Coherent States in Quantum Mechanics

Gavin Rockwood

June 14, 2019

Overview

1. Coherent States of the Quantum Harmonic Oscillator
   - Introduction
   - Minimal Uncertainty States
   - The Displacement Operator
2. General Coherent States
   - Definition
   - Heisenberg-Weyl Group
   - Coherent Spin States
3. Applications
   - Coherent States in Optics
   - Coherent States in Superfluids

Coherent States of the Quantum Harmonic Oscillator

The Quantum Harmonic Oscillator

For our discussion, we will deal with the harmonic oscillator in 1 dimension!

The Hamiltonian

$$\hat{H} = -\hbar^2 \frac{\hat{P}^2}{2m} + \frac{1}{2} m\omega \hat{X}^2 \tag{1}$$

$$\begin{aligned} \hat{a} &:= \frac{1}{\sqrt{2\hbar m\omega}} \left(m\omega \hat{Q} + i\hat{P} \right) \\ \hat{a}^\dagger &:= \frac{1}{\sqrt{2\hbar m\omega}} \left(m\omega \hat{Q} - i\hat{P} \right) \end{aligned} \tag{2}$$

$$\implies \hat{H} = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) \tag{3}$$

Minimal Uncertainty States

For the QHO, lets calculate $\Delta X \Delta P$.

Show $\Delta X \Delta P =$

Starting with:

$$\hat{X} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a} + \hat{a}^{\dagger}\right) \qquad \hat{P} = -i\sqrt{\frac{\hbar m\omega}{2}}\left(\hat{a} - \hat{a}^{\dagger}\right) \tag{4}$$

With this we have:

$$\Delta x \Delta p = \sqrt{\langle n|\,\hat{X}^2\,|n\rangle - \langle n|\,\hat{X}\,|n\rangle^2}\sqrt{\langle n|\,\hat{P}^2\,|n\rangle - \langle n|\,\hat{P}\,|n\rangle^2} \tag{5}$$

$$= \frac{\hbar}{2}(2n+1) \tag{6}$$

We see that $n = 0$ saturates the Heisenberg Uncertainty relation: $\Delta x \Delta p \geq \frac{\hbar}{2}$.

Another Minimal Uncertainty State

Lets define the following state:

$$|z\rangle = e^{-\frac{1}{2}|z|^2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle \, ; \quad z \in \mathbb{C} \tag{7}$$

This state is exciting because $\hat{a}\,|z\rangle = z\,|z\rangle$. Also $\langle z|z\rangle = 1$. Its good to note that $|z\rangle$ can also be written as:

$$|z\rangle = e^{-\frac{1}{2}|z|^2 + z\hat{a}} |0\rangle \tag{8}$$

Note!

It is worth noting that there is a yet unmentioned way to write $|z\rangle$ that will be discussed shortly

Another Minimal Uncertainty State Continued

As mentioned, $|z\rangle$ is a minimal uncertainty state. To see this:

Proof that $|z\rangle$ is a minimal uncertainty state.

Starting with:

$$\begin{aligned}
\langle z|\hat{X}|z\rangle &= \sqrt{\frac{\hbar}{2m\omega}}\left(\langle z|\hat{a}|z\rangle + \langle z|\hat{a}^\dagger|z\rangle\right) = \sqrt{\frac{\hbar m\omega}{2}}(z+z^*) \\
\langle z|\hat{P}|z\rangle &= -i\sqrt{\frac{\hbar m\omega}{2}}(z-z^*) \\
\langle z|\hat{X}^2|z\rangle &= \sqrt{\frac{\hbar}{2m\omega}}\left((z+z^*)^2+1\right) \\
\langle 2|\hat{P}^2|z\rangle &= sqrt\frac{\hbar m\omega}{2}\left((z-z^*)^2-1\right)
\end{aligned} \tag{9}$$

With this:

$$\Delta x \Delta p = \frac{\hbar}{2} \tag{10}$$

The Translation Operator

Let us define the following operator, called the Heisenberg-Weyl Translation operator.

$$\hat{T}(z) = e^{\frac{i}{\hbar}(p\hat{Q} - q\hat{P})}, \quad \frac{1}{\sqrt{2\hbar m\omega}}(m\omega q + ip) \in \mathbb{C} \tag{11}$$

This operator can also be written in terms of creation and annihilation operators:

$$\hat{T}(z) = e^{z\hat{a}^\dagger - z^*\hat{a}} \tag{12}$$

This last version is the one most commonly seen in discussions about coherent states.

We will go back and explore the Heisenberg-Weyl group in greater depth when we talk about general coherent states

The Action of $\hat{T}(z)$ on $\hat{Q}$ and $\hat{P}$

We are after $\hat{T}(z)\hat{Q}\hat{T}^{-1}(z)$ and $\hat{T}(z)\hat{P}\hat{T}^{-1}(z)$.

Start with $\hat{T}(z)\hat{a}\hat{T}^{-1}(z)$.

$$\begin{aligned}\hat{T}(z)\hat{a}\hat{T}^{-1}(z) &= \hat{a} - \left[\hat{a}, z\hat{a}^\dagger - z^*\hat{a}\right] \\ &= \hat{a} - z\end{aligned}$$

By expanding $\hat{a}, z$ in terms of $\hat{Q}, \hat{P}, q, p$ we get:

$$\begin{aligned}\hat{T}(z)\hat{Q}\hat{T}^{-1}(z) &= \hat{Q} - q \\ \hat{T}(z)\hat{P}\hat{T}^{-1}(z) &= \hat{P} - p\end{aligned} \tag{13}$$

$|z\rangle$ from $\hat{D}(z)$

Earlier I mentioned that there was another way to write $|z\rangle$ and that is as:

$$|z\rangle = \hat{D}(z)\,|0\rangle \tag{14}$$

Some Quick Observations

If we let ϕ_z be the position wavefunction of $|z\rangle$ then:

- $\hat{T}(0)\,|0\rangle = |0\rangle \implies \hat{T}(0)\phi_0 = \phi_0$ where ϕ_0 is the groundstate wavefunction of the harmonic oscillator. In other words, the groundstate is a coherent state.
- $$\begin{aligned}\hat{T}(z)\phi_0(x) &= e^{-\frac{i}{2\hbar}qp}e^{\frac{i}{\hbar}xp}\phi_0(x-q)\\ \hat{T}(z)\mathcal{F}[\phi_0](k) &= e^{-\frac{i}{2\hbar}qp}e^{\frac{i}{\hbar}qk}\mathcal{F}[\phi_0](k-p)\end{aligned} \tag{15}$$

  Where $\mathcal{F}[\phi_0](k)$ is the Fourier transform of ϕ_0. Here, we see that $|z\rangle$ is now localized at $\hat{Q} = q$ and $\hat{P} = p$ in phase space.

Time Evolution!

The time evolution of $\hat{T}(z)$ is given by:

$$\hat{T}(z_t) = e^{-iH_{\text{HO}}t/\hbar}\hat{T}(z_0)e^{iH_{\text{HO}}t/\hbar} \tag{16}$$

With this, the time evolution of a coherent state is given by:

$$e^{-iH_{\text{HO}}t/\hbar}\left|z_0\right\rangle = \left|z(t)\right\rangle, \quad z(t) = e^{i\omega t}z_0 \tag{17}$$

Time Evolution of $\langle Q \rangle$ and $\langle P \rangle$

$$\begin{aligned} \langle z(t)| \hat{Q} |z(t)\rangle &= p_0 \sin(\omega t) - q_0 \cos(\omega t) \\ \langle z(t)| \hat{P} |z(t)\rangle &= -q_0 \sin(\omega t) - p_0 \cos(\omega t) \end{aligned} \tag{18}$$

General Coherent States

Definition of General Coherent State

Let G be an arbitrary Lie group and T be an irreducible representation acting on a Hilbert space $\mathcal{H}$. Pick a vector $|\psi_0\rangle \in \mathcal{H}$, then the coherent state system $\{T, |\psi_0\rangle\}$ is the set $\{|\psi\rangle \in \mathcal{H} \text{ s.t. } |\psi\rangle = T(g)|\psi_0\rangle, g \in G\}$.

Now, we will want to look at a subgroup $H = \left\{g \in G \text{ s.t. } T(g)|\psi_0\rangle = e^{i\alpha(g)}|\psi_0\rangle\right\}$. H is the isotropy group of $|\psi_0\rangle$.

Definition

Then a general coherent state $|\psi_g\rangle = g|\psi_0\rangle$ is determined by a point $x = x(g)$ in the coset space G/H such that $|\psi_g\rangle = e^{i\alpha(x)}|x\rangle$.

Looking at the Heisenberg - Weyl Group

They algebra of the Heisenberg-Weyl group in n dimensions (denoted as W_n is generated by I, $Q_1, ...Q_n$ and $P_1, ...P_n$.

$$\begin{gathered}[Q_i, Q_j] = [P_i, P_j] = [I, Q_i] = [I, P_i] = 0 \\ [Q_i, P_j] = \delta_{ij}\hbar I\end{gathered} \tag{19}$$

The Lie Algebra $\mathfrak{h}_n$ is generated by $I \to i\hat{I}$, $Q \to i\hat{Q}$ and $P \to i\hat{P}$. Thus, any element of $\mathfrak{h}_n$ can be written as:

$$\hat{W} = \frac{it}{2\hbar}I + \frac{i}{\hbar}\left(p \cdot \hat{Q} - q \cdot \hat{P}\right) = \frac{it}{2\hbar} + \frac{i}{\hbar}\hat{L}(z), \quad \hat{L}(z) := \left(p \cdot \hat{Q} - q \cdot \hat{P}\right), z = (q, p) \in \mathbb{R}^{2n} \tag{20}$$

So, $e^{\hat{W}} = e^{\frac{it}{2\hbar}} e^{\frac{i}{\hbar}\hat{L}(z)} = e^{\frac{it}{2\hbar}} \hat{T}(z)$

Some Properties of $\hat{T}$

The product of $\hat{T}(z)$ and $\hat{T}(z')$.

$$\hat{T}(z)\hat{T}(z') = e^{-\frac{i}{2\hbar}(q\cdot p' - p\cdot q')}\hat{T}(z+z') \in W_n \tag{21}$$

The identity operator is $e^{\hat{W}(t=0,z=0)}$ and for a given state $|\psi\rangle$, the isotropy group is given by $H_n = \left\{e^{\hat{W}(t,0)}\right\} \cong U(1)$.

The set of coherent operators is $W_n/H_n = \left\{\hat{T}(z), z = (q,p) \in \mathbb{R}^{2n}\right\}$

So Back to the Harmonic Oscillator

If we go back to the harmonic oscillator, we had a ground state $|0\rangle$, The group we are considering is W_1, the isotropy group is still $\left\{e^{\hat{W}(t=0,z=0)}\right\} \cong U(1)$. Thus coherent states of the Heisenberg-Weyl group are generated by $T(z), z = (q,p) \in \mathbb{R}^2$.

A Few More Notes on the Harmonic Oscillator

- The set $\left\{ \hat{T}(z)\left|0\right\rangle \right\}$ is complete but not necessarily orthogonal:

$$\left\langle z' | z \right\rangle = \left\langle 0 \right| \hat{T}^{-1}(z')\hat{T}(z)\left|0\right\rangle \tag{22}$$

$$= \left\langle 0 \right| T(-z')T(z)\left|0\right\rangle \tag{23}$$

$$= e^{\frac{i}{2\hbar}(q'p-p'q)}\left\langle 0 \right| \hat{T}(z'+z)\left|0\right\rangle \tag{24}$$

$$= \exp\left[\frac{i}{2\hbar}\left(q'p-p'q\right) - \frac{(q'-q)^2+(p'-p)^2}{4\hbar}\right] \tag{25}$$

$$\neq 0 \tag{26}$$

This last part comes from the fact $\left\langle 0 \right| \hat{T}(z)\left|0\right\rangle = \exp\left(-\frac{q^2+p^2}{4\hbar}\right)$. This implies $\left\{ \hat{T}(z)\left|0\right\rangle \right\}$ is in fact overcomplete.

Minimal Uncertainty States

Let $|j, m\rangle$ be a state of spin j and z projection m. Then $J_3 |j, m\rangle = m |j, m\rangle$ and $J^2 |j, m\rangle = j(j+1) |j, m\rangle$. The uncertainty relation $\Delta J_1 \Delta J_2 \geq \frac{1}{2} \langle J_3 \rangle$.

$$\Delta J_1 \Delta J_2 = \frac{1}{2} \left(j(j+1) - m^2 \right) \tag{27}$$

States of minimal uncertainty are given by $m = \pm j$.

Sooo, whats G and H?

Our Lie group for this system is $G = SU(2)$. With this, we can go back to our definition of a general coherent state. The isotropy group is $U(1)$ and thus the phase space is $SU(2)/U(1)$ which is topologically equivalent to S^2.

Now! Lets construct them!

~~Constructing Spin Coherent States~~ Defining Some Ladder Operators

Lets start by defining:

Ladder Operators!

$$\hat{J}_{\pm} = \hat{J}_1 \pm i\hat{J}_2 \tag{28}$$

where

$$\hat{J}_{\pm} |j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\, |j, m \pm 1\rangle \tag{29}$$

and

$$\begin{aligned} \left[\hat{J}_3, \hat{J}_{\pm}\right] &= \pm\hat{J}_{\pm} \\ \left[\hat{J}_+, \hat{J}_-\right] &= 2\hat{J}_3 \end{aligned} \tag{30}$$

Ok, Now Start to Construct Spin Coherent States

We begin by defining an operator:

$$\hat{D}(\xi) = e^{\xi \hat{J}_+ - \xi^* \hat{J}_-}, \quad \xi = -\tan\left(\frac{\theta}{2}\right) e^{-i\phi} \tag{31}$$

Then, in a similar fashion as earlier, a coherent state is defined as:

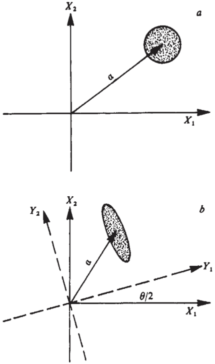
$$|\xi\rangle = \hat{D}(\xi)\,|j,-j\rangle = \sum_{m=-j}^{j} \left[\frac{(2j)!}{(j+m)(j-m)}\right]^{\frac{1}{2}} (1+|\xi|^2)^{-j} \xi^{j+\mu}\,|j,m\rangle \tag{32}$$

Applications

The Squeeze Operator

The squeeze operator is defined as:

$$S(\zeta) := e^{\frac{1}{2}\left(\zeta^* a^2 - \zeta a^{\dagger 2}\right)}, \quad \zeta = re^{i\theta} \tag{33}$$



Why are These Useful?

- Optical Communication: Helps improve signal to noise ratio
- Interferometry: LIGO!
- Some weird quantum information stuff

Coherent States of Fields

Superfluids are discussed in terms of bosonic fields. So with that in mind, lets define:

$$|z_k\rangle = e^{-\frac{1}{2}|z|^2} \sum_{n_k}^{\infty} \frac{z_k^n}{\sqrt{n_k!}} |n_k\rangle \tag{34}$$

Where n_k is the number of states in the k^{th} mode of the field. We have the completeness relation:

$$\frac{1}{\pi} \int dz_k \, |z_k\rangle \langle z_k| = 1 \tag{35}$$

The set of all states of the form $\prod_k |z_k\rangle$ form a complete set of states.

Free Energy of $|\{z\}\rangle$

The free energy of $|\{z\}\rangle$ is given by:

$$F\{z\} = -k_b \ln\left[\langle\{z\}|\exp\left(-\frac{\hat{H}^{(0)} - \mu\hat{N}}{k_b T}\right)|\{z\}\rangle\right] \tag{36}$$

Which to first order goes like:

$$F^{\{0\}}\{z\} = -k_b \sum_k \left(e^{-\frac{\frac{1}{2}\hbar\omega - \mu}{k_b T}} |z_k|^2\right) \tag{37}$$

This is a good approximation for a Bose-Einstein condensate. Adding a quadratic interaction term to 36 will approximate the free energy of a superfluid because it allows the BEC to "flow".

[1] J. Aasi, J. Abadie, B. P. Abbott, R. Abbott, T. D. Abbott, M. R. Abernathy, C. Adams, T. Adams, P. Addesso, R. X. Adhikari, and et al. Enhanced sensitivity of the LIGO gravitational wave detector by using squeezed states of light. *Nature Photonics*, 7:613–619, August 2013.

[2] Monique Combescure and Didier Robert. *Coherent States and Applications in Mathematical Physics*, volume 2012. 02 2012.

[3] K. Fujii. Introduction to Coherent States and Quantum Information Theory. *eprint arXiv:quant-ph/0112090*, December 2001.

[4] J. S. Langer. Coherent states in the theory of superfluidity. *Phys. Rev.*, 167:183–190, Mar 1968.

[5] A. I. Lvovsky. Squeezed light. *arXiv e-prints*, January 2014.

[6] A. Perelomov. Generalized coherent states and their applications.

[7] Henning Vahlbruch, Moritz Mehmet, Karsten Danzmann, and Roman Schnabel. Detection of 15 db squeezed states of light and their application for the absolute calibration of photoelectric quantum efficiency. *Phys. Rev. Lett.*, 117:110801, Sep 2016.

[8] D. F. Walls. Squeezed states of light. *Nature*, 306:141–146, 1983.