The Lorentz and Poincaré Groups

AND CLASSIFICATION OF RELATIVISTIC FIELDS

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Outline

- Spacetime and the Poincaré Group
- Multiplication law and subgroup structure
- Lorentz Group and Algebra
- Fields as Representations of Lorentz Group
- Examples of Fields
- Poincaré Algebra and Casimir Invariants
- Wigner's Classification of Relativistic Fields

Spacetime Isometries: a background

Let $(\mathbb{R}^4, \eta)$ be a pseudo-Riemannian manifold with metric η=diag(1,-1,-1,-1)

Maps ϕ: $\mathbb{R}^4 \rightarrow \mathbb{R}^4$ which preserve the metric ($\phi_* \eta = \eta$) are called <u>isometries</u>

(continuous isometries form a continuous group)

In this Minkowski spacetime, the isometry group is a 10 dimensional Lie group, called the <u>Poincaré Group</u> "P":

- 4 translations (3 spatial, 1 temporal) $\in T_4$
- 6 rotations (3 entirely spatial, 3 spatiotemporal "Boosts")$\in O(1,3)$

The Poincaré Group

We denote a typical element of P by $(R, \vec{a}) \in (O(1,3), T_4)$

and its action on elements of the tangent space T $\mathbb{R}^4$ (vectors) by

$$(R, \vec{a})\vec{x} = R\vec{x} + \vec{a}$$

Group multiplication rule: $(R_2, \vec{a_2}) \circ (R_1, \vec{a_1}) = (R_2 R_1, R_2 \vec{a_1} + \vec{a_2})$

Inverse: $(R, \vec{a})^{-1} = (R^{-1}, -R^{-1}\vec{a})$

Normal subgroup: $(R, \vec{a})^{-1}(1, \vec{t})(R, \vec{a}) = (1, -R^{-1}\vec{t}) \in T_4$

Remark: O(1,3) is **not** a normal subgroup of P

The Lorentz Group

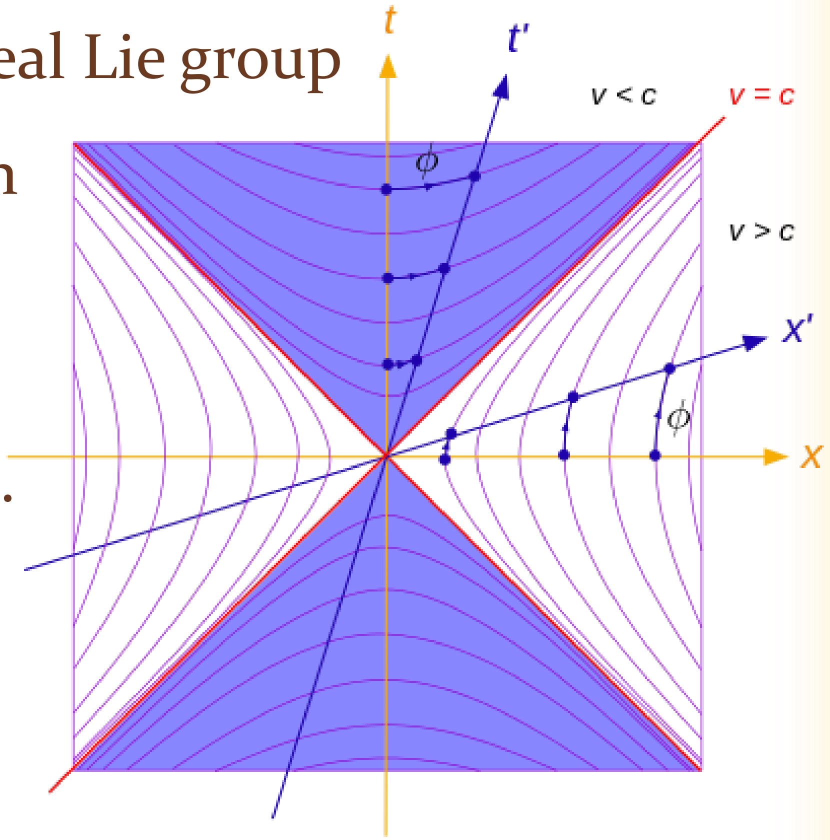
$$P/T_4 \cong O(1,3) \quad P \cong T_4 \rtimes O(1,3)$$

Six dimensional, non-compact, non-connected, real Lie group

It has four doubly-connected* components, which characterize the light cone structure

Boosts transport vectors along hyperbolas (right), confining them to their own side of the light cone.

Since a boost that rotates a time/space-like vector to the surface of the light cone does not exist, the Lorentz group is non-compact.



*to be discussed shortly

The Restricted Lorentz Group $SO^{+}(1,3) \cong O(1,3)/(Z_2 \otimes Z_2)$

If we mod out the discrete group composed of reflections and time reversal (isomorphic to the Klein four-group {e, T, P, PT}) we find the identity/connected component $SO^{+}(1,3)$

In matrix form, typical elements look like:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) & 0 \\ 0 & \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{\theta \in [0, 2\pi)}$$

Rotation about z axis

$$\begin{pmatrix} \cosh(\beta) & \sinh(\beta) & 0 & 0 \\ \sinh(\beta) & \cosh(\beta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{\beta \in [0, 1)}$$

Boost along x axis

Note: rotation subgroup compact: topology of circle; boost subgroup noncompact: open line

The Unitarian Trick (Hurwitz, Weyl)

The complexification of su(2) is isomorphic to sl(2,C), thus

$$\mathfrak{su}(2)_{\mathbb{C}} \oplus \mathfrak{su}(2)_{\mathbb{C}} \cong \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$$

The image of this algebra under the exp map is the group $SL(2,\mathbb{C}) \otimes SL(2,\mathbb{C})$ which contains the compact subgroup $SU(2) \otimes SU(2)$

Lorentz Group non-compact, so we lose the chance of Unitary representations of finite dimension.

Best chances of building a theory come from holomorphic representations of the complexified lie algebra of Lorentz Group

Remark: Holomorphic vector spaces

In a complex vector space spanned by $\left(\frac{\partial}{\partial x^\mu}, \ldots, \frac{\partial}{\partial y^\mu}, \ldots\right)$ it is appropriate to define a basis:

$$\frac{\partial}{\partial z^\mu} = \frac{1}{2}\left(\frac{\partial}{\partial x^\mu} - i\frac{\partial}{\partial y^\mu}\right)$$
$$\frac{\partial}{\partial \bar{z}^\mu} = \frac{1}{2}\left(\frac{\partial}{\partial x^\mu} + i\frac{\partial}{\partial y^\mu}\right)$$

and a linear map called the "almost complex structure"

$$J = i\left(\mathrm{d}z^\mu \otimes \frac{\partial}{\partial z^\mu} - \mathrm{d}\bar{z}^\mu \otimes \frac{\partial}{\partial \bar{z}^\mu}\right)$$
$$J^2 = -\mathrm{Id}_{\mathbb{C}^n}$$

This structure naturally splits the vector space into 2 disjoint vector spaces, (holomorphic and anti-holomorphic) $\mathbb{C}^n \cong H_+ \oplus H_-$

Maps from H+ to the field of complex numbers are analytic. (consider the exponential map on a complexified lie algebra)

The Lie Algebra $\mathfrak{so}^+(1,3)_\mathbb{C}$

Let **J** be the generators of the rotations, and **K** be the generators of the boosts. The commutators are:

$$[J_i, J_j] = i\epsilon_{ijk}J_k$$
$$[J_i, K_j] = i\epsilon_{ijk}K_k$$
$$[K_i, K_j] = -i\epsilon_{ijk}J_k$$

If we make linear combinations:

$$\vec{A}_\pm = \frac{1}{\sqrt{2}}(\vec{J} \pm i\vec{K})$$

$$[A_i^\pm, A_j^\pm] = i\epsilon_{ijk}A_k^\pm$$
$$[A_i^\pm, A_j^\mp] = 0$$

Hence: $\mathfrak{so}^+(1,3)_\mathbb{C} \cong \mathfrak{su}(2)_\mathbb{C} \oplus \mathfrak{su}(2)_\mathbb{C}$

And thus by the Unitarian trick, we can study reps of SO(1,3) via reps of SU(2)

(Holomorphic) Representations of $\mathfrak{so}^+(1,3)_{\mathbb{C}}$

This is a semi-simple Lie algebra (sl(2,C) is simple), so any rep can be built as a direct sum of irreps. We have choices of tensor product representation:

$$\pi_n \otimes \pi_m : \mathfrak{su}(2) \oplus \mathfrak{su}(2) \longrightarrow \mathfrak{gl}(U) \oplus \mathfrak{gl}(V)$$

We will use the representation that assigns the following:

$$\pi_n \otimes \pi_m(X) = \pi_n(X) \otimes \mathrm{Id}_m + \mathrm{Id}_n \otimes \pi_m(X) \qquad X \in \mathfrak{so}^+(1,3)$$

Where each π is (if we so choose) an irrep of su(2). The indices (n,m) will classify the representations by dimension and (as we will see) the fields by their spin.

$$\pi_{(n,m)}(J_i) = A_i^{(n)} \otimes \mathrm{Id}^{(m)} + \mathrm{Id}^{(n)} \otimes A_i^{(m)}$$

$$\pi_{(n,m)}(K_i) = i(A_i^{(n)} \otimes \mathrm{Id}^{(m)} - \mathrm{Id}^{(n)} \otimes A_i^{(m)})$$

Representation theory of su(2)

Let **S** be the generators of an arbitrary, finite dimensional rep of su(2).

The Cartan sub-algebra $\{S_z, S^2\}$ allows us to form a basis $|s, m\rangle$ such that

$$S^2|s,m\rangle = s(s+1)|s,m\rangle$$
$$S_z|s,m\rangle = m|s,m\rangle$$
$$S_\pm|s,m\rangle = \sqrt{(s\mp m)(s\pm m+1)}|s,m\pm 1\rangle \qquad S_\pm = \frac{1}{\sqrt{2}}(S_x \pm iS_y)$$

This implies that the dimension of the representation is <u>(2s+1)</u>=number of non-zero eigenvalues of S_z.

Hence the value "s" must come in half-integer form: $s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$

So the rep defined by $\pi_{(m,n)}$ is (2m+1)(2n+1) dimensional

Restrictions on the pair (n,m)

To consider full Lorentz invariance, we impose time reversal and parity symmetry:

One finds under parity transformation: $\mathcal{P} A_i^{\pm} \mathcal{P}^{-1} = A_i^{\mp}$

So that $(n, m) \longrightarrow (m, n)$ under this symmetry. Hence we need symmetric representations of the form: $(n, m) \oplus (m, n)$ or (m, m)

In general, the product representation is **not** irreducible:

$$\pi_{(n,m)} = \bigoplus_p \pi_{(p)}$$

(Clebsch-Gordon decomposition)

$$p = |n + m| \ldots |n - m|$$

Let’s now consider the most important representations:

The Scalar (0,0) *or* The Trivial Representation

Scalars transform trivially under action of the Lorentz group:

$$\Phi(x) \mapsto e^{i\theta\pi_0(J)}\Phi(x) = \Phi(x)$$

Thus any Lagrangian that goes as $\sim \Phi(x)^p$ (for some power p) has hope of being manifestly Lorentz invariant.

To make it Poincare invariant requires imposing extra transformation rules

Examples: $$\phi(x) = \phi'(x+a); \phi' = \phi - a^\mu \partial_\mu \phi$$

- Higgs
- Numerous excitations in 1-d condensed matter systems

The Dirac Fermion $\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)$

The representation decomposes into $\pi_{(1/2)} \oplus \pi_{(1/2)}$, a 4-dimensional rep (not the defining representation).

Weyl fermions transform as: $\psi'_{R/L} = \exp\left(\frac{i\sigma \cdot \theta \pm \sigma \cdot \omega}{2}\right) \psi_{R/L}$

where θ (ω) parametrizes the rotation(boost)

Define the Dirac spinor $\psi_{Dirac} = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}$

If we restrict θ=zero and evaluate the explicit form of these transformations using ω=arctanh(β) and the Einstein energy relations, we arrive at the Dirac eq.

$$(\gamma^\mu p_\mu - mc)\psi_{Dirac} = 0 \qquad \gamma^0 = \begin{pmatrix} 0 & \mathrm{Id}_{2\times 2} \\ \mathrm{Id}_{2\times 2} & 0 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

The Massive Vector Field (defining rep) $\left(\frac{1}{2}, \frac{1}{2}\right)$

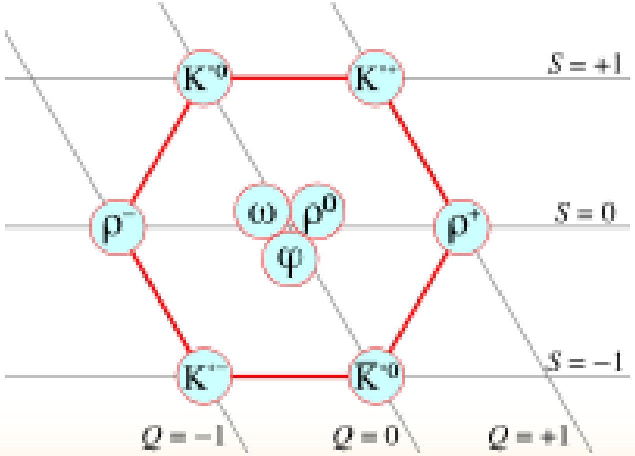
This representation decomposes into $\pi_{(1)} \oplus \pi_{(0)}$, which is the defining representation, of dimension 4.

One finds that the field equations are $(\partial^\nu \partial_\nu - m^2)B_\mu = 0$

This is distinct from the massless case, in which there are only two degrees of freedom. The massive field has longitudinal modes with no angular momentum. It has no gauge freedom.

<u>Examples:</u>

- W and Z bosons
- Composite particles (right)
- Cooper pairs in BCS theory



The Anti-Symmetric Tensor Field $(1,0)\oplus(0,1)$

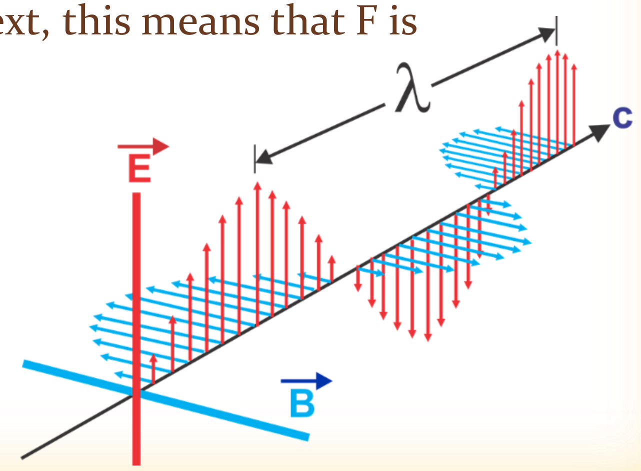
This rep decomposes into $\pi_{(1)} \oplus \pi_{(1)}$ which acts on the space of traceless, anti-symmetric tensor fields.

There are 6 degrees of freedom, just as with the tensor $F_{\mu\nu}$ of Maxwell's formalism.

The field satisfies the following relations:

$$\partial_\mu F^{\mu\nu} = 0 \qquad\qquad \partial_{[\alpha}F_{\mu\nu]} = 0 \quad \text{(Bianchi's Identity)}$$

The second relation is a statement that the 2-form is "closed" (dF=0), which by Poincare's lemma, means it can be expressed uniquely (locally) via an exact 1-form A (where F=dA). In our context, this means that F is the curvature tensor of a massless vector field with gauge freedom.



The field equations for A are much different than the massive case:

$$(\Box\delta^\mu_\nu - \partial^\mu\partial_\nu)A_\mu = 0$$

The Poincaré Algebra

Let P_μ be the generators of spacetime translations, and $M_{\mu\nu}$ an antisymmetric tensor (relativistic angular momentum) that satisfies the following:

$$M_{0i} = K_i \qquad \frac{1}{2}\epsilon_{ijk}M_{jk} = J_i$$

Then we have:

$$[P_\mu, P_\nu] = 0$$
$$[M_{\mu\nu}, P_\rho] = i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu)$$
$$[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} + \eta_{\nu\sigma}M_{\mu\rho})$$

There are two Casimir operators of the Poincare Algebra:

$P_\mu P^\mu$ and $W_\mu W^\mu$ where $W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}M^{\nu\rho}P^\sigma$ is the Pauli-Lubanski vector

Casimir invariants of the Poincare Algebra

One can evaluate W^2 in the rest frame to find that it is proportional to the squared angular momentum (and thus, the identity). In the rest frame, for this to be non-zero implies a definite spin of the particle/field.

$$\mathrm{Tr}(W^2) = m^2 j(j+1) \times \dim(\mathrm{rep})$$

For the arbitrary (n,m) rep, this involves calculating $W^2 = m^2 \sum_{i=1}^{3} [\pi_{(n,m)}(J_i)]^2$

Using the tracelessness of the individual J's, and the property:

$$\mathrm{Tr}(A \otimes B) = \mathrm{Tr}(A)\mathrm{Tr}(B)$$

One finds that **j=m+n**

Hence the spin of a relativistic field is specified uniquely by the finite dimensional representation of the lie algebra $\mathfrak{so}^+(1,3)$ **with which it is associated.**

Wigner's Classification

Casimir invariants of Poincaré group are: $P_\mu P^\mu \quad W_\mu W^\mu$

<u>3 cases:</u>

1. Massive
2. Massless, P_0 nonzero:
3. Massless, all components of P are zero:

Case 1: Massive fields

The first Casimir invariant is trivially $P_\mu P^\mu = m^2$

Consider the rest frame: P and W are well defined:

$$P = (m, \vec{0}) \quad W = (0, m\vec{J})$$

This form is obvious, since the form of P puts heavy restrictions on W

The eigenspace of the momentum operator is, as we have seen, a representation of SU(2)

Hence massive fields are classified by irreps of SU(2) that determines spin

Case 2: Massless fields with momentum

For a massless propagating field (like the photon) we may use a momentum operator: $P = (P_0, \vec{P}); P_0^2 - \vec{P} \cdot \vec{P} = 0$

We find that the Pauli-Lubanski vector has the exact form

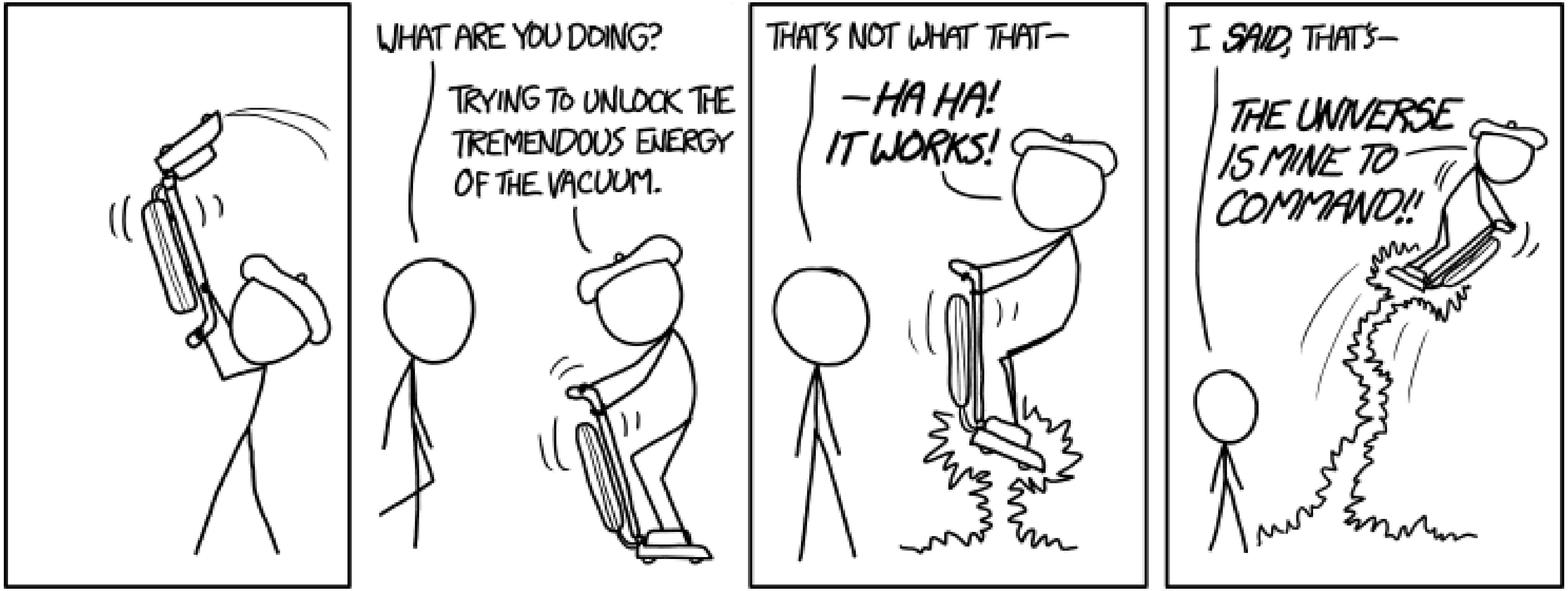
$$W = (\vec{J} \cdot \vec{P}, P_0 \vec{J} + \vec{K} \times \vec{P})$$

It follows that $W_\mu P^\mu = P_\mu P^\mu = 0$ and thus we may choose W to be proportional to P, with proportionality constant "h" dubbed the helicity operator.

We see immediately that $h = \frac{W_0}{P_0} = \frac{\vec{J} \cdot \vec{P}}{P_0} = \frac{\vec{S} \cdot \vec{P}}{P_0}$ and can verify it commutes with the Poincare algebra. Hence it's eigenvalues, which come in half-integer form, supply a classification of massless fields.

Case 3: Massless field with no momentum

The only finite dimensional unitary solution is the trivial one



References

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excellent review of mathematical subjects in physics

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(for the bit on complex vector spaces/holomorphic space, etc)

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(representation theory of SU(2))

Steven Weinberg – Quantum Field Theory vol. 1

Stack Exchange

(what would grad school be like if we couldn't use this?)