Applications of Permutation Groups to the System of Identical Particles

Yuzhan Zhao

University of California
Santa Cruz

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Overview

1. Motivation
   - The wave functions of system of identical particles.
2. General approach
   - A quick review of S_n
   - The Fock states
   - The symmetric and anti-symmetric states
   - The projection operators
   - The Young diagrams
3. Applications and examples
   - Example:S_3
   - Baryons: the flavor wave functions
   - Relation to $SU(N)$ through Young diagrams

Motivation

For a system of identical particles, we want to know the wave functions that describe the system. For example, in molecular physics, getting the wave functions of orbiting electrons can allow us to study the symmetry properties and chemical behaviors. Also, given three quarks with different flavors, we are interested in constructing the wave functions of a baryon.

In fact, the permutation groups are good tools to obtaining these wave functions.

A quick review of S_n

The elements P of the permutation groups S_n can be described as

$$P = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ a_1 & a_2 & a_3 & \dots & a_n \end{pmatrix}, \tag{1}$$

In cycle notation, e.g:

$$(1\,3\,2) \equiv \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}. \tag{2}$$

Every n-cycle $(n > 2)$ can be written as series of transpositions, e.g.

$$(1\,3\,2\,4) = (1\,3)(3\,2)(2\,4), \tag{3}$$

and the sign of $P \in S_n$ is given by

$$sgn(P) = \begin{cases} +1 & \text{even number of transposition.} \\ -1 & \text{odd number of transposition.} \end{cases} \tag{4}$$

The Fock states

Let h_i be the Hilbert space of a single particle, and we can construct a Hilbert space H of N particles through tensor products

$$H = h_1 \otimes h_2 \otimes h_3 \otimes ... \otimes h_n. \tag{5}$$

For simplicity, we describe the N-particles with Fock states

$$\Psi(\lambda_1, \lambda_2, ..., \lambda_n) = \prod_{i=1}^{n} \psi_{\lambda_i}(\vec{x}_i) = \langle \vec{x}_i \vec{x}_2 ... \vec{x}_n | \lambda_1 \lambda_2 ... \lambda_n \rangle \,, \tag{6}$$

$$\left\langle \Psi(\lambda') \middle| \Psi(\lambda) \right\rangle = \prod_{i=1}^{n} \delta_{\lambda_i \lambda_i'}$$

where $\vec{x}_i$ is the generalized coordinates of the correspoding λ_i single partilce state.

The symmetric and anti-symmetric states

If we permute two particles, a phase different will be introduced. Apply the same permutation again, we should obtain the original states. For example, we permute the second and third particles

$$\begin{aligned} (2\,3)\Psi(\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n) &= c\Psi(\lambda_1, \lambda_3, \lambda_2, ..., \lambda_n) \\ c(2\,3)\Psi(\lambda_1, \lambda_3, \lambda_2, ..., \lambda_n) &= c^2\Psi(\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n) \\ \Psi(\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n) &= c^2\Psi(\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n), \end{aligned} \tag{7}$$

where c is a complex number (or phase), and the above results suggest that $c = \pm 1$. In fact, the identicle bosons follow the $+1$ symmetric states for exchanging two particles, and fermions follows the -1 anti-symmetric states.

The projection operators

Now, to construct a total symmetric or anti-symmetric state, we need to build up the projection operators.

We defind the following operators

$$
\begin{aligned}
\hat{S} &= \frac{1}{\sqrt{N!}} \sum_{P \in S_n} P \\
\hat{A} &= \frac{1}{\sqrt{N!}} \sum_{P \in S_n} sgn(P) P.
\end{aligned}
\tag{8}
$$

The projection operators(conti.)

We can show that these two operators are idempotent

$$\hat{S}^2 = \hat{S}$$
$$\hat{A}^2 = \hat{A}, \tag{9}$$

and they also satisfy

$$P\hat{S} = \hat{S}$$
$$P\hat{A} = sgn(P)\hat{A}, \tag{10}$$

for any $P \in S_n$. Basically, with the above properties, these two operators project the Fock states into the bosons (fermions) subspace, and the projected states are totally symmetric (anti-symmetric) under any permutation.

The projection operators(conti.)

Apply these two projection operators to the Fock states we have in previous slides, we get

$$\begin{aligned} \text{Bosons, symmetric:} \Psi_B &= \hat{S}\Psi \\ \text{Fermions, anti-symmetric:} \Psi_A &= \hat{A}\Psi. \end{aligned} \tag{11}$$

Note: In fact, Ψ_A is the same as the Slater determinant.

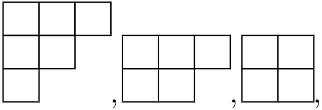
The Young diagrams

Recall that every S_n group is associated with Young diagrams. For the total symmetric states Ψ_B, these are corresponded to the symmetric normal Young diagram

$$\boxed{1}\boxed{2}\boxed{}\ldots\boxed{}\boxed{n},$$

and the anti-symmetric states Ψ_A are corresponded to the "transpose" of the above Young diagram.

Notice that we still have other type of Young diagrams, for example



$, etc$

In fact, these Young diagram are related to the mixed symmetry states.

Example: S_3

To see how it works, we first look at the simple example S_3. First, we construct the projection operators

$$\begin{aligned} \hat{S} &= \frac{1}{\sqrt{6}}(\, e + (1\,2) + (1\,3) + (2\,3) + (1\,2\,3) + (1\,3\,2)\,) \\ \hat{A} &= \frac{1}{\sqrt{6}}(\, e - (1\,2) - (1\,3) - (2\,3) + (1\,2\,3) + (1\,3\,2)\,) \end{aligned} \tag{12}$$

Let's say we have the Fock state $\langle \vec{x}_1\, \vec{x}_2\, \vec{x}_3\, | u\, d\, s \rangle$, and apply the above operators to get the total symmetric and anti-symmetric states

$$\begin{aligned} \psi_S &= \left\langle \vec{x}_1\, \vec{x}_2\, \vec{x}_3\, |\hat{S}| u\, d\, s \right\rangle \\ \psi_A &= \left\langle \vec{x}_1\, \vec{x}_2\, \vec{x}_3\, |\hat{A}| u\, d\, s \right\rangle \end{aligned} \tag{13}$$

Example: S_3 (cont.)

$$\begin{aligned}
\psi_S &= \frac{1}{\sqrt{6}}(\, uds + dus + sdu + usd + dsu + sud \,) \\
\psi_A &= \frac{1}{\sqrt{6}}(\, uds - dus - sdu - usd + dsu + sud \,)
\end{aligned} \tag{14}$$

In fact, these states can be thought as the basis vectors of the group S_3, and there are 4 more states that we need to find. To find the remaining four states, we can symmetrize with respect to two particles and anti-symmetrize another two but with one in common. At the end, we will find that 2 of those 4 states forms an invariant subspace, and the other two form another invariant subspace that is up to equivalent relation.

Example: S_3 (cont.)

This process of finding all the wave functions (basis states) and invariant subspaces is indeed constructing the irreducible representation of the group S_3. Hence, we can simply our previous works by imposing the Young's theorem

Theorem (Young's theorem)

The irreducible representations of S_n are generated by the Young's symmetrizer of associated normal Young tableaux (Recall from lecture that the Young's symmetrizers are constructed with permutation elements that keep the numbers stay within the same row (column) after the operation.)

Baryon: the flavor wave functions

One of the applications of permutation groups is that we can construct the flavor wave functions for baryons. Let's refer the uds from previous example as the quark flavor. For example, we can show one of the mixed symmetric states, which is referred to the normal Young tableaux

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\ , \text{and}\ \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\ .$$

Constructing the Young symmetrizer and using the Young's theorem, and replace s with u, we have (up to normalization)

$$\begin{aligned} \text{symmetric:}\quad & udu + duu - 2uud \\ \text{anti-symmetric:}\quad & udu - duu. \end{aligned} \tag{15}$$

Relation to $SU(N)$ through Young diagram

These are the flavor wave funtions for protons. In order to determine which one is the acutual wave function for protons, we need the taking spin into account. However, involving spin-flavor requires to find the irreducible representation of $6 \otimes 6 \otimes 6 \otimes$ of the $SU(6)$ group. In fact, each irreducible representation of the group $SU(N)$ can be associated with certain Young diagram (as we saw on the lecture), which is related to the permutation groups.

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