Group Theory of Magnetic Monopoles

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Dirac Monopoles

- A magnetic monopole has the vector potential as:

$$\overrightarrow{A_N} = \frac{g}{4\pi r}\frac{1-\cos\theta}{\sin\theta}\hat{\phi} \qquad \overrightarrow{A_S} = \frac{g}{4\pi r}\frac{1+cos\theta}{sin\theta}\hat{\phi}$$

- Then we have $\overrightarrow{B} = \nabla \times \overrightarrow{A} = \frac{g}{4\pi r^2}\hat{r}$

- There are singularities at θ=0 and θ=π correspond to Dirac string. Moving in a circle around string, particle wave function picks up a phase *exp{-iqg}.* The Dirac string is undetectable and so we require that the phase factor must be equal to 1, which lead to the Dirac quantization contition

$$qg = 2\pi n$$

General gauge theory formalism

- Consider the faithful representation of G, given by D(g) and a scalar field ϕ which transforms under the representation

$$\phi \to D(g)\phi$$

- g is a function of spacetime: g=g(x), under such transformations:

$$\partial^\mu \phi \to D(g)\partial^\mu \phi + \partial^\mu D(g)\phi$$

- To keep it covariant, we introduce gauge fields W_a^μ and associate with them a matrix in the Lie algebra of G:

$$W^\mu = W_a^\mu T^a \in L(G)$$

- If we specify the gauge transformation as

$$W^{\mu} \to gW^{\mu}g^{-1} + \frac{i}{e}(\partial^{\mu}g)g^{-1}$$

- Then the modified covariant derivative is given by

$$\mathrm{D}^{\mu}\phi = \partial^{\mu}\phi + ieD(W^{\mu})\phi \to D(g)\mathrm{D}^{\mu}\phi$$

- If we define the antisymmetric gauge field tensor as

$$G^{\mu\nu} = G^{\mu\nu}_{a}T^{a} = \partial\mu W^{\nu} - \partial^{\nu}W^{\mu} + ie[W^{\mu}, W^{\nu}]$$

- We will find that $[D^{\mu}, D^{\nu}]\phi = ieD(G^{\mu\nu})\phi$ and consequently

$$G^{\mu\nu} \to gG^{\mu\nu}g^{-1}$$

- Since we assume the group G is compact, we can always arrange that

$$\mathrm{Tr}(T^a T^b) = \kappa \delta^{ab}$$

- Then we will find the field tensor is invariant under the action of the gauge group

$$G_a^{\mu\nu} G_{a\mu\nu} = \frac{1}{\kappa} \mathrm{Tr}(G^{\mu\nu} G_{\mu\nu}) \to \frac{1}{\kappa} \mathrm{Tr}(g G^{\mu\nu} g^{-1} g G_{\mu\nu} g^{-1}) = \frac{1}{\kappa} \mathrm{Tr}(G^{\mu\nu} G_{\mu\nu})$$

The structure of the Higgs vacuum

- Consider the Lagrangian density

$$L = -\frac{1}{4}G_a^{\mu\nu}G_{a\mu\nu} + (\mathrm{D}^\mu\phi)^\dagger(\mathrm{D}_\mu\phi) - V(\phi)$$

- Where V(ϕ) is invariant under the action of gauge group G

$$V(D(g)\phi) = V(\phi)$$

- At any given time we expect the Lagrangian to satisfy the following equations everywhere in space apart from a finite number of compact regions which we call monopoles, this is ***the Higgs vacuum***.

$$V(\phi) = 0, \;\; \mathrm{D}^\mu\phi = 0$$

- We use M_0 to denote the manifold of Higgs field ϕ which minimize the potential function

$$M_0 = \{\phi : V(\phi) = 0\}$$

- A non-trivial vacuum manifold consists of more than one point, forms an orbit of G. Any two points which can be related by an element of G are said to be on the same orbit.

$$\phi_1 = D(g)\phi_2$$

- Here we assume that M_0 consists of a single orbit of the gauge group G, that is to say, given

- $$\phi_1, \phi_2 \in M_0, \exists g_{12} \in G, \text{such that } \phi_1 = \mathrm{D}(\mathrm{g}_{12})\phi_2$$

Little group

- For representation D of gauge group G, the little group of a point $\phi \in M_0$ is defined as

$$H_\phi = \{h \in G : D(h)\phi = \phi\}$$

- Lemma: Representations belonging to the same orbit have their little groups interrelated as

$$H = g^{-1} H' g$$

- Thus for single orbit H is isomorphic for different ϕ, if we choose any point $\phi_0 \in M_0$ we will have $H = H_{\phi_0}$ and the manifold structure is only determined by G.

- In fact,

$$M_0 = G/H$$

- Prove: Associate a point ϕ with $g \in G$ via $\phi = D(g)\phi_0$, the elements $g_1,\ g_2$ will be associated with the same ϕ if and only if g_1 and g_2 belong to the same right coset of H in G, That is, $D(g_1^{-1}g_2)\phi_0 = \phi_0$, or equivalently, $g_1^{-1}g_2 \in H$

- Thus we may identify M_0 with the right coset space G/H which means once H has been determined the other details associated with the Higgs field may be ignored.

Homotopy class

- Consider a compact monopole region *M*, surrounded by a large region S, in which the equations defining the Higgs vacuum hold to a good approximation.

- If $r \in S$ then $\phi(r) \in M_0$

- This implies that if we consider a closed surface Σ , enclosing M once, then $\phi : \Sigma \to M_0$

- As time evolves, if the map varies continuously with time we called such change a homotopy, and $\phi(r, t_1), \phi(r, t_2)$ are said to define homotopic maps.

- Generally, two continuous maps f_1 and f_2 between topological spaces X and Y are homotopic, if there exists F sending $(x, t) \to F(x, t) \in Y \ (t \in [0,1] \text{ and } \mathrm{x} \in \mathrm{X})$, such that

$$F(x,0) = f_1(x) \text{ and } F(x,1) = f_2(x)$$

- Thus F maps $X \times [0,1] \to Y$ and constitutes a continuous deformation of the map f_1 into the map f_2, we denote such classes of maps from n-sphere to Y by $\Pi_n(Y)$.

- Consider the magnetic flux through some closed surface, surrounding a region where $\mathrm{D}^\mu \phi = 0$ fails.

$$g_\Sigma = \int_\Sigma B \cdot dS$$

- Given the form of the gauge field outside the monopoles region as

$$W^\mu = \frac{1}{a^2 e} \phi \wedge \partial^\mu \phi + \frac{1}{a} \phi A^\mu$$

- In fact, g_Σ is time independent, gauge invariant, and also independent of ϕ on the surface, a small variation in the Higgs field produces no change in the flux. Thus the flux depends only on the homotopy classes of the maps

$$\lim_{r \to \infty} \phi(\vec{r}) : S^{d-1} \to M_0$$

- We are interested in $\Pi_{d-1}(M_0)$

- If G is simply connected as well as M, then we have

$$\Pi_0(G) = 0 \quad \Pi_1(G) = 0$$

- A theorem in homotopy theory tells us that:

$$\Pi_1(G/H) \simeq \Pi_0(H) \text{ and } \Pi_2(\mathrm{G/H}) \simeq \Pi_1(\mathrm{H})$$

- The first isomorphism tells us that assuming M is connected is equivalent to assuming that H is connected.

- The second isomorphism provides a description of the magnetic charges in terms of the first homotopy group of H since $\phi : \Sigma \to M_0$ defines an element of $\Pi_2(M_0)$ under the assumption that $\Pi_1(M_0) = 0$

't Hooft-Polyakov monopoles

- The gauge group is $G = SO(3)$
- Then we have the little group

$$H = SO(2) \simeq U(1)$$

- And the manifold of Higgs vacuum $M_0 = SO(3)/U(1) \simeq S^2$
- Thus in 3+1dimensions, $\Pi_{d-1}(G/H) = \Pi_2(S^2) = Z$
- The equivalence classes are characterized by the number of times N, that ϕ(r) covers the sphere M_0 as r covers a two-dimensional sphere once. The number N determine the homotopy class.

Charge quantization

- SO(3) is not simply connected, but we can replace it by SU(2) to obtain a simply connected group. Then the homotopically distinct closed paths in H=U(1) are

$$h(s) = \exp(iq\int_{\Sigma} B \cdot dS) = e^{iqg}$$

- It is obtained when we solve the equation $\mathrm{D}^{\mu}\phi = 0$ with the parameterized surface $\sum$ as the unit square with its perimeter identified to a single point $r_0 \in \Sigma$

$$\Sigma = \{r(s,t) : s \in [0,1],\ t \in [0,1]\}$$

- The closure requires that $h(0) = h(1) = 1$ and leads to the Dirac quantization condition $qg = 2\pi N,\ N \in Z$

Reference

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