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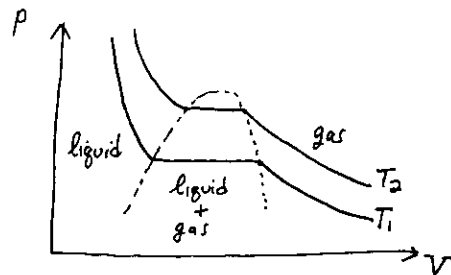
Physics 112

FINAL EXAM SOLUTIONS

Winter 2000

PART I

- ① FALSE. If one plots the liquid-gas phase transformation on a P vs. V graph, it looks as follows:



Two isotherms are drawn (with $T_1 < T_2$). Assuming that the liquid and gas are in thermodynamic equilibrium, we see that pushing a piston down on the system at fixed temperature (we assume that this process occurs quasi-statically, so that approximate thermodynamic equilibrium is maintained) does not change the pressure. The pressure remains constant until all of the gas has been transformed into liquid.

- ② TRUE. Consider the change of entropy when ΔN particles are removed from system 1 and transferred to system 2:

$$\Delta S \approx \left(\frac{\partial S}{\partial N_1} \right)_{E,V} (-\Delta N) + \left(\frac{\partial S}{\partial N_2} \right)_{E,V} (\Delta N)$$

By definition, $\left(\frac{\partial S}{\partial N} \right)_{E,V} = \frac{\mu}{T}$

②

Thus, noting that the total entropy of the isolated system must increase,

$$\Delta S = \frac{\Delta N}{T} (\mu_1 - \mu_2) > 0$$

which implies that $\mu_1 > \mu_2$.

- ③ FALSE.

The correct statement is: at $T=0$, the total energy of a Bose gas is zero while the total energy of a Fermi gas is non-zero. This is a consequence of the Pauli exclusion principle which is obeyed by fermions but not obeyed by bosons.

For bosons at $T=0$, all particles can simultaneously occupy their single particle ground states. Defining the ground state energy to be zero, the total energy of the system is then zero.

For fermions at $T=0$, only $2s+1$ particles can occupy their single particle ground states. Here, s is the spin of the fermion and $2s+1$ is the degeneracy of each single-particle energy level. Thus, if the total number of fermions in the gas $N > 2s+1$, then some must occupy excited energy levels.

An explicit calculation gives $E = \frac{3}{5} N E_F$ for the energy of an ideal Fermi gas at $T=0$, where the Fermi energy is

$$E_F = \frac{\hbar^2}{2m} \left(\frac{6\pi^2 N}{(2s+1)V} \right)^{2/3}$$

Clearly, $E \neq 0$.

2

3

④ TRUE.

Consider an isolated system at fixed temperature and pressure. The Gibbs free energy is defined as

$$G = E + PV - TS$$

For an isolated system, the total energy E is fixed and the total volume is fixed. Thus, at constant P and T ,

$$\Delta G = -T\Delta S$$

But, in the approach to equilibrium, $\Delta S \geq 0$ for an isolated system. Thus,

$$\Delta G \leq 0$$

That is, the Gibbs free energy is minimized in the approach to thermodynamic equilibrium at constant T and P .

Note: the utility of this result is that it also applies to non-isolated systems as long as T and P of the system are held constant (say, by placing the system in thermal contact with a heat reservoir and movable piston).

⑤ TRUE

For a classical ideal gas, the entropy is given by

$$S = N \left[\ln \left(\frac{V \lambda_{th}^{-3}}{N} \right) + \frac{5}{2} \right]$$

$$\text{where } \lambda_{th} = \left(\frac{2\pi\hbar^2}{m k T} \right)^{1/2}$$

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The third law of thermodynamics states that as $T \rightarrow 0$ the entropy must vanish (if the ground state is non-degenerate).

But, as $T \rightarrow 0$, $\lambda_{th} \rightarrow \infty$ and we see that $S \rightarrow -\infty$.

Thus, the classical ideal gas does not satisfy the third law of thermodynamics.

The reason for this behavior is that the classical limit is viable only when

$$\frac{V}{N} \gg \lambda_{th}^3$$

This inequality must be violated as $T \rightarrow 0$.

(5)

PART II

- ⑥ During the process where thermal equilibrium is reached, system 1 gains energy

$$\Delta E_1 = \int_{T_1}^{T_f} C_V dT$$

while system 2 loses energy

$$\Delta E_2 = \int_{T_2}^{T_f} C_V dT$$

where T_1 and T_2 are the initial temperatures of systems 1 and 2 respectively and T_f is the common final temperature.

Conservation of energy requires that

$$\Delta E_1 + \Delta E_2 = 0$$

Plugging in $C_V = bT^3$, we have

$$\frac{b}{4}(T_f^4 - T_1^4) + \frac{b}{4}(T_f^4 - T_2^4) = 0$$

Solving for T_f ,

$$T_f = \left(\frac{T_1^4 + T_2^4}{2} \right)^{1/4}$$

Plugging in $T_1 = 100\text{K}$ and $T_2 = 200\text{K}$, one finds $T_f = 170.7\text{K}$.

Thus, the correct answer is (d).

(6)

- ⑦ The average speed of the molecules of an ideal gas is [RB eq (13.19) on p 313]

$$\langle v \rangle = \sqrt{\frac{8kT}{\pi m}}$$

Noting that $m_{D_2} = 2m_{H_2}$,

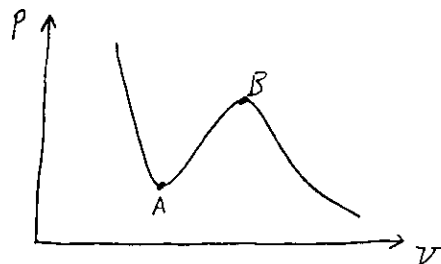
$$\frac{\langle v \rangle_{H_2}}{\langle v \rangle_{D_2}} = \sqrt{\frac{m_{D_2}}{m_{H_2}}} = \sqrt{2}$$

Thus, the correct answer is (d).

- ⑧ In class, I showed that a necessary condition for mechanical stability is:

$$\left(\frac{\partial V}{\partial P} \right)_T < 0$$

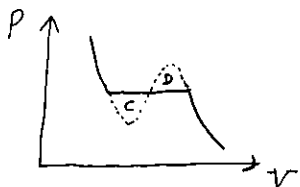
For a van der Waals gas, the isotherms below the critical temperature look as follows:



Between the points A and B labeled above, $(\partial V / \partial P)_T > 0$. Thus, in this region, the van der Waals equation of state cannot represent a state of mechanical equilibrium.

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This defect can be "repaired" by using the Maxwell construction.
The resulting iso-therm indicated by the solid line



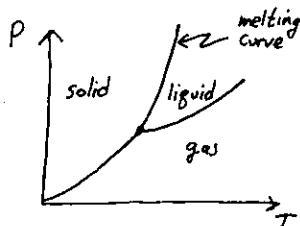
(where the areas enclosed by region C and D are equal) provides a good model for the liquid-gas phase transformation.

All the other statements of the problem are correct. Thus, the correct answer is (c).

⑨ Given a solid and liquid co-existing in thermal equilibrium, the typical behavior is that the liquid freezes completely when the pressure increases. The water/ice phase transformation is one of the few counterexamples to this behavior.

The question posed is immediately answered by examining the Clausius-Clapeyron equation:

$$\left(\frac{dP}{dT}\right)_{\text{melting curve}} = \frac{L_{\text{fusion}}}{T(V_{\text{liquid}} - V_{\text{solid}})}$$



If the liquid freezes when the pressure is raised, this means that $\left(\frac{dP}{dT}\right)_{\text{melting curve}} > 0$. Since the latent heat of fusion,

$L_{\text{fusion}} > 0$ (it takes positive heat energy to melt the solid), it follows that $V_{\text{liquid}} > V_{\text{solid}}$. Thus, during the freezing process, the specific volume decreases.

The correct answer is (b).

8

⑩ The Fermi temperature is related to the Fermi energy by $E_F = k T_F$. Thus, for a spin- $1/2$ particle ($s = \frac{1}{2}$), we have $2s+1 = 2$ and so

$$E_F = k T_F = \frac{\hbar^2}{2m} \left(\frac{3\pi^2 N}{V} \right)^{2/3}$$

First, we need to compute N/V . For ${}^3\text{He}$, $\rho = 0.081 \text{ g cm}^{-3}$.

$$\rho = \frac{M}{V} = \frac{m N}{V}$$

$$\begin{aligned} \text{where } m = m_{{}^3\text{He}} &\approx \frac{3}{4} m_{{}^4\text{He}} = \frac{3}{4} (6.649 \times 10^{-27} \text{ kg}) \\ &= 4.987 \times 10^{-27} \text{ kg} \end{aligned}$$

Convert ρ to kg m^{-3} :

$$1 \text{ g cm}^{-3} = 10^6 \text{ g m}^{-3} = 10^3 \text{ kg m}^{-3}$$

Thus,

$$\frac{N}{V} = \frac{\rho}{m} = \frac{81 \text{ kg m}^{-3}}{4.987 \times 10^{-27} \text{ kg}} = 1.624 \times 10^{28} \text{ m}^{-3}$$

Finally,

$$\begin{aligned} T_F &= \frac{(1.0546 \times 10^{-34} \text{ J s})^2 [3\pi^2 (1.624 \times 10^{28} \text{ m}^{-3})]^{2/3}}{2(4.987 \times 10^{-27} \text{ kg})(1.381 \times 10^{-23} \text{ J K}^{-1})} \\ &= 4.96 \text{ K} \end{aligned}$$

Thus, the correct answer is (d).

(9)

PART III: PROBLEMS

① Use the Clausius-Clapeyron equation:

$$\frac{dP}{dT} = \frac{L_{\text{vap}}}{T(v_{\text{vap}} - v_{\text{ig}})}$$

[RB eq. (12.27) on p.281]. To a very good approximation, $v_{\text{vap}} \gg v_{\text{ig}}$.
If we use the ideal gas law

$$Pv_{\text{vap}} = kT$$

(where $v_{\text{vap}} = V/N$), then

$$\frac{dP}{dT} = \frac{P}{kT^2} L_{\text{vap}}$$

Assume that L_{vap} is constant over the range $98^\circ\text{C} < T < 103^\circ\text{C}$.
Then, this is an easy differential equation to solve. RB eq. (12.30)
on p.281 provides the solution:

$$P = \frac{P_0}{\exp(-L_{\text{vap}}/kT_0)} \exp(-L_{\text{vap}}/kT)$$

$$= P_0 \exp\left[-\frac{L_{\text{vap}}}{k}\left(\frac{1}{T} - \frac{1}{T_0}\right)\right]$$

Solving for L_{vap} ,

$$L_{\text{vap}} = k\left(\frac{1}{T_0} - \frac{1}{T}\right)^{-1} \ln\left(\frac{P}{P_0}\right)$$

(10)

We are given:

$$\begin{aligned} T &= 103^\circ\text{C} & \text{at } z &= 0 \\ T &= 98^\circ\text{C} & \text{at } z &= 0.5 \text{ km} \end{aligned}$$

where z is the altitude. Let us determine the corresponding pressures.
At $z=0$, $p = 1 \text{ atm} = 1.013 \times 10^5 \text{ N m}^{-2}$. At $z=0.5 \text{ km}$, if we
employ the approximation of the isothermal atmosphere, then

$$P(z) = P(0) e^{-mgz/kT}$$

[see problem 4 on problem set #6: RB Chapter 7, problem 1.]

Taking the mass of an air molecule to be [RB, Table S.2 on p.107]

$$m = 4.8 \times 10^{-26} \text{ kg}$$

and the atmospheric temperature to be $T = 300 \text{ K}$,

$$\begin{aligned} P(z=0.5 \text{ km}) &= (1 \text{ atm}) \exp \frac{-(4.8 \times 10^{-26} \text{ kg})(9.8 \text{ m s}^{-2})(500 \text{ m})}{(1.381 \times 10^{-23} \text{ J K}^{-1})(300 \text{ K})} \\ &= 0.945 \text{ atm} \end{aligned}$$

Plugging in:

$$\begin{aligned} T_0 &= 103^\circ\text{C} = 376 \text{ K} & \text{at } P_0 &= 1 \text{ atm} \\ T &= 98^\circ\text{C} = 371 \text{ K} & \text{at } P_0 &= 0.945 \text{ atm} \end{aligned}$$

into the equation for L_{vap} :

$$\begin{aligned} L_{\text{vap}} &= (1.381 \times 10^{-23} \text{ J K}^{-1}) \left(\frac{1}{376} - \frac{1}{371}\right)^{-1} \ln\left(\frac{0.945}{1}\right) \\ &= 2.18 \times 10^{-20} \text{ J/molecule} \end{aligned}$$

(11)

Note that L_{vap} corresponds to the heat required to convert one molecule of the liquid into gas. In terms of eV,

$$L_{\text{vap}} = \frac{2.18 \times 10^{-20} \text{ J}}{1.602 \times 10^{-19} \text{ J/eV}}$$

$$L_{\text{vap}} = 0.136 \text{ eV/molecule}$$

This can be compared with, e.g. $L_{\text{vap}} = 0.42 \text{ eV/molecule}$ for water [see RB Table 12.2 on p 274].

(12)

(2) (a) At constant temperature and number of molecules, the multiplicity is proportional to V^N [see RB section 2.9, pp 31-35]. The ratio of the final to the initial multiplicity is:

$$\frac{g_f}{g_i} = \left(\frac{V_f}{V_i}\right)^N = \left(\frac{0.1 \text{ m}^3}{0.3 \text{ m}^3}\right)^N = 3^{-N}$$

We can write:

$$3^{-N} = (10^{\log_{10} 3})^{-N} = (10^{0.4771})^{-N}$$

For $N = 3 \times 10^{24}$ molecules,

$$\frac{g_f}{g_i} = 10^{-1.43 \times 10^{24}}$$

(b) The change in entropy of the gas is

$$\begin{aligned} \Delta S &= k \ln g_f - k \ln g_i \\ &= k \ln \left(\frac{g_f}{g_i}\right) \\ &= k \ln 3^{-N} \\ &= -Nk \ln 3 \\ &= -(1.381 \times 10^{-23} \text{ J K}^{-1})(3 \times 10^{24}) \ln 3 \\ &= -45.5 \text{ J/K} \end{aligned}$$

(13)

(c) By assumption, the compression takes place slowly, so

$$\Delta S_{\text{gas}} + \Delta S_{\text{environment}} = 0$$

Thus, from part (b),

$$\begin{aligned}\Delta S_{\text{environment}} &= -\Delta S_{\text{gas}} \\ &= 45.5 \text{ J/K}\end{aligned}$$

In addition, at constant temperature ($T = 280 \text{ K}$), the heat energy added to the environment is

$$\begin{aligned}q &= T \Delta S_{\text{environment}} \\ &= (280 \text{ K})(45.5 \text{ J/K}) \\ &= 1.27 \times 10^4 \text{ J}\end{aligned}$$

(d) There are two methods to compute the work done on the gas during compression.

Method 1:

The work done by the gas is equal to

$$w = \int_{V_i}^{V_f} P dV$$

So the work done on the gas is equal to $-w$. If we use the ideal gas law: $P = \frac{NkT}{V}$, then since T is constant,

$$-w = - \int_{V_i}^{V_f} \frac{NkT}{V} dV = -NkT \ln\left(\frac{V_f}{V_i}\right)$$

(14)

Using $\frac{V_f}{V_i} = \frac{1}{3}$, we obtain

$$\begin{aligned}-w &= NkT \ln 3 \\ &= (3 \times 10^{24})(1.381 \times 10^{-23} \text{ J K}^{-1})(280 \text{ K}) \ln 3 \\ &= 1.27 \times 10^4 \text{ J}\end{aligned}$$

Method 2:

For an ideal gas, the energy remains constant if the temperature is fixed. Using the first law of thermodynamics,

$$q = \Delta E + w$$

where q is the heat energy added to the system (i.e. the gas) and w is the work done by the gas. Since $\Delta E = 0$,

$$-w = -q_{\text{gas}}$$

But q_{gas} is equal to the negative of the heat added to the environment as computed in part (b), since this is heat removed from the gas. Hence,

$$q_{\text{gas}} = -1.27 \times 10^4 \text{ J}$$

and indeed we confirm that the work done on the gas is

$$-w = 1.27 \times 10^4 \text{ J}.$$

③ (a) The "statistical" definition of temperature is:

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E} \right)_{\text{fixed external parameters}}$$

[see RB ex. (4.21) on p82].

We are given for the black hole

$$S = \frac{\kappa c^3 A}{4G\kappa}$$

where $A = 4\pi R_s^2$ and $R_s = \frac{2GM}{c^2}$. Thus,

$$S = \frac{4\pi \kappa G M^2}{\kappa c}$$

Now, M is related to the total internal energy of the black hole via $E = Mc^2$. Thus,

$$S = \frac{4\pi \kappa G E^2}{\kappa c^5}$$

We can now apply the statistical definition of temperature:

$$\frac{1}{T} = \frac{8\pi \kappa G E}{\kappa c^5}$$

Expressing this result in terms of the black hole mass (using $E = Mc^2$ again),

$$T = \frac{\kappa c^3}{8\pi \kappa G M}$$

(b) Let us return to the expression for E in terms of the temperature T . We can write this as:

$$E = \frac{\kappa c^5}{8\pi \kappa G T}$$

The heat capacity is

$$C = \frac{\partial E}{\partial T} = \frac{-\kappa c^5}{8\pi \kappa G T^2}$$

Thus, the heat capacity is negative! This means that as the black hole loses energy, its temperature increases! How bizarre!

(c) The Stefan-Boltzmann law states that the energy flux of a blackbody radiator is given by $\sigma_B T^4$. [See RB ex (6.30) on p124].

$$\text{Energy flux} = \frac{dE}{dA dt}$$

If we integrate over the surface area of the black hole, we obtain the power radiated at the surface is equal to

$$\frac{dE}{dt} = -A \sigma_B T^4$$

(the minus sign indicates that energy is being lost in radiation)

Plug in $E = Mc^2$ on the left hand side and

$$A = 4\pi R_s^2 = 4\pi \left(\frac{2GM}{c^2} \right)^2 = \frac{16\pi G^2 M^2}{c^4}$$

$$T = \frac{\kappa c^3}{8\pi \kappa G M} \quad [\text{obtained in part (a)}]$$

on the right hand side

$$c^2 \frac{dM}{dt} = -\frac{16\pi G^2 M^2}{c^4} \sigma_B \left(\frac{\hbar c^3}{8\pi \hbar G M} \right)^4$$

Simplifying,

$$\frac{dM}{dt} = \frac{\sigma_B \hbar^4 c^6}{256 \pi^3 \hbar^4 G^2} \frac{1}{M^2}$$

We can simplify this expression by noting that [RB eq (6.29) on p124] that the Stefan-Boltzmann constant can be expressed in terms of fundamental constants:

$$\sigma_B = \frac{\pi^2 \hbar^4}{60 \hbar^3 c^2}$$

Thus,

$$\frac{dM}{dt} = \frac{-\hbar c^4}{15 \cdot 2^{10} \pi G^2 M^2}$$

(d) Solving the differential equation,

$$\int M^2 dM = \frac{-\hbar c^4}{15 \cdot 2^{10} \pi G^2} \int dt$$

$$\frac{1}{3} (M_f^3 - M_i^3) = \frac{-\hbar c^4}{15 \cdot 2^{10} \pi G^2} (t_f - t_i)$$

where i and f refer to the initial and final states.

If the black hole completely evaporates, then $M_f = 0$. This will occur in a time interval

$$\Delta t = \frac{5 \cdot 2^{10} \pi G^2 M^3}{\hbar c^4}$$

where M is the mass of the initial black hole.

If $M = 2 \times 10^{11} \text{ kg}$,

$$\Delta t = \frac{5 \cdot 2^{10} \pi (6.673 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2})^2 (2 \times 10^{11} \text{ kg})^3}{(1.0546 \times 10^{-34} \text{ J s}) (3 \times 10^8 \text{ ms}^{-1})^4}$$

or

$$\Delta t = 6.71 \times 10^{17} \text{ sec.}$$

Remark: In black hole physics, it is often useful to express the fundamental constants in terms of the Planck mass, which is defined as

$$M_{\text{Pl}} = \sqrt{\frac{\hbar c}{G}} = 2.18 \times 10^{-8} \text{ kg}$$

Then, it is convenient to replace Newton's constant G in favor of M_{Pl} . For example, the temperature of a black hole is

$$T = \frac{\hbar c^3}{8\pi \hbar M} \frac{M_{\text{Pl}}^2}{\hbar c} = \frac{M_{\text{Pl}}^2 c^2}{8\pi \hbar M}$$

Note that if $M = M_{\text{Pl}}$, then $\hbar T = \frac{M_{\text{Pl}} c^2}{8\pi}$. That is for a black hole the total energy is of $O(\hbar T)$ only when $M \sim O(M_{\text{Pl}})$.