This talk is based on work that appears in:


Outline

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  — The Higgs basis and the Higgs-mass basis
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Motivation

The Higgs sector of the minimal supersymmetric extension of the Standard Model (MSSM) is a constrained 2HDM. However, at one-loop all possible 2HDM interactions allowed by gauge invariance are generated (due to SUSY-breaking interactions).

Thus, the Higgs sector of the MSSM is in reality a general 2HDM model (albeit with certain relations among the Higgs sector parameters determined by the fundamental parameters of the broken supersymmetric model).

The general 2HDM consists of two identical (hypercharge-one) scalar doublets $\Phi_1$ and $\Phi_2$. Since one can always redefine the basis of scalar fields, the parameter $\tan \beta \equiv v_2/v_1$ is not meaningful and hence not a physical observable!

To perform model-independent studies of 2HDM phenomena at future colliders, basis-independent techniques are essential for identifying the physical Higgs observables.
Consider the 2HDM potential in a *generic* basis:

\[
\mathcal{V} = m_{11}^2 \Phi_1^\dagger \Phi_1 + m_{22}^2 \Phi_2^\dagger \Phi_2 - [m_{12}^2 \Phi_1^\dagger \Phi_2 + \text{h.c.}] + \frac{1}{2} \lambda_1 (\Phi_1^\dagger \Phi_1)^2 \\
+ \frac{1}{2} \lambda_2 (\Phi_2^\dagger \Phi_2)^2 + \lambda_3 (\Phi_1^\dagger \Phi_1)(\Phi_2^\dagger \Phi_2) + \lambda_4 (\Phi_1^\dagger \Phi_2)(\Phi_2^\dagger \Phi_1) \\
+ \left\{ \frac{1}{2} \lambda_5 (\Phi_1^\dagger \Phi_2)^2 + [\lambda_6 (\Phi_1^\dagger \Phi_1) + \lambda_7 (\Phi_2^\dagger \Phi_2)] \Phi_1^\dagger \Phi_2 + \text{h.c.} \right\}
\]

A basis change consists of a $U(2)$ transformation $\Phi_a \rightarrow U_{a\bar{b}} \Phi_{\bar{b}}$ (and $\Phi_{\bar{a}}^\dagger = \Phi_a^\dagger U_{b\bar{a}}^\dagger$). Rewrite $\mathcal{V}$ in a $U(2)$-covariant notation:

\[
\mathcal{V} = Y_{a\bar{b}} \Phi_{\bar{a}}^\dagger \Phi_{\bar{b}} + \frac{1}{2} Z_{a\bar{b}c\bar{d}} (\Phi_{\bar{a}}^\dagger \Phi_{\bar{b}})(\Phi_{\bar{c}}^\dagger \Phi_{\bar{d}})
\]

where $Z_{a\bar{b}c\bar{d}} = Z_{c\bar{d}a\bar{b}}$ and hermiticity implies $Y_{a\bar{b}} = (Y_{b\bar{a}})^*$ and $Z_{a\bar{b}c\bar{d}} = (Z_{b\bar{d}a\bar{c}})^*$. The barred indices help keep track of which indices transform with $U$ and which transform with $U^\dagger$. For example, $Y_{a\bar{b}} \rightarrow U_{a\bar{c}} Y_{c\bar{d}} U_{d\bar{b}}^\dagger$ and $Z_{a\bar{b}c\bar{d}} \rightarrow U_{a\bar{e}} U_{c\bar{g}} U_{d\bar{h}} U_{b\bar{f}}^\dagger Z_{e\bar{f}g\bar{h}}$. 

**The General Two-Higgs-Doublet Model**
Explicitly (in a generic basis),

\[ Y_{11} = m_{11}^2, \quad Y_{12} = -m_{12}^2, \]
\[ Y_{21} = -(m_{12}^2)^*, \quad Y_{22} = m_{22}^2, \]

and

\[ Z_{1111} = \lambda_1, \quad Z_{2222} = \lambda_2, \]
\[ Z_{1122} = Z_{2211} = \lambda_3, \quad Z_{1221} = Z_{2112} = \lambda_4, \]
\[ Z_{1212} = \lambda_5, \quad Z_{2121} = \lambda_5^*, \]
\[ Z_{1112} = Z_{1211} = \lambda_6, \quad Z_{1121} = Z_{2111} = \lambda_6^*, \]
\[ Z_{2212} = Z_{1222} = \lambda_7, \quad Z_{2221} = Z_{2122} = \lambda_7^*. \]
The most general U(1)_{EM}-conserving vacuum expectation value (vev) is:

\[ \langle \Phi_a \rangle = \frac{v}{\sqrt{2}} \begin{pmatrix} 0 \\ \hat{v}_a \end{pmatrix}, \quad \text{with} \quad \hat{v}_a \equiv e^{i\eta} \begin{pmatrix} c_\beta \\ s_\beta \, e^{i \xi} \end{pmatrix}, \]

where \( v \equiv 2m_W/g = 246 \) GeV. The overall phase \( \eta \) is arbitrary (and can be removed with a U(1)_{Y} hypercharge transformation). If we define the hermitian matrix \( V_{ab} \equiv \hat{v}_a \hat{v}_b^* \), then the scalar potential minimum condition is given by the invariant condition:

\[ \text{Tr} \, (VY) + \frac{1}{2} v^2 Z_{abcd} V_{b\bar{a}} V_{d\bar{c}} = 0. \]

The orthonormal eigenvectors of \( V_{a\bar{b}} \) are \( \hat{v}_b \) and \( \hat{w}_b \equiv \hat{v}_c^* \epsilon_{cb} \) (with \( \epsilon_{12} = -\epsilon_{21} = 1 \), \( \epsilon_{11} = \epsilon_{22} = 0 \)). Note that \( \hat{v}_b^* \hat{w}_b = 0 \). Under a U(2) transformation, \( \hat{v}_a \rightarrow U_{ab} \hat{v}_b \), but:

\[ \hat{w}_a \rightarrow (\text{det} \, U)^{-1} U_{ab} \hat{w}_b, \]

where \( \text{det} \, U \equiv e^{i\chi} \) is a pure phase. That is, \( \hat{w}_a \) is a pseudo-vector with respect to U(2).

One can use \( \hat{w}_a \) to construct a proper second-rank tensor: \( W_{a\bar{b}} \equiv \hat{w}_a\hat{w}_b^* \equiv \delta_{a\bar{b}} - V_{a\bar{b}}. \)

Remark: \( U(2) \cong SU(2) \times U(1)_{Y}/\mathbb{Z}_2 \). The parameters \( m_{11}^2, m_{22}^2, m_{12}^2 \), and \( \lambda_1, \ldots, \lambda_7 \) are invariant under \( U(1)_{Y} \) transformations, but are modified by a “flavor”-SU(2) transformation; whereas \( \hat{v} \) transforms under the full U(2) group.
Define new Higgs doublet fields:

\[ H_1 = (H_1^+, H_1^0) \equiv \hat{\nu}_a^* \Phi_a, \quad H_2 = (H_2^+, H_2^0) \equiv \hat{\omega}_a^* \Phi_a. \]

Equivalently, \( \Phi_a = H_1 \hat{\nu}_a + H_2 \hat{\omega}_a \). Since \( \hat{\nu}_a^* \hat{\nu}_a = 1 \) and \( \hat{\nu}_a^* \hat{\omega}_a = 0 \), it follows that

\[ \langle H_1^0 \rangle = \frac{v}{\sqrt{2}}, \quad \langle H_2^0 \rangle = 0. \]

The field \( H_1 \) defined above is invariant. However, under a U(2) transformation,

\[ H_2 \rightarrow (\det U)H_2. \]

For example, under the U(2) transformation \( U = \text{diag} (1, e^{i\chi}) \), one can transform among different Higgs bases that are related by a rephasing of the field \( H_2 \). Quantities that are invariant under SU(2) but not under U(2) will henceforth be called pseudo-invariants.

If we rewrite the Higgs potential \( V \) in the Higgs basis, we find:
\[
\mathcal{V} = Y_1 H_1^\dagger H_1 + Y_2 H_2^\dagger H_2 + [Y_3 H_1^\dagger H_2 + \text{h.c.}] + \frac{1}{2}Z_1 (H_1^\dagger H_1)^2 \\
+ \frac{1}{2}Z_2 (H_2^\dagger H_2)^2 + Z_3 (H_1^\dagger H_1)(H_2^\dagger H_2) + Z_4 (H_1^\dagger H_2)(H_2^\dagger H_1) \\
+ \left\{ \frac{1}{2}Z_5 (H_1^\dagger H_2)^2 + [Z_6 (H_1^\dagger H_1) + Z_7 (H_2^\dagger H_2)] H_1^\dagger H_2 + \text{h.c.} \right\},
\]

where

\[
Y_1 \equiv \text{Tr} \left( YV \right), \quad Y_2 \equiv \text{Tr} \left( YW \right),
\]

\[
Z_1 \equiv Z_{abcd} V_{ba} V_{dc}, \quad Z_2 \equiv Z_{abcd} W_{ba} W_{dc},
\]

\[
Z_3 \equiv Z_{abcd} V_{ba} W_{dc}, \quad Z_4 \equiv Z_{abcd} V_{bc} W_{da}
\]

are invariant quantities, whereas the following (potentially complex) pseudo-invariants

\[
Y_3 \equiv Y_{ab} \hat{v}_a^* \hat{w}_b, \quad Z_5 \equiv Z_{abcd} \hat{v}_a^* \hat{w}_b \hat{v}_c^* \hat{w}_d,
\]

\[
Z_6 \equiv Z_{abcd} \hat{v}_a^* \hat{v}_b \hat{v}_c^* \hat{w}_d, \quad Z_7 \equiv Z_{abcd} \hat{v}_a^* \hat{w}_b \hat{w}_c^* \hat{w}_d.
\]

transform as \([Y_3, Z_6, Z_7] \rightarrow (\det U)^{-1}[Y_3, Z_6, Z_7]\) and \(Z_5 \rightarrow (\det U)^{-2}Z_5\).
The invariants and pseudo-invariants in the generic basis are given by:

\[
Y_1 = m_{11}^2 c_\beta + m_{22}^2 s_\beta - \text{Re}(m_{12}^2 e^{i\xi}) s_{2\beta},
\]

\[
Y_2 = m_{11}^2 s_\beta + m_{22}^2 c_\beta + \text{Re}(m_{12}^2 e^{i\xi}) s_{2\beta},
\]

\[
Y_3 e^{i\xi} = \frac{1}{2}(m_{22}^2 - m_{11}^2) s_{2\beta} - \text{Re}(m_{12}^2 e^{i\xi}) c_{2\beta} - i \text{Im}(m_{12}^2 e^{i\xi}),
\]

\[
Z_1 = \lambda_1^4 c_\beta + \lambda_2^4 s_\beta + \frac{1}{2} \lambda_345 s_{2\beta} + 2 s_{2\beta} \left[ c_\beta^2 \text{Re}(\lambda_6 e^{i\xi}) + s_\beta^2 \text{Re}(\lambda_7 e^{i\xi}) \right],
\]

\[
Z_2 = \lambda_1^4 s_\beta + \lambda_2^4 c_\beta + \frac{1}{2} \lambda_345 s_{2\beta} - 2 s_{2\beta} \left[ s_\beta^2 \text{Re}(\lambda_6 e^{i\xi}) + c_\beta^2 \text{Re}(\lambda_7 e^{i\xi}) \right],
\]

\[
Z_3 = \frac{1}{4} s_{2\beta}^2 [\lambda_1 + \lambda_2 - 2 \lambda_345] + \lambda_3 - s_2\beta c_2\beta \text{Re}[(\lambda_6 - \lambda_7)e^{i\xi}],
\]

\[
Z_4 = \frac{1}{4} s_{2\beta}^2 [\lambda_1 + \lambda_2 - 2 \lambda_345] + \lambda_4 - s_2\beta c_2\beta \text{Re}[(\lambda_6 - \lambda_7)e^{i\xi}],
\]

\[
Z_5 e^{2i\xi} = \frac{1}{4} s_{2\beta}^2 [\lambda_1 + \lambda_2 - 2 \lambda_345] + \text{Re}(\lambda_5 e^{2i\xi}) + ic_{2\beta} \text{Im}(\lambda_5 e^{2i\xi}),
\]

\[-s_2\beta c_2\beta \text{Re}[(\lambda_6 - \lambda_7)e^{i\xi}] - is_2\beta \text{Im}[(\lambda_6 - \lambda_7)e^{i\xi}],
\]

\[
Z_6 e^{i\xi} = -\frac{1}{2} s_{2\beta} \left[ \lambda_1 c_\beta^2 - \lambda_2 s_\beta^2 - \lambda_345 c_{2\beta} - i \text{Im}(\lambda_5 e^{2i\xi}) \right] + c_\beta c_3\beta \text{Re}(\lambda_6 e^{i\xi}),
\]

\[+ s_\beta s_3\beta \text{Re}(\lambda_7 e^{i\xi}) + ic_\beta^2 \text{Im}(\lambda_6 e^{i\xi}) + is_\beta^2 \text{Im}(\lambda_7 e^{i\xi}),
\]

\[
Z_7 e^{i\xi} = -\frac{1}{2} s_{2\beta} \left[ \lambda_1 s_\beta^2 - \lambda_2 c_\beta^2 + \lambda_345 c_{2\beta} + i \text{Im}(\lambda_5 e^{2i\xi}) \right] + s_\beta s_3\beta \text{Re}(\lambda_6 e^{i\xi})
\]

\[+ c_\beta c_3\beta \text{Re}(\lambda_7 e^{i\xi}) + is_\beta^2 \text{Im}(\lambda_6 e^{i\xi}) + ic_\beta^2 \text{Im}(\lambda_7 e^{i\xi}).
\]

where \(\lambda_{345} \equiv \lambda_3 + \lambda_4 + \text{Re}(\lambda_5 e^{2i\xi}).\)
Starting in the Higgs basis,

\[ H_1 = \begin{pmatrix} G^+ \\ \frac{1}{\sqrt{2}} (v + \varphi_1^0 + iG^0) \end{pmatrix}, \quad H_2 = \begin{pmatrix} H^+ \\ \frac{1}{\sqrt{2}} (\varphi_2^0 + ia^0) \end{pmatrix}, \]

where \( \varphi_1^0, \varphi_2^0 \) and \( a^0 \) are neutral scalar fields, and \( H^+ \) is the physical charged Higgs boson, with mass \( m_{H^\pm}^2 = Y_2 + \frac{1}{2}Z_3v^2 \). If the Higgs sector is CP-violating, then \( \varphi_1^0, \varphi_2^0, \) and \( A \) all mix to produce three physical neutral Higgs states of indefinite CP. After employing the potential minimum conditions: \( Y_1 = -\frac{1}{2}Z_1v^2 \) and \( Y_3 = -\frac{1}{2}Z_6v^2 \), the resulting neutral Higgs squared-mass matrix is:

\[
\mathcal{M} = \begin{pmatrix}
Z_1v^2 & \text{Re}(Z_6)v^2 & -\text{Im}(Z_6)v^2 \\
\text{Re}(Z_6)v^2 & Y_2 + \frac{1}{2} [Z_3 + Z_4 + \text{Re}(Z_5)] v^2 & -\frac{1}{2} \text{Im}(Z_5)v^2 \\
-\text{Im}(Z_6)v^2 & -\frac{1}{2} \text{Im}(Z_5)v^2 & Y_2 + \frac{1}{2} [Z_3 + Z_4 - \text{Re}(Z_5)] v^2
\end{pmatrix}.
\]

Note that \( Z_7 \) does not appear above. The real symmetric matrix \( \mathcal{M} \) is diagonalized by an orthogonal transformation. That is, \( R\mathcal{M}R^T = \mathcal{M}_D = \text{diag} \left( m_1^2, m_2^2, m_3^2 \right) \), where \( RR^T = I \).
A convenient form for $R$ is:

$$
R = R_{12} R_{13} R_{23} = \begin{pmatrix}
    c_{12} & -s_{12} & 0 \\
    s_{12} & c_{12} & 0 \\
    0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
    c_{13} & 0 & -s_{13} \\
    0 & 1 & 0 \\
    s_{13} & 0 & c_{13}
\end{pmatrix}
\begin{pmatrix}
    1 & 0 & 0 \\
    0 & c_{23} & -s_{23} \\
    0 & s_{23} & c_{23}
\end{pmatrix}

= \begin{pmatrix}
    c_{12} c_{13} & -c_{23} s_{12} - c_{12} s_{13} s_{23} & -c_{12} c_{23} s_{13} + s_{12} s_{23} \\
    c_{13} s_{12} & c_{12} c_{23} - s_{12} s_{13} s_{23} & -c_{23} s_{12} s_{13} - c_{12} s_{23} \\
    s_{13} & c_{13} s_{23} & c_{13} c_{23}
\end{pmatrix},
$$

where $c_{ij} \equiv \cos \theta_{ij}$ and $s_{ij} \equiv \sin \theta_{ij}$. The neutral Higgs mass eigenstates are denoted by $h_1$, $h_2$ and $h_3$:

$$
\begin{pmatrix}
    h_1 \\
    h_2 \\
    h_3
\end{pmatrix} = R
\begin{pmatrix}
    \varphi_1^0 \\
    \varphi_2^0 \\
    a^0
\end{pmatrix}.
$$

Since the mass-eigenstates $h_i$ do not depend on the initial basis choice, they are $U(2)$-invariant fields. We have seen that Higgs basis parameters are either invariant or pseudo-invariant. In particular, one can show that under a $U(2)$ transformation,

$$
\theta_{12}, \theta_{13} \text{ are invariant, and } \quad e^{i\theta_{23}} \rightarrow (\det U)^{-1} e^{i\theta_{23}}.
$$
We can eliminate the middle man by expressing the mass eigenstate neutral Higgs directly in terms of the original shifted neutral fields, \( \Phi_a^0 \equiv \Phi_a^0 - v \hat{\nu}_a / \sqrt{2} \):

\[
h_k = \frac{1}{\sqrt{2}} \left[ \Phi_a^0 \left( q_{k1} \hat{\nu}_a + q_{k2} \hat{\omega}_a e^{-i\theta_{23}} \right) + (q_{k1}^* \hat{\nu}_a^* + q_{k2}^* \hat{\omega}_a^* e^{i\theta_{23}}) \Phi_a^0 \right],
\]

for \( k = 1, \ldots, 4 \), where \( h_4 = G^0 \) and the invariant quantities \( q_{kj} \) are given by:

<table>
<thead>
<tr>
<th>( k )</th>
<th>( q_{k1} )</th>
<th>( q_{k2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( c_{12}c_{13} )</td>
<td>( -s_{12} - ic_{12}s_{13} )</td>
</tr>
<tr>
<td>2</td>
<td>( s_{12}c_{13} )</td>
<td>( c_{12} - is_{12}s_{13} )</td>
</tr>
<tr>
<td>3</td>
<td>( s_{13} )</td>
<td>( ic_{13} )</td>
</tr>
<tr>
<td>4</td>
<td>( i )</td>
<td>0</td>
</tr>
</tbody>
</table>

Since \( \hat{\omega}_a e^{-i\theta_{23}} \) is a proper U(2)-vector, we see that the mass-eigenstate fields are indeed U(2)-invariant fields. We can now invert the above result to obtain:

\[
\Phi_a = \begin{pmatrix} G^+ \hat{\nu}_a + H^+ \hat{\omega}_a \\ \frac{v}{\sqrt{2}} \hat{\nu}_a + \frac{1}{\sqrt{2}} \sum_{k=1}^{4} \left( q_{k1} \hat{\nu}_a + q_{k2} e^{-i\theta_{23}} \hat{\omega}_a \right) h_k \end{pmatrix}.
\]
If $\text{Im } (Z_5^* Z_6^2) = 0$, then the neutral scalar squared-mass matrix can be transformed into block diagonal form, which contains the squared-mass of a CP-odd neutral mass-eigenstate Higgs field $A$ and a $2 \times 2$ sub-matrix that yields the squared-masses of two CP-even neutral mass-eigenstate Higgs fields $h$ and $H$.

If $\text{Im } (Z_5^* Z_6^2) \neq 0$, we can write $Z_6 \equiv |Z_6| e^{i\theta_6}$. Then the neutral scalar mass-eigenstates do not possess definite CP quantum numbers, and the three invariant mixing angles $\theta_{12}$, $\theta_{13}$ and $\phi_6 \equiv \theta_6 - \theta_{23}$ are non-trivial.

The angles $\theta_{13}$ and $\phi_6$ are determined modulo $\pi$ from

$$\tan \theta_{13} = \frac{\text{Im}(Z_5 e^{-2i\theta_{23}})}{2 \text{Re}(Z_6 e^{-i\theta_{23}})}, \quad \tan 2\theta_{13} = \frac{2 \text{Im}(Z_6 e^{-i\theta_{23}})}{Z_1 - A^2/v^2},$$

where $A^2 \equiv Y_2 + \frac{1}{2}[Z_3 + Z_4 - \text{Re}(Z_5 e^{-2i\theta_{23}})]v^2$. These equations exhibit multiple solutions (modulo $\pi$) corresponding to different orderings of the $h_k$ masses. Finally,

$$\tan 2\theta_{12} = \frac{2 \cos 2\theta_{13} \text{Re}(Z_6 e^{-i\theta_{23}})}{c_{13} [c_{13}^2 (A^2/v^2 - Z_1) + \cos 2\theta_{13} \text{Re}(Z_5 e^{-2i\theta_{23}})]}.$$

For a given solution of $\theta_{13}$ and $\phi_6$, the two solutions for $\theta_{12}$ (modulo $\pi$) correspond to the two possible relative mass orderings of $h_1$ and $h_2$. 
It is now a simple matter to insert the $U(2)$-covariant expression for $\Phi_\alpha$ in terms of the mass-eigenstate Higgs fields into the Higgs Lagrangian to obtain a $U(2)$-covariant form for the physical Higgs boson and Goldstone boson interactions. [Note: the Goldstone boson and neutral Higgs fields are invariant fields, whereas $H^{\pm} \to (\det U)^{\pm 1} H^{\pm}$.

The gauge boson–Higgs boson interactions are governed by the following interaction Lagrangians:

$$L_{VVH} = \left( g m_W W_\mu^+ W_\mu^- + \frac{g}{2c_W} m_Z Z_\mu Z^\mu \right) \Re(q_{k1}) h_k + e m_W A^\mu (W_\mu^+ G^- + W_\mu^- G^+),$$

$$-g m_Z s_W Z^\mu (W_\mu^+ G^- + W_\mu^- G^+) ,$$

$$L_{VVHH} = \left[ \frac{1}{4} g^2 W_\mu^+ W_\mu^- + \frac{g^2}{8c_W^2} Z_\mu Z^\mu \right] \Re(q_{j1}^* q_{k1} + q_{j2}^* q_{k2}) h_j h_k$$

$$+ \left[ e^2 A_\mu A^\mu + \frac{g^2}{c_W} \left( \frac{1}{2} - s_W^2 \right)^2 Z_\mu Z^\mu + \frac{2gc}{c_W} \left( \frac{1}{2} - s_W^2 \right) A_\mu Z^\mu \right] (G^+ G^- + H^+ H^-)$$

$$+ \left\{ \left( \frac{1}{2} e A_\mu W_\mu^+ - \frac{g s_W^2}{2c_W} Z^\mu W_\mu^+ \right) (q_{k1} G^- + q_{k2} e^{-i\theta_{23} H^-}) h_k + \text{h.c.} \right\} ,$$

$$L_{VHH} = \frac{g}{4c_W} \Im(q_{j1}^* q_{k1} + q_{j2}^* q_{k2}) Z^\mu h_j \overleftrightarrow{\partial_\mu} h_k - \frac{1}{2} g \left\{ i W_\mu^+ \left[ q_{k1} G^- \overleftrightarrow{\partial_\mu} h_k + q_{k2} e^{-i\theta_{23} H^-} \overleftrightarrow{\partial_\mu} h_k \right] + \text{h.c.} \right\}$$

$$+ \left[ i e A_\mu + \frac{ig}{c_W} \left( \frac{1}{2} - s_W^2 \right) Z^\mu \right] (G^+ \overleftrightarrow{\partial_\mu} G^- + H^+ \overleftrightarrow{\partial_\mu} H^-) .$$
Likewise, the cubic and quartic Higgs couplings are given by (with $h_4 = G^0$):

$$\mathcal{L}_{3h} = -\frac{1}{2}v h_j h_k h_l \left[ q_j^* q_k^* q_{l1}^* \text{Re}(q_{l1}) Z_1 + q_j^* q_k^* q_{l2}^* \text{Re}(q_{l2}) (Z_3 + Z_4) + \text{Re}(q_j^* q_k^* q_{l2}^* Z_5 e^{-2i\theta_{23}}) \right]$$

$$+ \text{Re} \left( [2q_j^* + q_j^* q_{k1}^* q_{l2}^* Z_6 e^{-i\theta_{23}}] + \text{Re}(q_j^* q_k^* q_{l2}^* Z_7 e^{-i\theta_{23}}) \right)$$

$$- v h_k G^+ G^- \left[ \text{Re}(q_{k1}) Z_1 + \text{Re}(q_{k2} e^{-i\theta_{23}} Z_6) \right] + v h_k H^+ H^- \left[ \text{Re}(q_{k1}) Z_3 + \text{Re}(q_{k2} e^{-i\theta_{23}} Z_7) \right]$$

$$- \frac{1}{2} v h_k \left\{ G^- H^+ e^{i\theta_{23}} \left[ q_j^* q_k^* Z_4 + q_j^* q_k^* Z_5 e^{-2i\theta_{23}} Z_5 + 2\text{Re}(q_{k1}) Z_6 e^{-i\theta_{23}} \right] + \text{h.c.} \right\} ,$$

and

$$\mathcal{L}_{4h} = -\frac{1}{8} h_j h_k h_l h_m \left[ q_j^* q_k^* q_{l1}^* q_{m1}^* Z_1 + q_j^* q_k^* q_{l2}^* q_{m2}^* Z_2 + 2q_j^* q_k^* q_{l2}^* q_{m2}^* (Z_3 + Z_4) \right]$$

$$+ 2\text{Re}(q_j^* q_k^* q_{l2}^* q_{m2}^* Z_5 e^{-2i\theta_{23}}) + 4\text{Re}(q_j^* q_k^* q_{l1}^* q_{m2}^* Z_6 e^{-i\theta_{23}}) + 4\text{Re}(q_j^* q_k^* q_{l2}^* q_{m2}^* Z_7 e^{-i\theta_{23}}) \right]$$

$$- \frac{1}{2} h_j h_k G^+ G^- \left[ q_j^* q_k^* Z_1 + q_j^* q_k^* Z_3 + 2\text{Re}(q_j^* q_k^* Z_6 e^{-i\theta_{23}}) \right]$$

$$- \frac{1}{2} h_j h_k H^+ H^- \left[ q_j^* q_k^* Z_2 + q_j^* q_k^* Z_3 + 2\text{Re}(q_j^* q_k^* Z_7 e^{-i\theta_{23}}) \right]$$

$$- \frac{1}{2} h_j h_k \left\{ G^- H^+ e^{i\theta_{23}} \left[ q_j^* q_k^* Z_4 + q_j^* q_k^* Z_5 e^{-2i\theta_{23}} + q_j^* Z_6 e^{-i\theta_{23}} + q_j^* q_k^* Z_7 e^{-i\theta_{23}} \right] + \text{h.c.} \right\}$$

$$- \frac{1}{2} Z_1 G^+ G^- G^- - \frac{1}{2} Z_2 H^+ H^- H^+ H^- - (Z_3 + Z_4) G^+ G^- H^+ H^-$$

$$- \frac{1}{2} (Z_5 H^+ H^- G^- + Z_5^* H^- H^+ G^+) - G^+ G^- (Z_6 H^+ G^- + Z_6^* H^- G^+) - H^+ H^- (Z_7 H^+ G^- + Z_7^* H^- G^+) .$$
Example: Higgs self-couplings

Lightest neutral Higgs boson cubic self-coupling:

\[
g(h_1 h_1 h_1) = -3v \left\{ Z_1 c_{12}^3 c_{13}^3 + (Z_3 + Z_4) c_{12} c_{13} |s_{123}|^2 + c_{12} c_{13} \text{Re}(s_{123}^2 Z_5 e^{2i\theta_{23}}) \right. \\
-3 c_{12}^2 c_{13}^2 \text{Re}(s_{123} Z_6 e^{i\theta_{23}}) - |s_{123}|^2 \text{Re}(s_{123} Z_7 e^{i\theta_{23}}) \left. \right\}
\]

Lightest neutral Higgs boson quartic self-coupling:

\[
g(h_1 h_1 h_1 h_1) = -3 \left\{ Z_1 c_{12}^4 c_{13}^4 + Z_2 |s_{123}|^4 + 2(Z_3 + Z_4) c_{12} c_{13}^2 |s_{123}|^2 \\
+2 c_{12}^2 c_{13}^2 \text{Re}(s_{123}^2 Z_5 e^{2i\theta_{23}}) - 4 c_{12} c_{13}^3 |s_{123}|^2 \text{Re}(s_{123} Z_6 e^{i\theta_{23}}) \\
-4 c_{12} c_{13} |s_{123}|^2 \text{Re}(s_{123} Z_7 e^{i\theta_{23}}) \right\}
\]

where \( s_{123} \equiv s_{12} + i c_{12} s_{13} \).

Note that these quantities depend on U(2)-invariants. In particular \( Z_5 e^{-2i\theta_{23}} \), \( Z_6 e^{-i\theta_{23}} \) and \( Z_7 e^{-i\theta_{23}} \) are U(2)-invariants!
In the generic basis, the Higgs-fermion Yukawa Lagrangian is:

\[ -\mathcal{L}_Y = \overline{Q}_L^0 \tilde{\Phi}_1 \eta_{1}^{U,0} U_R^0 + \overline{Q}_L^0 \Phi_1 (\eta_1^{D,0})^\dagger D_R^0 + \overline{Q}_L^0 \tilde{\Phi}_2 \eta_{2}^{U,0} U_R^0 + \overline{Q}_L^0 \Phi_2 (\eta_2^{D,0})^\dagger D_R^0 + \text{h.c.}, \]

where \( \tilde{\Phi}_i \equiv i\sigma_2 \Phi_i^* \), \( Q_L^0 \) is the weak isospin quark doublet and \( U_R^0, D_R^0 \) are weak isospin quark singlets in an interaction eigenstate basis, and \( \eta_{1}^{U,0}, \eta_{2}^{U,0}, \eta_{1}^{D,0}, \eta_{2}^{D,0} \) are 3 \( \times \) 3 matrices in quark flavor space.

Identify the fermion mass eigenstates by employing the appropriate bi-unitary transformation of the quark mass matrices involving unitary matrices \( V_L^U, V_L^D, V_R^U, V_R^D \), where \( K \equiv V_L^U V_L^D \) is the CKM matrix. Then, define the U(2)-vector \( \eta^Q \equiv (\eta_1^Q, \eta_2^Q) \), where

\[
\eta_a^U \equiv V_L^U \eta_a^{U,0} V_R^U \dagger, \quad \eta_a^D \equiv V_R^D \eta_a^{D,0} V_L^D \dagger.
\]

In terms of the quark mass-eigenstate fields and the transformed couplings,

\[-\mathcal{L}_Y = \overline{Q}_L \tilde{\Phi}_a \eta_a^U U_R + \overline{Q}_L \Phi_a \eta_a^D \dagger D_R + \text{h.c.} \]
We can construct basis-independent couplings by transforming to the Higgs basis.

\[-\mathcal{L}_Y = \overline{Q}_L(\tilde{H}_1 \kappa^U + \tilde{H}_2 \rho^U)U_R + \overline{Q}_L(H_1 \kappa^D + H_2 \rho^D)D_R + \text{h.c.},\]

where

\[\kappa^Q \equiv \tilde{\nu}^*_a \eta^Q_a, \quad \rho^Q \equiv \tilde{\omega}^*_a \eta^Q_a.\]

Inverting these equations yields: \(\eta^Q_a = \kappa^Q \tilde{\nu}_a + \rho^Q \tilde{\omega}_a\). Under a U(2) transformation, \(\kappa^Q\) is invariant, whereas \(\rho^Q \rightarrow (\det U) \rho^Q\).

By construction, \(\kappa^U\) and \(\kappa^D\) are proportional to the (real non-negative) diagonal quark mass matrices \(M_U\) and \(M_D\), respectively. In particular,

\[M_U = \frac{v}{\sqrt{2}} \kappa^U = \text{diag}(m_u, m_c, m_t) = V^U_L M^0_U V^U_R,\]
\[M_D = \frac{v}{\sqrt{2}} \kappa^D = \text{diag}(m_d, m_s, m_b) = V^D_L M^0_D V^D_R,\]

where \(M^0_U \equiv (v / \sqrt{2}) \tilde{\nu}^*_a \eta^U_{a,0} \) and \(M^0_D \equiv (v / \sqrt{2}) \tilde{\omega}_a \eta^D_{a,0} \). That is, we have chosen the unitary matrices \(V^U_L, V^U_R, V^D_L\) and \(V^D_R\) such that \(M_D\) and \(M_U\) are diagonal matrices with real non-negative entries. In contrast, the \(\rho^Q\) are independent complex \(3 \times 3\) matrices.
The final form for the Yukawa couplings of the mass-eigenstate Higgs bosons and the Goldstone bosons to the quarks is:

\[-\mathcal{L}_Y = \frac{1}{v} D \left\{ M_D (q_{k1} P_R + q_{k1}^* P_L) + \frac{v}{\sqrt{2}} \left[ q_{k2}^* e^{i\theta_{23}} \rho^D \right]^\dagger P_R + q_{k2} e^{i\theta_{23}} \rho^D P_L \right\} D h_k \]

\[\left. + \frac{1}{v} U \left\{ M_U (q_{k1} P_L + q_{k1}^* P_R) + \frac{v}{\sqrt{2}} \left[ q_{k2} e^{i\theta_{23}} \rho^U \right]^\dagger P_R + q_{k2}^* e^{i\theta_{23}} \rho^U P_L \right\} U h_k \]

\[\left. + \left\{ \overline{U} \left[ K [\rho^D]^\dagger P_R - [\rho^U]^\dagger K P_L \right] D H^+ + \frac{\sqrt{2}}{v} \overline{U} \left[ K M_D P_R - M_U K P_L \right] D G^+ + \text{h.c.} \right\} \right].\]

By writing \([\rho^Q]^\dagger H^+ = [\rho^Q e^{i\theta_{23}}]^\dagger [e^{i\theta_{23}} H^+]\), we see that the Higgs-fermion Yukawa couplings depend only on invariant quantities: the diagonal quark mass matrices, \(\rho^Q e^{i\theta_{23}}\), and the invariant angles \(\theta_{12}\) and \(\theta_{13}\).

The couplings of the neutral Higgs bosons to quark pairs are generically CP-violating as a result of the complexity of the \(q_{k2}\) and the fact that the matrices \(e^{i\theta_{23}} \rho^Q\) are not generally hermitian or anti-hermitian. \(\mathcal{L}_Y\) also exhibits Higgs-mediated flavor-changing neutral currents (FCNCs) at tree-level by virtue of the fact that the \(\rho^Q\) are not flavor-diagonal. Thus, for a phenomenologically acceptable theory, the magnitudes of the off-diagonal elements of \(\rho^Q\) must be small.
Conditions for CP-invariance

The general 2HDM is CP-violating. The requirement of a CP-conserving bosonic sector is equivalent to the requirement that the scalar potential is explicitly CP-conserving and that the Higgs vacuum is CP-invariant. The bosonic sector is CP-conserving if and only if:

\[ \text{Im}[Z_6 Z^*_7] = \text{Im}[Z_5^* Z_6^2] = \text{Im}[Z_5^* (Z_6 + Z_7)^2] = 0. \]

Note that \( \text{Im}[Z_5^* Z_6^2] = 0 \) implies that there is no CP-even/CP-odd scalar mixing in the diagonalization of the neutral scalar squared-mass matrix. Nevertheless, this is not a sufficient condition for CP-conserving Higgs couplings.

Additional constraints arise when the Higgs-fermion couplings are included. If \( Z_5[\rho^Q]^2 \), \( Z_6 \rho^Q \), and \( Z_7 \rho^Q \) (\( Q = U, D, E \)) are hermitian matrices, then the couplings of the neutral Higgs bosons to fermion pairs are CP-invariant. Thus, if all the above conditions are satisfied, then the neutral Higgs bosons are eigenstates of CP, and the only possible source of CP-violation in the 2HDM is the unremovable phase in the CKM matrix \( K \) that enters via the charged current interactions mediated by either \( W^\pm \) or \( H^\pm \) exchange.

*Since one of the scalar potential minimum conditions yields \( Y_3 = -\frac{1}{2} Z_6 v^2 \), no separate condition involving \( Y_3 \) is required.
The significance of $\tan \beta$

So far, $\tan \beta$ has been completely absent from the Higgs couplings. This must be so, since $\tan \beta$ is basis-dependent in a general 2HDM. However, a particular 2HDM may single out a preferred basis, in which case $\tan \beta$ would be promoted to an observable. To simplify the discussion, we focus on a one-generation model, where the Yukawa coupling matrices are simply numbers.

As an example, the MSSM Higgs sector is a type-II 2HDM, i.e., $\eta_1^U = \eta_2^D = 0$. A basis-independent condition for type-II is: $\eta_{\tilde{a}}^D \eta_\tilde{a}^U = 0$. In the preferred basis, $\hat{v} = (\cos \beta, \sin \beta e^{i \xi})$ and $\hat{w} = (-\sin \beta e^{-i \xi}, \cos \beta)$. Evaluating $\kappa^Q = \hat{v}^* \cdot \eta^Q$ and $\rho^Q = \hat{w}^* \cdot \eta^Q$ in the preferred basis, it follows that:

$$e^{-i \xi} \tan \beta = -\frac{\rho^D}{\kappa^D} = \frac{\kappa^U}{\rho^U},$$

where $\kappa^Q = \sqrt{2} m_Q / v$. These two definitions are consistent if $\kappa^D \kappa^U + \rho^D \rho^U = 0$ is satisfied. But this is equivalent to the type-II condition, $\eta_{\tilde{a}}^D \eta_\tilde{a}^U = 0$. 
Since $\rho^Q$ is a pseudo-invariant, we can eliminate $\xi$ by rephasing $\Phi_2$. Hence,

$$\tan \beta = \frac{|\rho^D|}{\kappa_D} = \frac{\kappa^U}{|\rho^U|},$$

with $0 \leq \beta \leq \pi/2$. Indeed, $\tan \beta$ is now a physical parameter, and the $|\rho^Q|$ are no longer independent:

$$|\rho^D| = \frac{\sqrt{2} m_d \tan \beta}{v}, \quad |\rho^U| = \frac{\sqrt{2} m_u \cot \beta}{v}.$$

In the more general (type-III) 2HDM, $\tan \beta$ is not a meaningful parameter. Nevertheless, one can introduce three $\tan \beta$-like parameters:

$$\tan \beta_d \equiv \frac{|\rho^D|}{\kappa_D}, \quad \tan \beta_u \equiv \frac{\kappa^U}{|\rho^U|}, \quad \tan \beta_e \equiv \frac{|\rho^E|}{\kappa^E},$$

the last one corresponding to the Higgs-lepton interaction. In a type-III 2HDM, there is no reason for the three parameters above to coincide.

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$\dagger$ Interpretation: In the Higgs basis, up and down-type quarks interact with both Higgs doublets. But, clearly there exists some basis (i.e., a rotation by angle $\beta_u$ from the Higgs basis) for which only one of the two up-type quark Yukawa couplings is non-vanishing. This defines the physical angle $\beta_u$. 
The MSSM Higgs sector is a type-III 2HDM

The tree-level Higgs potential of the MSSM satisfies:

\[ \lambda_1 = \lambda_2 = -\lambda_{345} = \frac{1}{4}(g^2 + g'^2), \lambda_4 = -\frac{1}{2}g^2, \lambda_5 = \lambda_6 = \lambda_7 = 0. \]

But, one-loop radiative corrections generate corrections to these relations, due to SUSY-breaking. E.g., at one-loop, \( \lambda_5, \lambda_6, \lambda_7 \neq 0 \).

For MSSM Higgs couplings to fermions, Yukawa vertex corrections modify the effective Lagrangian that describes the coupling of the Higgs bosons to the third generation quarks:

\[ -\mathcal{L}_{\text{eff}} = e_{ij} \left[ (h_b + \delta h_b)\bar{b}_R H_d^i Q_L^j + (h_t + \delta h_t)\bar{t}_R Q_L^i H_u^j \right] + \Delta_h b \bar{b}_R Q_L H_u^{k*} + \Delta_h t \bar{t}_R Q_L^{k*} + \text{h.c.} \]

Indeed, this is a general type-III model. For example, in some MSSM parameter regimes (corresponding to large \( \tan \beta \) and large supersymmetry-breaking scale compared to \( v \)),

\[ \Delta h_b \cong h_b \left[ \frac{2\alpha_s}{3\pi} \mu M_{\tilde{g}} I(M_{b_1}^2, M_{b_2}^2, M_{\tilde{g}}^2) + \frac{h_t^2}{16\pi^2} \mu A_t I(M_{t_1}^2, M_{t_2}^2, \mu^2) \right]. \]

This leads to a modification of the tree-level relation between \( m_b \) and \( h_b \). In addition, it leads to a splitting of “effective” \( \tan \beta \)-like parameters \( \tan \beta_b \) and \( \tan \beta_t \).

\[ I(a, b, c) = [ab \ln(a/b) + bc \ln(b/c) + ca \ln(c/a)]/(a - b)(b - c)(a - c). \]
For illustrative purposes, we neglect CP violation in the following simplified discussion. The tree-level relation between $m_b$ and $h_b$ is modified:

$$m_b = \frac{h_b v}{\sqrt{2}} \cos \beta (1 + \Delta_b),$$

where $\Delta_b \equiv (\Delta h_b / h_b) \tan \beta$. That is, $\Delta_b$ is $\tan \beta$-enhanced, and governs the leading one-loop correction to the physical Higgs couplings to third generation quarks. In typical models at large $\tan \beta$, $\Delta_b$ can be of order 0.1 or larger and of either sign.

In the approximation scheme above (keeping only the $\tan \beta$-enhanced terms),

$$\tan \beta_b \equiv \frac{v \rho^D}{\sqrt{2} m_b} \simeq \frac{\tan \beta}{1 + \Delta_b}, \quad \tan \beta_t \equiv \frac{\sqrt{2} m_t}{v \rho^U} \simeq \frac{\tan \beta}{1 - \tan \beta (\Delta h_t / h_t)}.$$

Thus, supersymmetry-breaking loop-effects can yield observable differences between $\tan \beta$-like parameters that are defined in terms of basis-independent quantities. In particular, the leading one-loop $\tan \beta$-enhanced corrections are automatically incorporated into:

$$g_{A.bb} = \frac{m_b}{v} \tan \beta_b, \quad g_{A.t\bar{t}} = \frac{m_t}{v} \cot \beta_t.$$
The decoupling limit

The decoupling limit corresponds to the limiting case in which one of the two Higgs doublets of the 2HDM receives a very large mass and is therefore decoupled from the theory. This can be achieved by assuming that $Y_2 \gg v^2$ and $|Z_i| \lesssim \mathcal{O}(1)$ [for all $i$]. The effective low energy theory is a one-Higgs-doublet model that corresponds to the Higgs sector of the Standard Model. We shall order the neutral scalar masses according to $m_1 \ll m_{2,3}$ and define the invariant Higgs mixing angles accordingly. Thus, we expect one light CP-even Higgs boson, $h_1$, with couplings identical (up to small corrections) to those of the Standard Model (SM) Higgs boson. In particular,\(^8\)

$$m_1^2 = Z_1 v^2 + \mathcal{O} \left( \frac{|Z_6| v^2}{m_3^2} \right),$$

$$m_2^2 = A^2 + v^2 \text{Re}(Z_5 e^{-2 i \theta_{23}}) + \mathcal{O} \left( \frac{|Z_6| v^2}{m_3^2} \right),$$

$$m_3^2 = A^2 + \mathcal{O} \left( \frac{|Z_6| v^2}{m_3^2} \right),$$

$$m_{H^\pm}^2 = Y_2 + \frac{1}{2} Z_3 v^2.$$

Hence, $m_1 \ll m_2 \simeq m_3 \simeq m_{H^\pm}$.

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\(^8\)Recall that: $A^2 \equiv Y_2 + \frac{1}{2} [Z_3 + Z_4 - \text{Re}(Z_5 e^{-2 i \theta_{23}})] v^2$. 
Finally, the invariant mixing angles are given by:

\[ s_{12} = \frac{v^2 \text{Re}(Z_6 e^{-i\theta_{23}})}{m_2^2 - m_1^2} \left[ 1 + O \left( \frac{|Z_6|^2 v^4}{m_3^4} \right) \right] \ll 1, \]

\[ s_{13} = \frac{-v^2 \text{Im}(Z_6 e^{-i\theta_{23}})}{m_3^2 - m_1^2} \left[ 1 + O \left( \frac{|Z_6|^2 v^4}{m_3^4} \right) \right] \ll 1, \]

\[ \text{Im}(Z_5 e^{-2i\theta_{23}}) = \frac{-v^2 \text{Im}(Z_6^2 e^{-2i\theta_{23}})}{m_3^2 - m_1^2} \left[ 1 + O \left( \frac{|Z_6|^2 v^4}{m_3^4} \right) \right] \ll 1. \]

In the exact decoupling limit, these quantities are all zero. However, the identity:\footnote{Another identity, \( \text{Im}(Z_5^* Z_6^2) v^6 = 2 s_{13} c_{13} s_{12} c_{12} (m_2^2 - m_1^2)(m_3^2 - m_1^2)(m_3^2 - m_2^2) \), yields the same conclusion.}

\[ \text{Im}(Z_5^* Z_6^2) = 2 \text{Re}(Z_5 e^{-2i\theta_{23}}) \text{Re}(Z_6 e^{-i\theta_{23}}) \text{Im}(Z_6 e^{-i\theta_{23}}) \]

\[ - \text{Im}(Z_5 e^{-2i\theta_{23}}) \left\{ [\text{Re}(Z_6 e^{-i\theta_{23}})]^2 - [\text{Im}(Z_6 e^{-i\theta_{23}})]^2 \right\} \]

implies that \( \text{Im}(Z_5^* Z_6^2) \) need not be particularly small in the decoupling limit. Therefore in the decoupling limit, the properties of \( h_1 \) approach those of the neutral CP-even Standard Model Higgs boson. In contrast, \( h_2 \) and \( h_3 \) are states of indefinite CP \((i.e.,\, strongly-mixed\, linear\, combinations\, of\, H \, and \, A)\).
Lessons and future work

• If phenomena consistent with the 2HDM are found, we will not know a priori the underlying structure that governs the model. In this case, one needs a model-independent analysis of the data that allows for the most general CP-violating Model-III.

• Instead of claiming that you have measured $\tan \beta$ (which can only be done in the context of a simplified version of the model), measure the physical parameters of the model. Examples include the $\tan \beta$-like parameters introduced in the one-generation model. (For three generations, the formalism becomes more complicated. However, one has good reason to assume that the third generation quark–Higgs Yukawa couplings dominate.)

• Which $\tan \beta$-like parameters will be measured in precision Higgs studies at the LHC and ILC? How can one best treat the full three-generation model to one-loop order? What simplifications can be exploited in the MSSM?

• Compute the one-loop radiative corrections to various Higgs processes in terms of the physical $\tan \beta$-like parameters in the MSSM.