*DUE: THURSDAY, JUNE 15, 2017*

---

*FINAL PROJECTS ALERT:* The presentations of the final projects will take place during final exams week. To accommodate the twelve presentations, we will need to hold at least two sessions. Stay tuned for further details. You are strongly encouraged to attend all sessions. The slides from your presentation will be posted to the class website.

---

1. The two-dimensional Poincaré group P(2) is the group consisting of two-dimensional Lorentz transformations [i.e., transformations on 2-vectors $\binom{ct}{x}$ that preserve $x^2 - c^2t^2$] and translations in time and space. P(2) can be represented by $3 \times 3$ matrices acting homogeneously on the column vector, $\begin{pmatrix} ct \\ x \\ 1 \end{pmatrix}$, in analogy with the two-dimensional Euclidean group, E(2), worked out in class.

(a) Find the infinitesimal generators (i.e., differential operators) of the corresponding Lie algebra, $\mathfrak{p}(2)$. Work out the commutation relations of $\mathfrak{p}(2)$.

(b) Compute the Cartan-Killing form. Show that P(2) is noncompact and non-semisimple.

(c) Express the Lie algebra $\mathfrak{p}(2)$ as a semidirect sum of two abelian subalgebras.

2. The Lie algebra of U(2) can be written as a direct sum, $\mathfrak{u}(2) \cong \mathfrak{su}(2) \oplus \mathfrak{u}(1)$. As for the corresponding Lie groups, show that $\mathrm{U}(2) \cong \mathrm{SU}(2) \otimes \mathrm{U}(1)/\mathbb{Z}_2$. How do these results generalize to $\mathrm{U}(n)$ and its Lie algebra $\mathfrak{u}(n)$?

*HINT:* Consider the homomorphism of $(A, e^{i\theta}) \longmapsto e^{i\theta}A$, where $A \in \mathrm{SU}(n)$ and $e^{i\theta} \in \mathrm{U}(1)$.

3. This problem concerns the Lie group SO(4) and its Lie algebra $\mathfrak{so}(4)$.

(a) Work out the Lie algebra $\mathfrak{so}(4)$ and verify that $\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$.

*HINT:* Show that there is a choice of basis for $\mathfrak{so}(4)$ consisting of $4 \times 4$ antisymmetric matrices that contain precisely two non-zero entries: 1 and $-1$. Evaluate the commutation relations of these $\mathfrak{so}(4)$ generators. Then, by choosing a new basis consisting of sums and differences of pairs of the old $\mathfrak{so}(4)$ generators, show that the resulting commutation relations are isomorphic to the commutation relations of the Lie algebra $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$.

(b) What is the universal covering group of SO(4)? What is the center of SO(4)? Identify the adjoint group $\mathrm{Ad}\big(\mathrm{SO}(4)\big)$.

(c) Calculate the Killing form of $\mathfrak{so}(4)$ and verify that this Lie algebra is semisimple and compact.

4. A Lie algebra $\mathfrak{g}$ is defined by the commutation relations of the generators,

$$[e_a, e_b] = f_{ab}^{c} e_c \,.$$

Consider the finite-dimensional matrix representations of the $e_a$. We shall denote the corresponding generators in the adjoint representation by $F_a$ and in an arbitrary irreducible representation $R$ by $R_a$. The dimension of the adjoint representation, $d$, is equal to the dimension of the Lie algebra $\mathfrak{g}$, while the dimension of $R$ will be denoted by $d_R$.

(a) Show that the Cartan-Killing metric $g_{ab}$ can be written as $g_{ab} = \mathrm{Tr}(F_a F_b)$.

(b) If $\mathfrak{g}$ is a simple real compact Lie algebra, prove that for any irreducible representation $R$,

$$\mathrm{Tr}(R_a R_b) = c_R g_{ab} \,,$$

where $c_R$ is called the *index* of the irreducible representation $R$.

*HINT:* Choose a basis where $g_{ab}$ is proportional to $\delta_{ab}$. Then the $f_{ab}^{c}$ are antisymmetric in all three indices. Show that $\mathrm{Tr}\,[R_a, R_b]R_c = \mathrm{Tr}\,R_a[R_b, R_c]$ and argue that this implies that $\mathrm{Tr}\,R_a R_b$, viewed as the $ab$ element of a $d \times d$ matrix, commutes with all Lie algebra elements in the adjoint representation. Finally, invoke Schur's lemma.[1]

(c) The quadratic Casimir operator is defined as $C_2 \equiv g^{ab} e_a e_b$ where $g^{ab}$ is the inverse of $g_{ab}$. Recall that $C_2$ commutes with all elements of the Lie algebra. Hence, by Schur's lemma, $C_2$ must be a multiple of the identity operator. Let us write $C_2 = C_2(R)\mathbf{I}$ where $\mathbf{I}$ is the $d_R \times d_R$ identity matrix and $C_2(R)$ is the eigenvalue of the Casimir operator in the irreducible representation $R$. Show that $C_2(R)$ is related to the index $c_R$ by

$$C_2(R) = \frac{d c_R}{d_R} \,,$$

where $d$ is the dimension of the Lie algebra $\mathfrak{g}$. Check the above formula in the case that $R$ is the adjoint representation.

*HINT:* The matrix elements of the $R_a$ are $(R_a)_{ij}$, where $i, j = 1, \ldots, d_R$. If you keep the matrix element indices explicit, then the derivation of the above result is straightforward.

(d) Compute the index of an arbitrary irreducible representation of $\mathfrak{su}(2)$.

(e) Compute the index of the defining representation of $\mathfrak{su}(3)$. Generalize this result to $\mathfrak{su}(n)$.

5. Various subalgebras of $\mathfrak{su}(3)$ may be identified with specific subsets of the $\mathfrak{su}(3)$ generators.

(a) Show that the Gell-Mann matrices $\lambda_1$, $\lambda_2$, and $\lambda_3$ generate an $\mathfrak{su}(2)$ subalgebra.

(b) Show that the Gell-Mann matrices $\lambda_2$, $\lambda_5$, and $\lambda_7$ generate an $\mathfrak{so}(3)$ subalgebra. (Why do you think I called this an $\mathfrak{so}(3)$ subalgebra rather than an $\mathfrak{su}(2)$ subalgebra?)

---

[1] Note that by complexifying the simple real compact Lie algebra, one can easily show that the above result also holds for any simple complex Lie algebra.

(c) Decompose (if necessary) the three-dimensional irreducible representation of $\mathfrak{su}(3)$ into representations that are irreducible under the subalgebras of parts (a) and (b).

6. Consider the simple Lie algebra $\mathfrak{g}$ generated by the ten $4\times4$ matrices: $\sigma_a\otimes\mathbf{I}$, $\sigma_a\otimes\tau_1$, $\sigma_a\otimes\tau_3$ and $\mathbf{I}\otimes\tau_2$, where $(\mathbf{I},\sigma_a)$ and $(\mathbf{I},\tau_a)$ are the $2\times2$ identity and Pauli matrices in orthogonal spaces. For example, since $\tau_3=\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$, we obtain in block matrix form:

$$\sigma_a\otimes\tau_3=\left(\begin{array}{c|c}\sigma_a & \mathbf{0}\\ \hline \mathbf{0} & -\sigma_a\end{array}\right)\,,\qquad (a=1,2,3)\,,$$

where $\mathbf{0}$ is the $2\times2$ zero matrix. The remaining seven matrices can be likewise obtained. Take $H_1=\sigma_3\otimes\mathbf{I}$ and $H_2=\sigma_3\otimes\tau_3$ as the generators of the Cartan subalgebra. Note that if $A$, $B$, $C$, and $D$ are $2\times2$ matrices, then $(A\otimes B)(C\otimes D)=AC\otimes BD$.

(a) Find the roots of $\mathfrak{g}$. Normalize the roots such that the shortest root vector has length 1. What is the rank of $\mathfrak{g}$?

(b) Determine the simple roots and evaluate the corresponding Cartan matrix. Deduce the Dynkin diagram for this Lie algebra and identify it by name.

(c) The fundamental weights $\boldsymbol{m}_i$ are defined in terms of the simple roots $\boldsymbol{\alpha}_j\in\Pi$ such that

$$\frac{2(\boldsymbol{m}_i,\boldsymbol{\alpha}_j)}{(\boldsymbol{\alpha}_j,\boldsymbol{\alpha}_j)}=\delta_{ij}\,,\qquad \text{for } i,j=1,2,\ldots,r\,,$$

where $r\equiv\operatorname{rank}\mathfrak{g}$. Using the results of part (b), determine all the fundamental weights of $\mathfrak{g}$.

*HINT:* Expand the $\boldsymbol{m}_i$ as a linear combination of the simple roots and solve for the coefficients.

(d) [*EXTRA CREDIT*] Each of the $r$ fundamental weights is the highest weight for an irreducible representation of $\mathfrak{g}$. Collectively, these are called the fundamental (or basic) representations of $\mathfrak{g}$. For each fundamental representation of $\mathfrak{g}$, compute the complete set of weights and draw the corresponding weight diagrams.[2] What are the corresponding dimensions of the fundamental representations of $\mathfrak{g}$.

*HINT:* In this example, all weights of the fundamental representations of $\mathfrak{g}$ appear with multiplicity equal to one. The complete set of weights for the irreducible representations of $\mathfrak{sp}(2,\mathbb{C})\cong\mathfrak{so}(5,\mathbb{C})$ corresponding to the highest weights $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$, respectively, can be obtained by the method of block weight diagrams described in Robert N. Cahn, *Semi-Simple Lie Algebras and Their Representations* (Dover Publications, Inc., Mineola, NY, 2006).[3]

[2]The weight diagrams should be plotted on a two dimensional plane, where the axes correspond to the diagonalized generators normalized such that the shortest root vector has length 1.

[3]However, note that Cahn defines the Cartan matrix that is the transpose of our definition.