \exp\left(\frac{i\sqrt{2}}{F_0}\Pi_a\hat{T}^a\right)$$

Π_a are the NBGs, one for each broken generator.

Transformations Properties of Fields under $\mathcal{G}$

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- We thus find that the Π fields transform as

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- This obeys the group multiplication law. Transforming by $g_1 g_2$

$$\begin{aligned} g_1 g_2 \mathbf{\Phi} = g_1 g_2 U[\Pi]\boldsymbol{F} &= g_1 U[\Pi^{(g_2)}]h[\Pi, g_2]\boldsymbol{F} \\ &= U[\Pi^{(g_1 g_2)}]h[\Pi^{(g_2)}, g_2]h[\Pi, g_2]\boldsymbol{F} \\ &= U[\Pi^{(g_1 g_2)}]h[\Pi, g_1 g_2]\boldsymbol{F} \\ &= U[\Pi^{(g_1 g_2)}]\boldsymbol{F} \end{aligned}$$

- With $h[\Pi, g_1 g_2] = h[\Pi^{(g_2)}, g_1]h[\Pi, g_2]$ and

$$U[\Pi^{(g_1 g_2)}] = g_1 g_2 U[\Pi]h[\Pi, g_2]^{-1}h[\Pi^{(g_2)}, g_1]^{-1} = g_1 g_2 U[\Pi]h[\Pi, g_1 g_2]^{-1}$$

- $U[\Pi]$ is called a non-linear realization of $\mathcal{G}$ (called a realization instead of representation since it is non-linear)

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$$\left[T^a, T^b\right] = i f^{ab}_{c} T^c + \cancel{i f^{ab}_{\hat{c}} \hat{T}^{\hat{c}}} \equiv T^c \left({t_{\text{Ad}}}^a\right)_c^{\ b}$$

$f^{ab}_{\hat{c}} = 0$ since $\mathcal{H}$ is a subgroup

t_{Ad} is adjoint representation of $\mathcal{H}$ generators

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$$\left[T^a, \hat{T}^{\hat{b}}\right] = \cancel{i f^{a\hat{b}}_{c} T^c} + i f^{a\hat{b}}_{\hat{c}} \hat{T}^{\hat{c}} \equiv \hat{T}^{\hat{c}} \left(t^a_\pi\right)_{\hat{c}}^{\ \hat{b}}$$

$f^{a\hat{b}}_{c} = 0$ since $f^{ab}_{\hat{c}} = 0$ and f is totally anti-symmetric

t^a_π is some yet unknown representation we call $\boldsymbol{r_\pi}$

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Transformation Properties of Fields Under $\mathcal{H}$

- To determine how $U[\Pi]$ transforms under $\mathcal{H}$, we need the commutation relations between generators: $T^a, \hat{T}^{\hat{a}}$
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- Next, we note the following identity:

$$\exp(i\alpha_a T^a)\hat{T}^{\hat{a}} \exp(-i\alpha_a T^a) = \hat{T}^{\hat{b}} \left[\exp(i\alpha_a t^a_{\pi})\right]_{\hat{b}}^{\ \hat{a}}$$

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&= \hat{T}^{\hat{a}} + i\alpha_a\hat{T}^{\hat{b}}\,(t^a_\pi)_{\hat{b}}{}^{\hat{a}} + \mathcal{O}(\alpha^2)\\
&= \hat{T}^{\hat{b}}\left(\delta^{\hat{a}}_{\hat{b}} + i\alpha_a\,(t^a_\pi)_{\hat{b}}{}^{\hat{a}}\right) + \mathcal{O}(\alpha^2)
\end{aligned}$$

Transformation Properties of Fields Under $\mathcal{H}$

- Using the previous identity, we find that for $g = g_{\mathcal{H}} = \exp(i\alpha_a T^a)$ and $c = i\sqrt{2}/F_0$

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\end{aligned}
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using previous result

$$
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&= \exp\Big(cT^{\hat{b}}\left[\exp(i\alpha_a t_\pi^a)\right]_{\hat{b}}^{\ \hat{a}}\,\Pi_{\hat{a}}\Big) g_{\mathcal{H}} \\
&= U[\exp(i\alpha_a t_\pi^a)\Pi] g_{\mathcal{H}}
\end{aligned}
$$

Transformation Properties of Fields Under $\mathcal{H}$

- The NGB transform under $\mathcal{H}$ as

NGB Transformation Under $\mathcal{H}$

$$\left(\Pi^{(g_{\mathcal{H}})}\right)_{\hat{b}} = \left[\exp(i\alpha_a t_\pi^a)\right]_{\hat{b}}^{\ \hat{a}}\, \Pi_{\hat{a}}$$

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- Therefore, Π transforms as a shift

NGB Transformation Under $\mathcal{G}/\mathcal{H}$

$$\Pi_{\hat{a}} \to \Pi_{\hat{a}} + \frac{F_0}{\sqrt{2}}\alpha_{\hat{a}} + \cdots$$

Construction of Invariants

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- In terms of d_μ and e_μ, this is

$$\begin{aligned} d_\mu + e_\mu &= h\left(d_\mu + e_\mu\right)h^{-1} + ih\partial_\mu h^{-1} \\ &= hd_\mu h^{-1} + h\left(e_\mu + i\partial_\mu\right)h^{-1} \end{aligned}$$

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- Thus, d_μ and e_μ transform under an arbitrary group element $g \in \mathcal{G}$ as

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Lowest Order Lagrangian

- Since d_μ transforms for a general group element $g \in \mathcal{G}$ as $d_\mu \to h[\Pi, g] d_\mu h^{-1}[\Pi, g]$, we can see that

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$$\begin{aligned} iU^{-1}\partial_\mu U &= i\left(I - ic\Pi_{\hat{a}}\hat{T}^{\hat{a}} + \cdots\right)\partial_\mu\left(I + ic\Pi_{\hat{a}}\hat{T}^{\hat{a}} + \cdots\right) \\ &= i\left(I - ic\Pi_{\hat{a}}\hat{T}^{\hat{a}} + \cdots\right)\left(ic\partial_\mu\Pi_{\hat{a}}\hat{T}^{\hat{a}} + \cdots\right) \end{aligned}$$

Lowest Order Lagrangian

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- The lowest order Lagrangian is thus

$$\mathcal{L}^{(2)} = \frac{F_0^2}{4}\mathrm{Tr}(d_\mu d^\mu) = \frac{1}{2}\left(\partial_\mu \Pi_{\hat{a}}\right)\left(\partial^\mu \Pi_{\hat{a}}\right) + \cdots$$

Part III
Chiral Perturbation Theory (ChPT)

Goldstone Boson Matrix

- Symmetry group is $\mathcal{G} = \mathrm{SU}(3)_L \otimes \mathrm{SU}(3)_R$ which is broken down to $\mathcal{H} = \mathrm{SU}(3)_V$

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$$g = (L, R) \qquad L \in \mathrm{SU}(3)_L, R \in \mathrm{SU}(3)_R$$

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- Note that a generic element of g can be written as

$$g = (L, R) = (L, RL^\dagger L) = (1, RL^\dagger)(L, L)$$

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- We ca ... i

$RL^\dagger$ = Goldstones!

This is similar to $e^{i\sqrt{2}\Theta_a T^a/F_0} = U[\Pi]h[\Pi, g]$

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- We identify the Goldstone matrix as $\Sigma = RL^\dagger \in \mathrm{SU}(3)$

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π^0 : Neutral pion

$\pi^\pm$: Charged pions

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$K^0, \bar{K}^0$: Neutral Kaon and anit-neutral Kaon

$K^\pm$: Charged Kaons

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- Under a generic group element $g = (\tilde{L}, \tilde{R})$, Σ transforms as

$$\Sigma \to \tilde{R}\Sigma\tilde{L}^\dagger$$

- Under $h = (V, V) \in \mathcal{H}$, Σ transforms under adjoint!

$$\Pi^c \xrightarrow{h} \Pi^c - f^{abc}\alpha^a\Pi^b$$

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Typical element in $(L, R)\mathcal{H}$ can be written as

$$\begin{aligned}(L, R)(V, V) = (LV, RV) = (LV, RL^\dagger LV) &= (I, RL^\dagger)(LV, LV)\\ &= (I, RL^\dagger)(V', V')\end{aligned}$$

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Thus, $(1, RL^\dagger)(V', V') = (L, R)(V, V)$, hence

$$(1, RL^\dagger)\mathcal{H} = (L, R)\mathcal{H}$$

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- Under an element of the coset space $(I, RL^\dagger)$, Σ transforms as a shift!

$$\Pi^a \overset{g^{\mathcal{G}/\mathcal{H}}}{\to} \Pi^a + \frac{f_\pi}{\sqrt{2}}\alpha^a$$

Chiral Lagrangian

- To compute the lowest order chiral Lagrangian, we need to compute $i\Sigma^\dagger\partial_\mu\Sigma$. Turns out that $i\Sigma^\dagger\partial_\mu\Sigma = d_\mu$

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Quark Masses and Symmetry Breaking

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- Can modify chiral Lagrangian to include symmetry breaking

$$\mathcal{L}^{(2)} = \frac{f_\pi^2}{4}\,\mathrm{Tr}\Big(\partial_\mu \Sigma \partial^\mu \Sigma^\dagger\Big) + \frac{f_\pi^2 B_0}{2}\,\mathrm{Tr}\Big(\Sigma M^\dagger + M \Sigma^\dagger\Big)$$

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Quark Masses and Symmetry Breaking

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- Treat M as a field which transforms as $M \to RML^\dagger$ (*Spurion field*)
- $\mathcal{H}$ symmetry is broken by the expectation value of M

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- This can be done for a general group $\mathcal{G}$ by modifying the Maurer-Cartan form. $iU^{-1}\partial_\mu U$ is replaced with

$$\bar{A}_\mu = U[\Pi]^{-1}\,(A_\mu + i\partial_\mu)\,U[\Pi] = d_\mu + e_\mu$$

Summary

- When we have a theory invariant under a Lie group $\mathcal{G}$ which is broken down to a subgroup $\mathcal{H}$, need a method to describe dynamics of the NBG
- Found that a smart way to parameterize the NBG was through

$$\exp\left(\frac{i\sqrt{2}}{F_0}\Pi_{\hat{a}}\hat{T}^{\hat{a}}\right)$$

- Can construct a term d_μ from Maurer-Cartan form $iU[\Pi]^{-1}\partial_\mu U[\Pi]$ which transformed under g as

$$d_\mu \to h[\Pi, g]d_\mu h[\Pi, g]^{-1}$$

- Lowest order Lagrangian can be constructed using

$$\mathcal{L}^{(2)} = \frac{F_0^2}{4}\,\mathrm{Tr}(d_\mu d^\mu)$$

The End

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