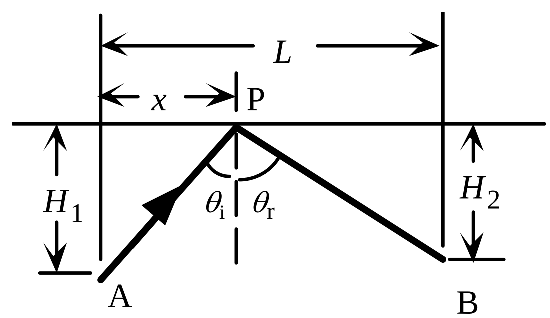
Fermat's Principle and the Laws of Reflection and Refraction

Fermat's principle states that "light travels between two points along the path that requires the least time, as compared to other nearby paths." From Fermat's principle, one can derive the law of reflection (the angle of incidence is equal to the angle of reflection) and the law of refraction (Snell's law).

(*a*) Proof of the law of reflection

The ray travels from point A, a distance H_1 from the surface, to point B, a distance H_2 from the surface, after reflecting from the surface. The component of the distance between A and B parallel to the surface is L. If x is the component of the distance between A and the point P, where the ray meets the surface, parallel to the surface, the time of travel is:



$t_{AB} = t_{AP} + t_{PB}$

$= [(x^2 + H_1^2)^{1/2}/c] + \{[(L - x)^2 + H_2^2]^{1/2}/c\}$.

We find the value of x for the minimum time from

$dt_{AB}/dx = [(1/c)\ x / (x^2 + H_1^2)^{1/2}] - \{(1/c)(L - x) / [(L - x)^2 + H_2^2]^{1/2}\} = 0$,

which reduces to

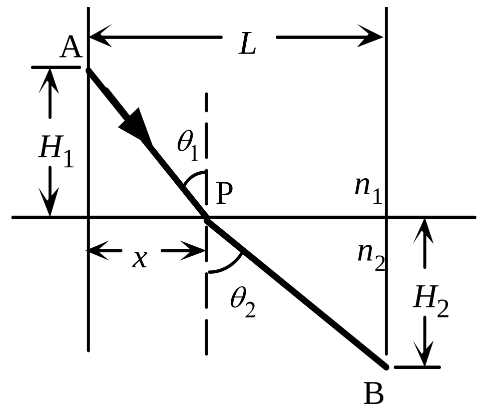
$$x / (x^2 + H_1^2)^{1/2} = (L - x) / [(L - x)^2 + H_2^2]^{1/2} .$$

From the diagram above, we see that

$$x / (x^2 + H_1^2)^{1/2} = \sin\theta_i \text{ and } (L - x) / [(L - x)^2 + H_2^2]^{1/2} = \sin\theta_r ,$$

so we have $\sin\theta_i = \sin\theta_r$, or $\theta_i = \theta_r$.

(b) Proof of the law of refraction



The ray travels from point A in the top medium, a distance H_1 from the surface, to point B in the bottom medium, a distance H_2 from the surface. The component of the distance between A and B parallel to the surface is L. If x is the component of the distance between A and the point P, where the ray meets the surface, parallel to the surface, the time of travel is:

$$t_{AB} = t_{AP} + t_{PB} = [(x^2 + H_1^2)^{1/2} / (c/n_1)] + \{[(L - x)^2 + H_2^2]^{1/2} / (c/n_2)\} .$$

We find the value of x for the minimum time from

$$\mathrm{d}t_{AB}/\mathrm{d}x = [(n_1/c)\, x / (x^2 + H_1^2)^{1/2}] - \{(n_2/c)(L - x) / [(L - x)^2 + H_2^2]^{1/2}\} = 0 ,$$

which reduces to

$$n_1\, x / (x^2 + H_1^2)^{1/2} = n_2\, (L - x) / [(L - x)^2 + H_2^2]^{1/2} .$$

From the diagram above, we see that

$$x / (x^2 + H_1^2)^{1/2} = \sin\theta_1 \quad \text{and} \quad (L - x) / [(L - x)^2 + H_2^2]^{1/2} = \sin\theta_2 ,$$

so we have

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 .$$

which is Snell's law.