Spinor and Spin Group

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SO(3) as an example

- SU(2) is the universal covering group of SO(3)
$$SO(3) \simeq SU(2)/Z_2$$
- SU(2) transformation of 2 $\times$ 2 matrices induces a rotation
$$U\sigma_i U^{-1} = R_{ij}\sigma_j \;\Rightarrow\; \textit{define}\ X = \vec{\sigma}\cdot\mathbf{x},\ X \to X' = \vec{\sigma}\cdot(\vec{R}\mathbf{x})$$
- Spinor is fundamental representation of SU(2) & a double valued rep of SO(3)

Spin group and Spinor

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We will be focusing on one example of spinor — Lorentz spinor

$Spin(1,3) \simeq SL(2,C)$

Extend Lorentz transformation to complex plane $L(R) \to L(C)$

- $L(R)$: $x \cdot x = (x^0)^2 - \vec{x} \cdot \vec{x}$, $x \in R^4$
- $L(C)$: $z \cdot z = (z^0)^2 - \vec{z} \cdot \vec{z}$, $z \in C^4$

Similar to before, define $Z \equiv z^\mu \cdot \sigma_\mu$, where $\sigma_\mu = (1, \vec{\sigma})$. Then, $det(Z) = z \cdot z$

Now suppose we have a matrix transformation:

$$Z' = AZB^T,$$

where A and B are complex 2×2 matrices. If such transformation preserve the determinant of Z, then this induce a Lorentz transformation $L(C)$ on z.

$Spin(1,3) \simeq SL(2,C)$

$$det(Z') = det(Z) \;\; \Rightarrow \;\; det(A)det(B) = 1$$

However, the pair of matrices (A, B) is not unique, for example, $(cA, B/c)$ with c to be any complex number, represents the same matrix transformation. Let's choose c such that

$$det(cA) = det(B/c) = 1$$

This does not eliminate the choice of sign for (A, B).
Now we have a pair of 2×2 matrices both with determinant 1, so such matrix transformation form the group $SL(2, C) \times SL(2, C)$.

$Spin(1,3) \simeq SL(2,C)$

$SL(2,C) \times SL(2,C)$ induce a Lorentz transformation $L(C)$

$$A\sigma_\nu B^T = \sigma_\mu \Lambda^\mu_\nu(A,B).$$

Now let's prove the 2 to 1 correspondence of
$SL(2,C) \times SL(2,C) \to L(C)$.
Suppose exist two pairs of matrices (A,B) and (A',B') such that

$$A\sigma_\mu B^T = A'\sigma_\mu B'^T.$$

Then we have:

$$\sigma_\mu = A^{-1}A'\sigma_\mu B'^T(B^T)^{-1} \equiv C\sigma_\mu D^T \tag{1}$$

$Spin(1,3) \simeq SL(2,C)$

Now, if we have $C = \lambda_1 I$, $D = \lambda_2 I$, then since A, B, A', B' are determinant 1, we have $\lambda_1 = \pm 1$, $\lambda_2 = \pm 1$. But in order to satisfy equation (1), we must have $\lambda_1 \lambda_2 = +1$, so $\lambda_1 = \lambda_2 = \pm 1$. Thus we have a 2 to 1 homeomorphism.
Let's introduce a lemma to show $C = \lambda_1 I$, $D = \lambda_2 I$.

- Lemma: $\tilde{\sigma}_\mu M \sigma^\mu = 2Tr(M)I$

Times $\tilde{\sigma}_\mu$ on the left side of equation (1), we have

$$\tilde{\sigma}_\mu \sigma^\mu = \tilde{\sigma}_\mu C \sigma^\mu D^T.$$

Use the lemma above,

$$4I = 2Tr(C)D^T \ \Rightarrow \ D = \lambda I.$$

$Spin(1,3) \simeq SL(2,C)$

Similarly we can show $C = \lambda_1 I$. And with the argument in last slides, we have:

$$SL(2,C) \times SL(2,C) \to L(C)$$

is a $2 \to 1$ homeomorphism.

$Spin(1,3) \simeq SL(2,C)$

Since we are clear with $SL(2,C) \times SL(2,C) \to L(C)$, let's get back to $L(R)$. For a pair (A,B) to give real Lorentz transformation, we need to have

$$(AXB^T)^\dagger = AXB^T$$

for any Hermitiean X. By this we have,

$$X = A^{-1}B^*XA^\dagger(B^T)^{-1}.$$

Use the lemma again, we have

$$A^{-1}B^* = A^*B^{-1} = \pm I.$$

So $(A,B) \to (A,\bar{A})$ or $(A,-\bar{A})$.

$$SL(2,C) \to L_+^\uparrow$$

is a $2 \to 1$ homeomorphism.

Representation of $SL(2, C)$

We know that Lie algebra $sl(2, C)$ is actually the complexification of $su(2)$, so we can imagine that the representation of $sl(2, C)$ is a tensor product of representation of $su(2)$. Let's introduce a theorem to help us figure out the representation.

- The irrep of $sl(2, C)$ are labeled by (s_1, s_2) for $s_j = 0, \frac{1}{2}, 1, \cdots,$ with dimension $(2s_1 + 1)(2s_2 + 1)$.

Use this theorem, we can introduce some representations that are of most physical interest.

Representation of $SL(2, C)$

- $(0, 0)$: The trivial representation on C, also called scalar representation
- $(\frac{1}{2}, 0)$: Often called left-handed Weyl spinors. The representation space is C^2
- $(0, \frac{1}{2})$: Often called right-handed Weyl spinors. Representation space is also C^2, this transform independently with $(\frac{1}{2}, 0)$
- $(\frac{1}{2}, \frac{1}{2})$: Called the "vector" representation. Transforms as $X \to \Omega X \Omega^\dagger$, and this induce a Lorentz transformation on R^4

Through $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ are transform independently under $L_+^\uparrow$, they transform into each other under parity transformation. So we usually use reducible the 4 complex dimensional representation $(\frac{1}{2}, 0) + (0, \frac{1}{2})$, which is known as "Dirac spinors".

Summary

- Spin group *Spin*(n) is the universal covering group of SO(n). It is also a double covering
- Spinors are fundamental representations of *Spin*(n)
- $Spin(1,3) \simeq SL(2, \mathbf{C}) \simeq SO(1,3)/Z_2$

Thank you for your attention!

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