Group Theory of Spontaneous Symmetry Breaking

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Physics 251

June 7, 2019

Classical Analogy

- Imagine trying to balance a rotationally symmetric pencil on a point. Any tiny perturbation would destabilize the unstable equilibrium and cause it to topple over.
- A collision with an air molecule or, even if in vacuum, a stray cosmic ray or quantum fluctuation would cause the pencil to fall over.
- In a perfect mathematical world, this rotational symmetry would be realized, but in the real and messy universe we exist in, we say these would-be symmetries are spontaneously broken.

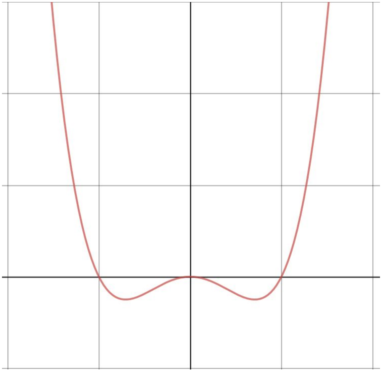
Quantum Mechanically

- Similar to the classical analogy, one could imagine an electron's magnetic moment anti-aligned with an external magnetic field. This exemplifies another unstable equilibrium.
- We, however, want to generalize our search for spontaneously broken symmetries to symmetries beyond what we can intuitively imagine. Hence, for arbitrary potentials, we search for local maxima.

Discrete Symmetry Example

- Consider the Lagrangian $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$
- We then have a potential $V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4$
- Consider a scenario in which we have $m^2 < 0$ and $\lambda > 0$. More obvious if we let $m^2 \rightarrow -\mu^2 < 0$
- Then our potential appears quartic with a local maximum and two minima. $V(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4!}\phi^4$

- $V(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4!}\phi^4$
- It is energetically favorable for ϕ to take the values that minimize the potential $V(\phi)$.



- We can find the minima by solving $\frac{\partial \mathcal{L}}{\partial \phi}\Big|_{\phi=v} = 0$
- $\implies \left(\mu^2 - \frac{\lambda}{6} v^2\right) v \implies v = \pm\sqrt{\frac{6\mu^2}{\lambda}}$
- What does this have to do with SSB? Recall how our Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!}\phi^4$$

is invariant under $\phi \to -\phi$. This is the spontaneously broken symmetry which, despite our Lagrangian being invariant under, is not obeyed by the ground state.

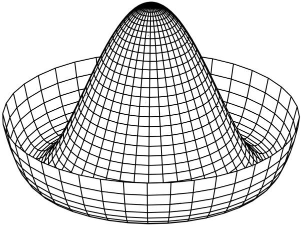
Mexican Hat

- Consider the Lagrangian $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^*(\partial^\mu \phi) - m^2\phi^*\phi - \frac{\lambda}{4}(\phi^*\phi)^2$
- We then have a potential $V(\phi) = m^2\phi^*\phi + \frac{\lambda}{4}(\phi^*\phi)^2$
- Again, suppose $m^2 = -\mu^2 < 0$ and $\lambda > 0$ We now look for the values of $\phi \in \mathbb{C}$ for which $V(\phi)$ is minimal.

- As in the previous example, we can find the minima: $|\phi|^2 = \frac{v^2}{2}$
- Recall how our Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^*(\partial^\mu \phi) - m^2\phi^*\phi - \frac{\lambda}{4}(\phi^*\phi)^2$$

is invariant under U(1). This is the spontaneously broken symmetry which, despite our Lagrangian being invariant under, is not obeyed by the ground state.



- We can now parameterize our field

$$\phi = \frac{1}{\sqrt{2}}\left(v+\rho\right) exp\left[i\pi/v\right]$$

  where ρ indicates perturbations about the equilibrium position and π is some complex phase. $(\rho, \pi \in \mathbb{R})$
- We can re-express our Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\rho)(\partial^\mu\rho) + \frac{1}{2}(\partial_\mu\pi)(\partial^\mu\pi)(1+\frac{\rho}{v})^2$$

$$+\frac{\lambda}{16}v^2 - \lambda\left(\frac{v}{2}\right)^2\rho^2 + \text{higher order } \rho \text{ terms}$$

- Note that while ρ has both a kinetic and a mass term, π only has a kinetic term. $\implies$ π is a massless particle.

Goldstone Theorem

- The spontaneous breaking of a continuous global symmetry implies the existence a massless particle. (Keep in mind for later examples)
- Mathematically: There corresponds a massless particle to every generator of a continuous global symmetry that does not annihilate the ground state.

Higgs Mechanism

- Let's now consider the spontaneous breaking of gauge symmetries.
- Let's complicate our "mexican hat" Lagrangian by including a gauge field term and including a covariant derivative.

$$\mathcal{L} = \frac{1}{2}(D_\mu \phi)^*(D^\mu \phi) - m^2 \phi^* \phi - \frac{\lambda}{4}(\phi^* \phi)^2 - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

- Substituting $\phi = \frac{1}{\sqrt{2}}(v + \rho) exp\left[i\frac{\pi}{v}\right]$ and performing the transformation of the gauge field: $A_\mu \rightarrow A_\mu + \frac{1}{e}\frac{\partial_\mu \pi}{v}$, we have:

$$\mathcal{L} = \frac{1}{2} e^2 v^2 A_\mu A^\mu + (\rho,\, A_\mu \text{ and their interactions terms})$$

- In words: Higgs Mechanism gives otherwise massless fields mass via spontaneous symmetry breaking.

Summary of Higgs Mechanism

- Say we have the spontaneous breaking of group $G \to H$.
- Gauge bosons corresponding to the generators that do not annihilate the ground state become massive. (Higgs Mechanism)
- The remaining gauge bosons stay massless.

Example: Triplet of Scalars Bosons

- Consider a triplet of scalar bosons interacting with a gauge field $\vec{A}_\mu$

$$\mathcal{L} = -\frac{1}{4}\vec{F_{\mu\nu}} \cdot \vec{F^{\mu\nu}} + \frac{1}{2}[(\partial_\mu - g\vec{t} \cdot \vec{A}_\mu)\phi]^2 + \frac{1}{2}\mu^2(\vec{\phi} \cdot \vec{\phi}) - \frac{1}{4}\lambda(\vec{\phi} \cdot \vec{\phi})^2$$

- This Lagrangian is invariant under the gauge transformations:

$$\vec{A}_\mu \to \vec{A}_\mu + \vec{\epsilon} \times \vec{A}_\mu + \frac{1}{g}\partial_\mu \vec{\epsilon}$$

$$\vec{\phi} \to \vec{\phi} + \vec{\epsilon} \times \vec{\phi}$$

- We can realize SSB if we let the third component of the scalar feild have a nonzero vacuum expectation value, that is:

$$\langle 0|\,\phi_3\,|0\rangle = v, \text{ and we define } \phi'_i \equiv \phi_i - \delta_{i3} v$$

- Therefore we have $<\phi'_i>= 0$ and the Lagrangian becomes:

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu)^2 + \frac{1}{2} g v^2 (A_{1\mu}^2 + A_{2\mu}^2) + \frac{1}{2}(\partial_\mu \vec{\phi'})^2 +$$

$$[\frac{1}{2}(\mu^2 - \lambda v^2)(\vec{\phi'} \cdot \vec{\phi'} - \lambda v^2 {\phi'_3}^2)] + (\mu^2 - \lambda v^2) v \phi'_3 + \text{interactions}$$

- We find, in a similar manner to above, that $v = \sqrt{\frac{\mu^2}{\lambda}}$
- We observe that that the gauge bosons $A_{\mu 1}$, $A_{\mu 2}$ get masses $g v^2$, while $A_{\mu 3}$ remains massless. Hence the gauge symmetry of the Lagrangian has been reduced from $SU(2)$ to $U(1)$.

- Furthermore one can read from the Lagrangian that ϕ_1' and ϕ_2' have no mass terms, and thus are the massless Goldstone bosons.
- The general feature of this type of symmetry breaking is to have as many zero-mass scalar bosons as massive gauge bosons.

- A more intuitive example is

$$V(\phi) = -\frac{1}{2}\mu^2(\vec{\phi}\cdot\vec{\phi}) + \frac{1}{4}\lambda(\vec{\phi}\cdot\vec{\phi})^2$$

- This potential is minimized for $\vec{\phi}\cdot\vec{\phi} = \frac{\mu^2}{\lambda}$. This is obviously invariant under $O(3)$. Furthermore we can construct any such vector $\vec{\phi}$ by rotating $\phi_3 = \sqrt{\mu^2/\lambda}$ using $O(3)$. From this form it is obvious that the symmetry group $O(2)$ is left unbroken.

General Procedure

1. Choose a particular representation for the scalar boson and write down the most general group-invariant potential $V(\phi)$ which is a fourth order polynomial of scalar fields.
2. Find the minimum of $V(\phi)$ by solving the equation $\frac{\partial V(\phi)}{\partial \phi} = 0$
3. Calculate the number of massless gauge bosons and therefrom determine the unbroken symmetry.

$O(n)$ Symmetry Breaking

Representation	$O(n)$
vector	$O(n)$
k-vector	$O(n-k)$
2nd-rank symmetric tensor	$O(n-1)$ or $O(l)(n-l), l=[\frac{1}{2}n]$
2nd-rank anti-symmetric tensor	$U(l)$ or $U(1)(n-2)$

Table 1: Summary of the Pattern of Symmetry Breaking of $O(n)$

$SU(n)$ Symmetry Breaking

Representation	$SU(n)$
vector	$SU(n-1)$
k-vector	$SU(n-k)$
2nd-rank symmetric tensor	$SU(n-1)$ or $O(n)$
2nd-rank anti-symmetric tensor	$O(2l+1)$
	or $SU(n-2), l = [\frac{1}{2}n]$
adjoint representation	$SU(l) \times SU(n-l) \times U(1)$
	or $SU(n-1), l = [\frac{1}{2}n]$

Table 2: Summary of the Pattern of Symmetry Breaking of $SU(n)$

Pseudo-Goldstone Bosons

- The quantum chromodynamics Lagrangian exhibits an approximate symmetry. In the limit of the quarks masses being approximately zero, a $SU(3)_L \times SU(3)_R$ symmetry is realized and spontaenously broken.
- This leads to "pseudo" Goldstone bosons with masses approximately zero.

References

- Group Structure of Gauge Theories - O'Raifeartaigh L.
- Group Theory of Spontaneously Broken Gauge Symmetries - Ling-Fong Li
- An Introduction to Quantum Field Theory - Peskin and Schroeder