The Car and the Goats

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1. THE PROBLEM. *A TV host shows you three numbered doors, one hiding a car (all three equally likely) and the other two hiding goats. You get to pick a door, winning whatever is behind it. You choose door #1, say. The host, who knows where the car is, then opens one of the other two doors to reveal a goat, and invites you to switch your choice if you so wish. Assume he opens door #3. Should you switch to #2?*

I'll call this Game I. It appeared in the *Ask Marilyn* column in *Parade* (a Sunday supplement) [**4**(a)]. Marilyn asserted that you should switch, arguing that the probability of winning, originally 1/3, had now gone up to 2/3. ("Marilyn" is standard terminology.) This led to an uproar featuring "thousands" of letters, nine-tenths of them insisting that with door # 3 now eliminated, #1 and #2 were equally likely; even the responses from college faculty voted her down two to one [**4**(b, c), **3**]. There is no denying that the problem is tricky (even though, technically speaking, it involves only undergraduate mathematics). The purpose of this article is to unravel it all.

2. GAME II. Marilyn's solution goes like this. The chance is 1/3 that the car is actually at #1, and in that case you lose when you switch. The chance is 2/3 that the car is either at #2 (in which case the host perforce opens #3) or at #3 (in which case he perforce opens #2)—and in these cases, the host's revelation of a goat shows you how to switch and win.

This is an elegant proof, but it does not address the problem posed, in which the host has shown you a goat at #3. Instead it is still considering the possibility that the car is at #3—whence the host cannot have already opened that door (much less to reveal a goat). In this game—Game II—you have to announce *before a door has been opened* whether you plan to switch.

3. GAME I. Game I is more complicated: What is the probability P that you win if you switch, *given that the host has opened door* #3? This is a *conditional* probability, which takes account of this extra condition. When the car is actually at #2, the host will open #3. But when it is at #1, he may open either #2 or #3. The answer to the question just asked *depends on his selection strategy when he has this choice* —on the *probability* q that he will then open door #3. (Marilyn did not address this question.)

In any case, it still pays you to switch (except in one extreme case, where it's fifty-fifty). The host has opened #3. It was *certain* he would do that if the car is at #2, but less than certain (except in the extreme case) if it is at #1. This gives the edge to #2. This argument is well known in the game of bridge as the "principle of restricted choice." A player holding both Queen and Jack of a suit will play them at random so as not to betray her holding. Hence when West plays the Queen, the Jack is now more likely to be with East, since if West had it she could have played it. (This ignores other information that may have come to light in the course of the play.)

We are interested in the following events:

C_i: the car is at door i; H_j: the host opens door j.

In this notation, the probability that you will win if you switch is the conditional probability

$$P = \mathbf{P}(C_2|H_3);$$

as noted, its value depends on the conditional probability

$$q = \mathbf{P}(H_3|C_1).$$

It turns out that *P can be any number between* $1/2$ *and* 1. (So the critics are still quite wrong.)

4. EXAMPLES. In the extreme case $q = 1$, the host's opening of #3 gives you no information, and $P = 1/2$. At the other extreme, $q = 0$, the host opens #3 only when the car is at #2, and $P = 1$.

When $q = 1/2$ the host is not differentiating between the two available doors, and you are essentially playing Game II. In fact, when the car is at #1 he opens #3 one time in two, but if it is at #2 he opens it two times in two. So when he actually does open #3, the car is at #2 two times out of three: $P = 2/3$. Similarly, if $q = m/n$ then $P = n/(n + m)$; thus,

$$P = \frac{1}{1 + q} \tag{1}$$

for any rational q. By Bayes's rule (Section 6), (1) holds for all real q, $0 \leqslant q \leqslant 1$. Note that these inequalities imply $1 \geqslant P \geqslant 1/2$.

To illustrate that the solution to Game I is consistent with that of Game II, consider the extreme case $q = 0$. Here the host would actually open #3, giving you the sure shot, only $1/3$ of the time. The remaining $2/3$ of the time, when he opens #2, your win probability is only $1/2$. Your net probability is $1/3$ in each case, for a total of $2/3$.

5. NOTATION AND TERMINOLOGY. Let

$$\mathbf{P}(C_i) = \textit{a priori} \text{ probability that the car is at door } i,$$

a priori referring to the state of our knowledge before any doors have been opened. It is given that $\mathbf{P}(C_i) = 1/3$ $(i = 1, 2, 3)$.

The host's choice of which door to open is made in response to the actual location of the car. We say, picturesquely, that the events C_i (the car is at i) are the *causes* that produce the *effects* H_j (the host opens door j). The probabilities of the *effects given the causes* we call the *productive* probabilities; these are the conditional probabilities $\mathbf{P}(H_j|C_i)$. What we have called q is the productivity probability $\mathbf{P}(H_3|C_1)$. We also let

$$\mathbf{P}(H_j) = \textit{a priori} \text{ probability that the host opens door } j,$$

a priori meaning without knowledge of the location of the car. Finally, we wish to know the probabilities of the *causes given the effects*—the *a posteriori* probabilities. These are the conditional probabilities $\mathbf{P}(C_i|H_j)$. What we have called P is the *a posteriori* probability $\mathbf{P}(C_2|H_3)$.

6. BAYES'S FORMULA. Bayes's formula is the fundamental equation relating the *a posteriori* to the productive probabilities:

$$\mathbf{P}(H_3)\mathbf{P}(C_i|H_3) = \mathbf{P}(C_i)\mathbf{P}(H_3|C_i). \tag{2}$$

(Technically, it is this equation solved for $\mathbf{P}(C_i|H_3)$.) It is an immediate consequence of the law of compound probability, according to which each side is equal to $\mathbf{P}(H_3 \cap C_i)$. In its general setting, the probabilities $\mathbf{P}(C_i)$ may be unequal for some choices of i; but $\mathbf{P}(H_3)$ is still independent of i. (We could write H instead of H_3). Therefore

$$\mathbf{P}(C_i|H_3) \sim \mathbf{P}(C_i)\mathbf{P}(H_3|C_i). \tag{3}$$

When all the $\mathbf{P}(C_i)$ *are* equal,

$$\mathbf{P}(C_i|H_3) \sim \mathbf{P}(H_3|C_i) \qquad (\mathbf{P}(C_i) = \text{const.}) \tag{4}$$

When the a priori probabilities are all equal, the a posteriori probabilities are proportional to the productive probabilities. The proposition seems intuitively clear (even when the hypothesis is not explicitly acknowledged). It underlies all the examples in Section 4. (To hammer this home, assign unequal *a priori* probabilities and rework an example using (3) instead of (4).)

In the example with $q = 1/2$, $\mathbf{P}(H_3|C_2) = 1$ and $\mathbf{P}(H_3|C_1) = q = 1/2$. So the host is twice as likely to open #3 when the car is at #2 as when it is at #1. By (4) when the host *does* open #3, the car is twice as likely to be at #2 as at #1; therefore $P = \mathbf{P}(C_2|H_3) = 2/3$. In general, for any q,

$$\mathbf{P}(C_2|H_3) : \mathbf{P}(C_1|H_3) = 1 : q,$$

and we get $P = 1/(1 + q)$.

We gave an example in Section 4 to illustrate that the solution of Game I is consistent with that of Game II. Here is a general proof:

$$\begin{aligned}
&\mathbf{P}(\text{You win Game II if you switch}) \\
&\quad = \mathbf{P}(H_3 \cap C_2) + \mathbf{P}(H_2 \cap C_3) \\
&\quad = \mathbf{P}(C_2)\mathbf{P}(H_3|C_2) + \mathbf{P}(C_3)\mathbf{P}(H_2|C_3) \\
&\quad = \frac{1}{3} \times 1 + \frac{1}{3} \times 1 = \frac{2}{3}.
\end{aligned}$$

7. THE PARADOX OF THE SECOND ACE. My interest in problems of conditional probability was sparked, many years ago, by a passage in *Mathematical Recreations and Essays*, by W. W. Rouse Ball (now Ball and Coxeter [**1**, p. 44]). When a bridge hand is dealt from a deck of cards (13 cards from 52), the probability that it contains at least two aces turns out to be .26. *Question*: What is the probability that it contains at least two aces given that it contains (a) an ace, (b) the ace of hearts? We expect the answers to be the same and of course greater than .26. We are half right: they turn out to be (a) .37, (b) .56. I was able to wade through the binomial coefficients, but I still wondered *why* (b) should be greater than (a).

Here is a way to see why without computation. In terms of the complementary events, we wish to show that the probability of exactly one ace, given that the hand contains an ace, is greater than the probability of exactly one ace given that the hand contains the ace of hearts. This means we want

$$\frac{\mathbf{N}(!A)}{\mathbf{N}(A)} > \frac{\mathbf{N}(!A_H)}{\mathbf{N}(A_H)},$$

where $\mathbf{N}(!A)$ is the number of hands containing exactly one ace, $\mathbf{N}(!A_H)$ is the number of such hands whose unique ace is the ace of hearts, and $\mathbf{N}(A)$ and $\mathbf{N}(A_H)$ are the numbers of hands containing an ace or the ace of hearts, respectively. A slightly more convenient form is

$$\frac{\mathbf{N}(!A)}{\mathbf{N}(!A_H)} > \frac{\mathbf{N}(A)}{\mathbf{N}(A_H)}. \tag{5}$$

Since a unique ace in a hand must be one of the four specific aces, the numerator of the first fraction is exactly four times the denominator, and the fraction is equal to four. But in the second fraction the numerator is *less* than four times the denominator, because of overlaps—e.g., a hand containing the aces of both hearts and spades should be counted only once. This establishes (5).

8. EXAMPLES. The situation may be clarified further by considering a deck of four cards, two aces and two jacks, from which you are dealt a hand of two cards. There are six possible hands, one of them consisting of the two aces, so the probability you have both aces is $1/6$. If it is given that the hand contains an ace we have eliminated the two jacks, and the probability for both aces goes up to $1/5$. But if it is given that you have the ace of hearts, then your other card is either the ace of spades or one of the jacks, and the probability that you are holding both aces is now $1/3$.

Carrying this to the extreme, consider a two-card hand from a deck of *three* cards, two aces and a jack. There are three possible hands, and the probability that you have the two aces is $1/3$. If you state that the hand contains an ace, I smirk. But if we are given that the hand contains the ace of hearts, the probability for both aces goes up to $1/2$. At this point (if not long since) your friend enters the picture with a "proof" that the probability of both aces is $1/2$, with or without any condition: "You have an ace. Either it is the ace of hearts or the ace of spades. If it is the ace of hearts, then as we have just proved, the probability of both aces is $1/2$. If it is the ace of spades, then, similarly, the probability for both aces is $1/2$. So in either case it is $1/2$. So it is $1/2$." It is easier to detect the flaw in this reasoning than to get your friend to understand it. A suggested response (guaranteed not to help) is printed upside down at the end of the article.

9. OTHER PROBLEMS. While preparing this article I looked through a number of books for related material but found very little other than the classic gold and silver coins distributed in three two-drawer boxes (Bertrand's box paradox), and the family with two children of whom one is a girl. I felt that the car-and-goats problem must surely have appeared somewhere. Eventually I was steered to a 1959 column of Martin Gardner [**2**], who presents the problem in terms of three prisoners, one of whom is to be paroled. It is noteworthy that he states explicitly, as part of the hypothesis, that the warden is to flip a coin when he has a choice between two prisoners to name (corresponding to the host's picking Door 3 with probability $q = 1/2$). Gardner mentions in his column that the problem "is now making the rounds"; but he told me recently he has no recollection of how he came to hear about it.

I deliberately misquoted Ball's problem when I asked for the probability "given that" your hand contains an ace (or the ace of hearts). Ball says you *assert* that the hand contains an ace. In such a case I would want to know how you decide what statement to assert. My present rule is that you are to state whether your hand contains an ace. Instead, suppose the rule in the four-card problem is that you are to pick a random card from your hand and tell whether it is an ace or a jack. Now when you pick an ace, the probability for both aces is higher than before, since if you had a jack you could have picked it. In fact the productive probabilities for picking an ace are in the proportions $2:1:1:1:1$, and the *a posteriori* probability that you have both aces, originally $1/5$, is now $1/3$.

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REFERENCES

1. W. W. Rouse Ball and H. S. M. Coxeter, *Mathematical Recreations and Essays*, 13th edition, Dover, New York, 1987 (Ball, first edition, 1892).
2. Martin Gardner, Mathematical Games, *Scientific American*, 201 (1959), October 180–182, November 188.
3. Leonard Gillman, The car and goats fiasco, *Focus* (the MAA newsletter), 11 (1991), June, 8.
4. Marilyn vos Savant, "Ask Marilyn," *Parade*, (a), September, 9 1990; (b), December 2 1990; (c), February 17 1991.

Suggested response:

Annoy your friend by asking, "What is your definition of 'it'?"

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> Mathematics not only demands straight thinking, it grants the student the satisfaction of knowing when he is thinking straight.
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> —*D. Jackson*