The Lorentz and Poincaré groups in relativistic field theory

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Generalized orthogonal groups

- The orthogonal groups O(n), which preserve the norm of a vector in $\mathbb{R}^n$, can be generalized to groups O(m,n) which preserve an indefinite metric $g = \mathbb{1}_{m,n} = \begin{bmatrix} \mathbb{1}_m & 0 \\ 0 & -\mathbb{1}_n \end{bmatrix}$.
- The defining relation for $\Lambda \in$ O(m,n) is

$$\Lambda^T g \, \Lambda = g$$

- For $\vec{x}, \vec{y} \in \mathbb{R}^{m+n}$, $\Lambda \in$ O(m,n), $\vec{y} = \Lambda\vec{x}$ implies

$$y_1^2 + \ldots + y_m^2 - y_{m+1}^2 - \ldots - y_{m+n}^2 = x_1^2 + \ldots + x_m^2 - x_{m+1} - \ldots - x_{m+n}^2$$

Lorentz transformations

- In special relativity we are interested in Lorentz transformations which preserve the ”lengths” of four-vectors in Minkowski space,

$$X^\mu X_\mu = g_{\mu\nu} X^\mu X^\nu = (X^0)^2 - (X^1)^2 - (X^2)^2 - (X^3)^2$$

  where $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ is the Minkowski metric.
- Equivalently, a Lorentz transformation Λ must satisfy $\Lambda^T g \, \Lambda = g$. That is, $\Lambda \in$ O(1,3).

The Lorentz group O(1,3)

- The defining relation $\Lambda^T g\, \Lambda = g$ implies that $|\text{det}\Lambda| = 1$ and $|\Lambda^0{}_0| \geq 1$ for any $\Lambda \in$ O(1,3).
- O(1,3) has 4 connected components corresponding to different possible signs of detΛ and $\Lambda^0{}_0$.
- The component connected to the identity is a subgroup, often called the proper orthochronous Lorentz group $\text{SO}(1,3)^+$.
- We can think of the 4 components as a group: $\{1, P, T, PT\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Relativistic field theory

- To construct a relativistic field theory, we would like our equations of motion for a particular field to hold true in all reference frames. This means the action $S = \int \mathcal{L}(x) d^4x$ should be invariant with respect to any transformation $\Lambda \in \mathrm{SO}(1,3)^+$
- The Lagrangian density $\mathcal{L}$ transforms like a Lorentz scalar :

$$\mathcal{L}(x) \to \mathcal{L}(\Lambda^{-1}x)$$

Relativistic field theory

- Consider a multiplet field Ψ_a with n components. It must transform as $\Psi_a(x) \to M_{ab}(\Lambda)\Psi_b(\Lambda^{-1}x)$, where $M_{ab}(\Lambda)$ is an n$\times$n matrix. This requires us to construct an n-dimensional representation of SO$(1,3)^+$.

The Lie algebra $\mathfrak{so}(1,3)$

- Linearizing the defining relation about the identity, we find that $x \in \mathfrak{so}(1,3)$ must satisfy $x^T g + gx = 0$. x is a 4×4 matrix, and this relation imposes 10 conditions on the elements of x. We can construct a basis $a_1, \ldots, a_6$ of the Lie algebra consisting of matrices J_i and K_i for $i = 1, 2, 3$.

The Lie algebra $\mathfrak{so}(1,3)$

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The J_i are clearly the generators of the SO(3) subgroup. The corresponding one-parameter subgroups are finite spatial rotations.

The Lie algebra $\mathfrak{so}(1,3)$

$$K_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad K_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

The K_i are the generators of Lorentz boosts.

The Lie algebra $\mathfrak{so}(1,3)$

In this basis, the algebra looks like :

$$[J_i, J_j] = \epsilon_{ijk} J_k$$

$$[J_i, K_j] = \epsilon_{ijk} K_k$$

$$[K_i, K_j] = -\epsilon_{ijk} J_k$$

for $i, j = 1, 2, 3$

The Lie algebra $\mathfrak{so}(1,3)$

- We can consider the complexification of the Lie algebra, denoted by $\mathfrak{so}(1,3)_{\mathbb{C}}$, in order to construct a basis that is equivalent to a direct sum of Lie algebras (that is, the complexification $\mathfrak{so}(1,3)_{\mathbb{C}}$ is semi-simple)
- Define a new basis through complex linear combinations of the original basis vectors :

$$\vec{\boldsymbol{J}}_{+} \equiv \frac{1}{2}(\vec{\boldsymbol{J}} + i\vec{\boldsymbol{K}})$$

$$\vec{\boldsymbol{J}}_{-} \equiv \frac{1}{2}(\vec{\boldsymbol{J}} - i\vec{\boldsymbol{K}})$$

The Lie algebra $\mathfrak{so}(1,3)$

This new basis satisfies the following commutation relations:

$$[J_+^i, J_+^j] = \epsilon^{ijk} J_+^k$$

$$[J_-^i, J_-^j] = \epsilon^{ijk} J_-^k$$

$$[J_+^i, J_-^j] = 0$$

In this basis, it is clear that $\mathfrak{so}(1,3)_{\mathbb{C}} \cong \mathfrak{su}(2)_{\mathbb{C}} \bigoplus \mathfrak{su}(2)_{\mathbb{C}}$. At the group level we have $\mathrm{SO}(1,3)^+ \cong \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$.

Finite dimensional reps. of $\mathrm{SO}(1,3)^+$

- The previous result implies we can characterize a representation of $\mathrm{SO}(1,3)^+$ by a pair of half-integer numbers, (s_1, s_2) where $s_1, s_2 = 0, \frac{1}{2}, 1, \ldots$
- The dimension of this representation is $(2s_1 + 1)(2s_2 + 1)$.
- These representations define the possible types of fields which can be described by our relativistically invariant field theory

Finite dimensional reps. of $\mathrm{SO}(1,3)^+$

- $(0,0) \rightarrow$ scalar fields, Φ
- $(\frac{1}{2},0) \rightarrow$ left chiral Weyl spinor, ψ_L
- $(0,\frac{1}{2}) \rightarrow$ right chiral Weyl spinor, ψ_R
- $(\frac{1}{2},\frac{1}{2}) \rightarrow$ four-vector, V^μ

Spin $\frac{1}{2}$ fields

- Note that there are two different 2-dimensional representations which are appropriate for describing the transformations of a spin $\frac{1}{2}$ field : $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$
- These representations describe Weyl fermions ψ_L and ψ_R of left and right chirality respectively. ψ_L and ψ_R are fundamentally different degrees of freedom - see neutrinos, and electroweak theory.
- Under infinitesimal Lorentz transformations,

$$\psi_L \to (1 - i\vec{\boldsymbol{\theta}} \cdot \vec{\boldsymbol{\sigma}}/2 - \vec{\boldsymbol{\zeta}} \cdot \vec{\boldsymbol{\sigma}}/2)\psi_L$$

$$\psi_R \to (1 - i\vec{\boldsymbol{\theta}} \cdot \vec{\boldsymbol{\sigma}}/2 + \vec{\boldsymbol{\zeta}} \cdot \vec{\boldsymbol{\sigma}}/2)\psi_R$$

- In the massless limit, a state of definite chirality is also a state of definite helicity.

Spin $\frac{1}{2}$ fields

- In QED, one often uses four component Dirac spinors which mix ψ_L and ψ_R
- This corresponds to the $(0, \frac{1}{2}) \bigoplus (\frac{1}{2}, 0)$ representation. The generators are

$$S^{0i} = -\frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$$

$$S^{ij} = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$$

for $i, j = 1, 2, 3$.

Finite dimensional reps. of $\mathrm{SO}(1,3)^+$

- One can show that the Klein-Gordon equation, $(\partial^2 + m^2)\Phi(x) = 0$ is invariant under the transformation $\Phi(x) \to \Phi(\Lambda^{-1}x)$
- Similarly, the Dirac equation $(i\gamma^\mu \partial\mu - m)\psi(x) = 0$ is invariant under the transformation $\psi(x) \to \exp(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu})\psi(\Lambda^{-1}x)$
- And, the Weyl equations $i\overline{\sigma}^\mu \partial_\mu \psi_L = 0$ and $i\sigma^\mu \partial_\mu \psi_R = 0$ are invariant under transformations of ψ_L and ψ_R under the 2-dimensional fundamental / anti-fundamental reps. of SU(2) respectively.

Finite dimensional reps. of $\mathrm{SO}(1,3)^+$

Important note : SO(1,3) is non-compact. This means that the finite-dimensional representations are not unitary - we can see this by looking at the boost component of the general transformation. In a quantum theory we need unitary representations. Basis states of the Hilbert space of a quantum field theory transform under unitary *infinite*-dimensional representations of the full Poincaré group.

Poincaré group

- The Poincaré group is also known as the inhomogeneous Lorentz group, or ISO(1,3). This group contains all of SO(1,3) as a subgroup, with 4 additional generators P^μ that generate translations in spacetime.
- We can outline the strategy for classifying the infinite-dimensional unitary representations of the Poincaré group.

Infinite dimensional reps. of the Poincaré group

- Consider the abelian invariant subgorup T_4 of ISO(1,3) (the translation group in four dimensions).
- Basis vectors of our representation are built out of eigenvectors of the generators of this group (P^μ), plus those of commuting operators from the Lie algebra of the little group.
- Basis vectors are characterized by their eigenvalues with respect to quadratic Casimir operators : $C_1 = P_\mu P^\mu$ (eigenvalue c_1 =mass) and $C_2 = W_\lambda W^\lambda$, where

$$W^\lambda \equiv \epsilon^{\lambda\mu\nu\sigma} J_{\mu\nu} P_\sigma / 2$$

is called the Pauli-Lubanski vector.

Infinite dimensional reps. of the Poincaré group

- Four possible cases :
  - $c_1 = 0, p^\mu = 0$ (vacuum)
  - $c_1 > 0$ (massive particle)
  - $c_1 = 0, p^\mu \neq 0$ (massless particle)
  - $c_1 < 0$
- Each case leads to a different little group of the factor group SO(1,3). Each irreducible unitary representation of the little group then induces an irreducible unitary representation of the full Poincaré group.

Case 1: $c_1 = 0, p^\mu = 0$

The little group is the subgroup of the factor group $ISO(1,3)/T_4 \cong \mathrm{SO}(1,3)$ which leaves the test vector invariant. In this case, the little group is the full homogeneous Lorentz group SO(1,3).

Case 2: $c_1 > 0$

For a massive particle, we can always boost to a rest frame where $p^\mu = (M, 0, 0, 0)$. In this case the little group is just the rotation group, SO(3). The basis vectors we will choose will satisfy:

$$\boldsymbol{P}^\mu \left|\mathbf{0}\lambda\right\rangle = p^\mu \left|\mathbf{0}\boldsymbol{\lambda}\right\rangle$$

$$\boldsymbol{J}^{\mathbf{2}} \left|\mathbf{0}\lambda\right\rangle = s(s+1) \left|\mathbf{0}\lambda\right\rangle$$

$$\boldsymbol{J}_3 \left|\mathbf{0}\lambda\right\rangle = \lambda \left|\mathbf{0}\lambda\right\rangle$$

Here, $\mathbf{0}$ is labeling the three-momentum $\vec{\boldsymbol{p}} = 0$.

Case 2: $c_1 > 0$

By acting on these basis vectors with elements of SO(1,3) not in the little group, we can obtain a general state $|\boldsymbol{p}\lambda\rangle$. The result is a representation of the Poincaré group labeled by (M, s) that is irreducible, unitary and infinite-dimensional.

Case 3: $c_1 = 0, p^\mu \neq 0$

This corresponds to a massless particle moving with some momentum ω, so we can write the four-momentum as $p^\mu = (\omega, 0, 0, \omega)$. The little group is even smaller: SO(2). The main difference is that in this case, the helicity λ is Lorentz invariant.

Infinite dimensional reps. of the Poincaré group

Result:

- Massless states are identified with basis vectors of infinite-dimensional irreps. labeled by a momentum $\vec{\boldsymbol{p}}$ and a helicity λ. There are two distinct possible states corresponding to either $\lambda = s$ or $\lambda = -s$ for some $s = 0, \frac{1}{2}, 1, \ldots$
- Massive states are identified with basis vectors of infinite-dimensional irreps. labeled by a mass M, a momentum $\vec{\boldsymbol{p}}$, and a helicity $\lambda = -s, \ldots, s$ for some $s = 0, \frac{1}{2}, 1, \ldots$

References

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