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Physics 112 Second Midterm Solutions

Winter 2000

PART I

① (b) is the correct answer.

Use the following result for the Helmholtz free energy of an ideal gas:

$$F = -kTN \left[\ln \left(\frac{V}{N} \lambda_{th}^{-3} \right) + 1 \right]$$

where $\lambda_{th}^{-3} \equiv \left(\frac{m k T}{2\pi \hbar^2} \right)^{3/2}$. Initially, each gas occupies volume $\frac{V}{2}$, so

$$F_{\text{initial}} = -kTN_1 \left[\ln \left(\frac{V}{2N_1} \lambda_{th}^{-3} \right) + 1 \right] - kTN_2 \left[\ln \left(\frac{V}{2N_2} \lambda_{th}^{-3} \right) + 1 \right]$$

After the mixing, each gas occupies volume V . Then,

$$F_{\text{final}} = -kTN_1 \left[\ln \left(\frac{V}{N_1} \lambda_{th}^{-3} \right) + 1 \right] - kTN_2 \left[\ln \left(\frac{V}{N_2} \lambda_{th}^{-3} \right) + 1 \right]$$

$$\Delta F = F_{\text{final}} - F_{\text{initial}} = -kTN_1 \ln 2 - kTN_2 \ln 2$$

$$= -kT(N_1 + N_2) \ln 2$$

In this problem, $N_1 = N_A$ and $N_2 = 4N_A$ where N_A is Avogadro's number. Since $R \equiv N_A k$, we end up with

$$\Delta F = -5RT \ln 2.$$

②

② (a) is the correct answer.

For example, the Helmholtz free energy of a photon gas is

$$F = -\frac{\pi^2 k^4 T^4 V}{45 \hbar^3 c^3}$$

$$\mu = \left(\frac{\partial F}{\partial N} \right)_{T,V} = 0$$

③ (e) is the correct answer.

The entropy of a photon gas is

$$S = \frac{4\pi^2 V k^4 T^3}{45 \hbar^3 c^3}$$

which we can write as

$$S = \frac{16\sigma_B V T^3}{3c}$$

For $T = 300\text{K}$ and $V = 1\text{m}^3$

$$S = \frac{16 (5.67 \times 10^{-8} \text{W m}^{-2} \text{K}^{-4}) (1\text{m}^3) (300\text{K})^3}{3 (3 \times 10^8 \text{m s}^{-1})}$$

$$= 2.72 \times 10^{-8} \text{JK}^{-1}$$

The number of microstates is related to S via $S = k \ln g$.

$$\text{Thus, } g = \exp \left(\frac{S}{k} \right) = \exp \left(\frac{2.72 \times 10^{-8} \text{JK}^{-1}}{1.381 \times 10^{-23} \text{JK}^{-1}} \right)$$

$$= \exp (1.97 \times 10^{15}) = 10^{8.56 \times 10^{14}}$$

③

④ (d) is the correct answer.

$T = 10,000 \text{ K}$ is a high enough temperature such that the vibrational and rotational degrees of freedom are unfrozen. For a diatomic molecule, there are two rotational degrees of freedom and one vibrational mode.

Each rotational degree of freedom adds $\frac{1}{2}k$ to the specific heat per molecule. Each vibrational mode adds k . Thus,

$$\frac{C_V}{N} = \frac{3}{2}k + k + k = \frac{7}{2}k$$

For an ideal gas, $C_p = C_v + Nk$. Thus,

$$\frac{C_p}{C_v} = \frac{9/2}{7/2} = \frac{9}{7}$$

⑤ (a) and (c) are true.

An equilibrium macrostate is characterized by an energy $E(S, V, N)$. Thus, the change of energy between two equilibrium states does not depend on how the change is made.

If S and N are fixed, then $dS = dN = 0$ and

$$(dE)_{S,N} = -P(dV)_{S,N}$$

Thus,

$$P = -\left(\frac{dE}{dV}\right)_{S,N} = -\left(\frac{\partial E}{\partial V}\right)_{S,N}$$

④

⑥ (d) is the correct answer.

The probability of having angular momentum J is

$$P(J) = \frac{(2J+1) \exp\left[-\frac{\hbar^2 J(J+1)}{2IkT}\right]}{Z_R} = \frac{(2J+1) \exp\left[-\frac{J(J+1)\Theta_r}{T}\right]}{Z_R}$$

where Z_R is the partition function

$$Z_R \equiv \sum_{J=0}^{\infty} (2J+1) \exp\left(-\frac{J(J+1)\Theta_r}{T}\right)$$

If we set $P(2) = P(3)$, we obtain:

$$5e^{-6\Theta_r/T} = 7e^{-12\Theta_r/T}$$

Solving for T ,

$$e^{6\Theta_r/T} = \frac{7}{5}$$

$$T = \frac{6\Theta_r}{\ln\left(\frac{7}{5}\right)} = \frac{6(12.2 \text{ K})}{\ln\left(\frac{7}{5}\right)} = 218 \text{ K}$$

⑦ (c) is correct.

From problem 3, we note that VT^3 is constant for an expanding photon gas at constant entropy. The blackbody temperature observed today for the microwave background radiation is $T \approx 3 \text{ K}$. Since the temperature has dropped by a factor of 10^3 , the volume must have expanded by 10^9 in order that VT^3 be constant.

PART II

① (a) The problem specifies that $Pv^\gamma = \text{constant}$, where $v \equiv V/N$. We use the ideal gas law to eliminate v . Since $PV = NkT$,

$$v = \frac{kT}{P}$$

and so

$$P \left(\frac{kT}{P} \right)^\gamma = \text{constant}$$

which we can re-write as

$$P^{1-\gamma} T^\gamma = \text{constant}$$

(the constant on the right hand side above differs from the previous constant by a factor of k^γ).

Differentiate this equation.

$$(1-\gamma) P^{-\gamma} T^\gamma + \gamma P^{1-\gamma} T^{\gamma-1} dT = 0$$

Multiply through by $P^\gamma T^{-\gamma}$:

$$(1-\gamma) dp + \frac{\gamma}{T} dT = 0$$

Solve for $\frac{dT}{dp}$.

$$\boxed{\frac{dT}{dp} = \left(\frac{\gamma-1}{\gamma} \right) \frac{T}{P}}$$

(b) In homework set #16, we saw that the mechanical equilibrium condition applied to an infinitesimal slice of the atmosphere (assumed to be an ideal gas) yields

$$\frac{dP}{dz} = -\frac{mg}{kT} P$$

We are interested in computing $\frac{dT}{dz}$. By the chain rule,

$$\frac{dT}{dz} = \frac{dT}{dp} \frac{dp}{dz}$$

$$= -\frac{mgP}{kT} \frac{dT}{dp}$$

Using the result obtained in part (a),

$$\boxed{\frac{dT}{dz} = -\frac{mg}{k} \left(\frac{\gamma-1}{\gamma} \right)}$$

which is indeed a constant.

(c) For N_2 gas at temperatures around $T = 300 \text{ K}$, the vibrational modes are frozen, but two rotational degrees of freedom are active and contribute to C_v . Thus,

$$C_v = \frac{5}{2} Nk$$

$$C_p = \frac{7}{2} Nk$$

and we conclude that $\gamma = \frac{C_p}{C_v} = \frac{7}{5}$.

⑤

⑥

Plugging in to the result of part (b),

$$\frac{dT}{dz} = - \frac{(4.653 \times 10^{-26} \text{ kg})(9.8 \text{ m s}^{-2})}{1.381 \times 10^{-23} \text{ J K}^{-1}} \left(\frac{2/5}{7/5} \right)$$

$$= -9.434 \times 10^{-3} \text{ K m}^{-1}$$

Since degrees K and degrees C are equal in size (only the zero points of each scale are shifted relative to one another), we can rewrite this result as

$$\frac{dT}{dz} = -9.434^\circ\text{C}/\text{km}$$

Mt. Everest has an altitude of 8.848 km. Thus, the temperature difference between sea level and the peak of Mt. Everest is

$$\Delta T = (9.434^\circ\text{C})(8.848)$$

$$= 83.5^\circ\text{C}$$

Clearly, the top of Mt. Everest is quite cold!

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③ (a) The Debye temperature is given by

$$\Theta_D = \frac{\hbar v_s}{k} (6\pi^2 n)^{1/3}$$

where $n \equiv \frac{N}{V}$ and v_s is the speed of sound

[RB expresses Θ_D in terms of $\hbar \equiv 2\pi\hbar$ in eq (6.50), where the average interatomic spacing $a \equiv (V/N)^{1/3} = n^{-1/3}$ (see bottom of p.133)].

We first must compute n , given the mass density $\frac{m}{V} = 0.145 \text{ g cm}^{-3}$.

Recall that there are 6.022×10^{23} atoms in 1 mole of ^4He , and the mass of this quantity is equal to its molecular weight in grams, i.e. 4 g. Thus,

$$n = \frac{N}{V} = \frac{N}{m} \frac{m}{V}$$

$$= \frac{6.022 \times 10^{23}}{4 \text{ g}} (0.145 \text{ g cm}^{-3})$$

$$= 2.18 \times 10^{22} \text{ cm}^{-3}$$

$$= 2.18 \times 10^{28} \text{ m}^{-3}$$

Thus,

$$\Theta_D = \frac{(1.0546 \times 10^{-34} \text{ J s})(238.3 \text{ m s}^{-1})}{1.381 \times 10^{-23} \text{ J K}^{-1}} \left[6\pi^2 (2.18 \times 10^{28} \text{ m}^{-3}) \right]^{1/3}$$

$$= 19.8 \text{ K}$$

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(b) Since $T \ll \Theta_D$, we can use the low temperature approximation for the average energy $\langle E \rangle$ and the heat capacity

$$C_V = \left(\frac{\partial \langle E \rangle}{\partial T} \right)_V$$

In deriving the expression for the energy on pp 133-134 of RB, both longitudinal and transverse vibrational modes were included. In this problem, only the one longitudinal mode is counted, while two transverse modes are absent. Thus, we must reduce the result for the energy (and correspondingly for the heat capacity) by a factor of 3.

Note that the computation of the Debye temperature is not affected. The Debye temperature is $\Theta_D = \hbar \nu_{\max} / k_B$ [RB eq (6.49)] where

$$\int_0^{\nu_{\max}} D_{\text{Debye}}(\nu) d\nu = 3N \quad [\text{RB eq. (6.46)}]$$

determines ν_{\max} . When there is only a longitudinal mode per vibrating atom, $D_{\text{Debye}}(\nu)$ [the density of states] is reduced by a factor of 3, but the total number of vibrational modes for N atoms is N , not $3N$. So the calculation of ν_{\max} remains the same. However, reducing $D_{\text{Debye}}(\nu)$ by a factor of 3 also reduces $\langle E \rangle$ by a factor of 3 [see RB eq. (6.51)] since we are summing over $1/3$ of all possible vibrational modes compared to the general case.

The upshot of these remarks is that we must simply modify RB eq. (6.57) for the heat capacity at $T \ll \Theta_D$ by multiplying by a factor of $1/3$. The end result:

$$C_V = \frac{4\pi^4}{5} Nk \left(\frac{T}{\Theta_D} \right)^3$$

The problem asks for the heat capacity per kilogram. In 1 kg there are $\frac{1}{4} (6.022 \times 10^{26}) = 1.5055 \times 10^{26}$ atoms since 4g constitutes 1 mole of ^4He . Thus,

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$$\frac{C_V}{1 \text{ kg}} = \frac{4\pi^4 (1.5055 \times 10^{26}) (1.381 \times 10^{-23} \text{ J K}^{-1})}{5 (19.8 \text{ K})^3} = 20.87 \times T^3 \text{ J kg}^{-1} \text{ K}^{-1}$$

Not bad, when compared with the experimentally observed value of $20.4 \times T^3 \text{ J kg}^{-1} \text{ K}^{-1}$.