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1 RB Chapter 1, problem 1

We assume that the ideal gas law applies. Then

$$PV = NkT$$

We have $V_{final} = \frac{1}{3}V_{initial}$ and $T_{final} = 5T_{initial}$, while N is held fixed.

(a) $\frac{PV}{T} = \text{constant}$, so

$$\frac{P_{initial} V_{initial}}{T_{initial}} = \frac{P_{final} V_{final}}{T_{final}}$$

Hence,

$$\frac{P_{final}}{P_{initial}} = \left(\frac{V_{initial}}{V_{final}} \right) \left(\frac{T_{final}}{T_{initial}} \right)$$

$$= (3)(5) = 15$$

(b) Using eq (1.9) of RB,

$$\frac{1}{2} m \langle v^2 \rangle = \frac{3}{2} kT$$

Thus,

$$\frac{\langle v_{final}^2 \rangle}{\langle v_{initial}^2 \rangle} = \frac{T_{final}}{T_{initial}} = 5$$

and we conclude that the typical speed of a molecule (which is roughly given by $\langle v^2 \rangle^{1/2}$) increases by a factor of $\sqrt{5} \approx 2.24$.

(c) By eq. (1.3), the number of collisions per second is given by

$$\frac{1}{2} v_x A \frac{N}{V}$$

Now, $\frac{1}{2} m v_x^2 = \frac{1}{2} kT$ so $v_x \propto T^{1/2}$. Thus, the number of collisions per second is proportional to

$$T^{1/2} \frac{N}{V}$$

Therefore, the collision rate changes by a factor of

$$\left(\frac{T_{final}}{T_{initial}} \right)^{1/2} \left(\frac{V_{initial}}{V_{final}} \right) = (\sqrt{5})(3) \approx 6.71$$

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④ We repeat the analysis given of pp 4-6 for a "gas" of photons.

$$\left(\begin{array}{l} \text{force on area } A \\ \text{due to photons} \end{array} \right) = \left(\begin{array}{l} \text{momentum transferred} \\ \text{to wall per collision} \end{array} \right) \left(\begin{array}{l} \text{number of collisions} \\ \text{in time } \Delta t \end{array} \right)$$

The momentum of a photon of frequency ν is $p = \frac{h\nu}{c}$.

Suppose the geometry of the collision is as pictured in Figure 1.2 of RB. Then,

$$\left(\begin{array}{l} \text{momentum transferred} \\ \text{to the wall per collision} \end{array} \right) = p_x - (-p_x) = 2p_x$$

and

$$\left(\begin{array}{l} \text{number of collisions} \\ \text{in time } \Delta t \end{array} \right) = (v_x \Delta t A) \frac{1}{2} \frac{N}{V}$$

To evaluate p_x and v_x , we use spherical coordinates. For example,

$$\begin{aligned} p_x &= p \sin \theta \cos \phi \\ p_y &= p \sin \theta \sin \phi \\ p_z &= p \cos \theta \end{aligned}$$

where $p = \frac{h\nu}{c}$. Likewise,

$$v_x = c \sin \theta \cos \phi$$

where c is the speed of light. Therefore,

$$\begin{aligned} \text{pressure} &= \frac{\text{force}}{A} = \frac{2p_x (v_x \Delta t A) \frac{1}{2} \frac{N}{V}}{A \Delta t} \\ &= h\nu \sin^2 \theta \cos^2 \phi \frac{N}{V} \end{aligned}$$

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This result assumes that all photons are moving with the same direction (θ, ϕ) . In reality, we must average over all directions, so

$$p = \frac{Nh\nu}{V} \langle \sin^2 \theta \cos^2 \phi \rangle$$

To evaluate the average value of $\sin^2 \theta \cos^2 \phi$, we can either compute it directly or resort to a trick.

First, the trick. It clearly does not matter which side of the box I used to compute the pressure. Thus, I should get the same answer if I replace $p_x v_x$ with $p_y v_y$ or $p_z v_z$. This means that:

$$\langle \sin^2 \theta \cos^2 \phi \rangle = \langle \sin^2 \theta \sin^2 \phi \rangle = \langle \cos^2 \theta \rangle$$

which then must equal $\frac{1}{3}$ the sum of the three terms:

$$\begin{aligned} &= \frac{1}{3} \langle \sin^2 \theta \cos^2 \phi + \sin^2 \theta \sin^2 \phi + \cos^2 \theta \rangle \\ &= \frac{1}{3} \langle \sin^2 \theta (\cos^2 \phi + \sin^2 \phi) + \cos^2 \theta \rangle \\ &= \frac{1}{3} \langle \sin^2 \theta + \cos^2 \theta \rangle \\ &= \frac{1}{3} \langle 1 \rangle \\ &= \frac{1}{3} \end{aligned}$$

Hence,

$$p = \frac{1}{3} \frac{Nh\nu}{V}$$

If we identify $E = Nh\nu$ as the total energy of the photons, then

$$p = \frac{1}{3} \frac{E}{V}$$

which is a famous result from the theory of classical electromagnetic radiation!

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⑤

For those of you who are uncomfortable with tricks, here is the explicit computation of $\langle \sin^2 \cos^2 \phi \rangle$. The average is taken over all possible solid angles. Since

$$d\Omega = \sin \theta d\theta d\phi$$

with $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$, with $\int d\Omega = 4\pi$, we have

$$\begin{aligned} \langle \sin^2 \theta \cos^2 \phi \rangle &= \frac{1}{4\pi} \int d\Omega \sin^2 \theta \cos^2 \phi \\ &= \frac{1}{4\pi} \int_0^{2\pi} \cos^2 \phi d\phi \int_0^\pi \sin^3 \theta d\theta \end{aligned}$$

A few more tricks. If you look at the graphs of $\cos^2 \phi$ and $\sin^2 \phi$, it is clear that

$$\int_0^{2\pi} \cos^2 \phi d\phi = \int_0^{2\pi} \sin^2 \phi d\phi$$

so each integral is equal to

$$\frac{1}{2} \int_0^{2\pi} (\cos^2 \phi + \sin^2 \phi) d\phi = \frac{1}{2} \int_0^{2\pi} d\phi = \pi$$

For the integral over θ , let $w = \cos \theta$, $dw = -\sin \theta d\theta$ and so

$$\int_0^\pi \sin^3 \theta d\theta = \int_{-1}^1 (1-w^2) dw = w - \frac{w^3}{3} \Big|_{-1}^1 = \frac{4}{3}$$

where I've used $\sin^2 \theta = 1 - \cos^2 \theta = 1 - w^2$. Thus,

$$\langle \sin^2 \theta \cos^2 \phi \rangle = \frac{1}{4\pi} (\pi) \left(\frac{4}{3} \right) = \frac{1}{3}$$

⑥

Likewise, you can easily show that

$$\langle \sin^2 \theta \sin^2 \phi \rangle = \langle \cos^2 \theta \rangle = \frac{1}{3}$$

An astute student might argue that I shouldn't average over 4π steradians, since only photons moving in the positive x-direction will hit the wall of the box. But, I can just as well compute the averages over a hemisphere (2π steradians) to obtain:

$$\begin{aligned} \langle \sin^2 \theta \cos^2 \phi \rangle &= \frac{1}{2\pi} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} d\Omega \sin^2 \theta \cos^2 \phi \\ &= \frac{1}{3} \end{aligned}$$

and obtain the same result.

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⑦ RB, Chapter 1, problem 4

For adiabatic compression of an ideal gas, $PV^\delta = \text{constant}$.
 Since $PV = NkT$ for an ideal gas, we can eliminate P in favor of T :

$$\frac{NkT}{V} V^\delta = \text{constant}$$

or since N and k are constant (the number of gas molecules N is fixed in this problem),

$$TV^{\delta-1} = \text{constant}$$

Thus,

$$T_f V_f^{\delta-1} = T_i V_i^{\delta-1}$$

or

$$T_f = T_i \left(\frac{V_i}{V_f} \right)^{\delta-1}$$

(a) For $\gamma = 1.4$, $V_i/V_f = 15$ and $T_i = 20^\circ\text{C} = 293\text{K}$,

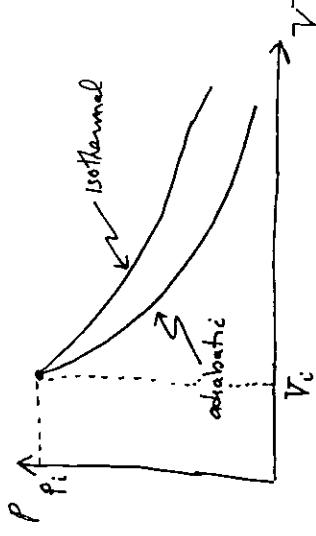
$$\begin{aligned} T_f &= 293 (15)^{0.4} \\ &= (293)(2.954) \\ &= 866\text{K} \\ &= 593^\circ\text{C} \end{aligned}$$

(b) $P_i V_i^\delta = P_f V_f^\delta$

So,

$$\frac{P_f}{P_i} = \left(\frac{V_i}{V_f} \right)^\delta = (15)^{1.4} = 44.3$$

⑧ RB Chapter 1, problem 5



The curve labeled isothermal is $PV = \text{constant}$.

The curve labeled adiabatic is $PV^\delta = \text{constant}$, with $\delta > 1$.

Since the curves intersect at the point (V_i, P_i) , we know the constant in each case:

isothermal curve: $P = \frac{P_i V_i}{V}$

adiabatic curve: $P = \frac{P_i V_i^\delta}{V^\delta}$

(a) To compute the slope, m , of each curve, take the derivative:

$$M_{\text{isothermal}} = -\frac{P_i V_i}{V^2}$$

$$M_{\text{adiabatic}} = -\delta \frac{P_i V_i^\delta}{V^{\delta+1}}$$

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(b) Consider the pressure at some value of $V > V_i$.

$$\frac{P_{\text{adiabatic}}}{P_{\text{isothermal}}} = \frac{P_i V_i^\gamma / V^\gamma}{P_i V_i / V} = \left(\frac{V_i}{V}\right)^{\gamma-1}$$

Since $\gamma > 1$, for all $V > V_i$ we have $\left(\frac{V_i}{V}\right)^{\gamma-1} < 1$.

This, $P_{\text{adiabatic}} < P_{\text{isothermal}}$ for all $V > V_i$.

Note: by the same reasoning, it is easy to verify that $P_{\text{adiabatic}} > P_{\text{isothermal}}$ for all $V < V_i$. Thus, the two curves only intersect at one point.

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Thus,

Adiabatic = Isothermal

$$\text{if } \frac{-P_i V_i}{V^2} = \frac{-\gamma P_i V_i^\gamma}{V^{\gamma+1}}$$

That is,

$$\gamma \left(\frac{V_i}{V}\right)^{\gamma-1} = 1$$

This is equivalent to:

$$\left(\frac{V}{V_i}\right)^{\gamma-1} = \gamma$$

or

$$(\gamma-1) \ln\left(\frac{V}{V_i}\right) = \ln \gamma$$

or

$$\ln\left(\frac{V}{V_i}\right) = \frac{1}{\gamma-1} \ln \gamma$$

or

$$\frac{V}{V_i} = \exp\left[\frac{1}{\gamma-1} \ln \gamma\right]$$

Since $\gamma > 1$, the right hand side is some number larger than 1. Thus, there exists a value of $V > V_i$ where the slopes of the two curves are equal.

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⑤ RB, Chapter 1, problem 8

(a) Isothermal expansion

As we worked out in class, using $PV = NkT$,

$$W = \int_{V_0}^{V_f} P dV = \int_{V_0}^{V_f} NkT \frac{dV}{V}$$

$$= NkT \int_{V_0}^{V_f} \frac{dV}{V}$$

Since T is constant during an isothermal process

$$= NkT \ln \left(\frac{V_f}{V_0} \right)$$

For $T = T_0$ and $V_f/V_0 = 7$,

$$W = NkT_0 \ln 7.$$

The energy of an ideal monatomic ideal gas is

$$E = \frac{3}{2} NkT$$

Thus, E does not change during an isothermal process.

From the first law of thermodynamics, $Q = \Delta E + W$.
Since $\Delta E = 0$, we have

$$Q = W = NkT_0 \ln 7.$$

(b) isobaric expansion

$$W = \int_{V_0}^{V_f} P dV = P \int_{V_0}^{V_f} dV = P(V_f - V_0)$$

since P is constant in an isobaric process. Putting $P = P_0$, $V_f = 7V_0$

$$W = 6P_0V_0$$

$$= 6NkT_0$$

where T_0 is the initial temperature. In the last step we have used the ideal gas law.

The final temperature T_f is given by

$$P_f V_f = NkT_f$$

But $P_f = P_0$ and $V_f = 7V_0$. Thus,

$$P_0 V_f = 7P_0 V_0 = 7NkT_0$$

and we conclude that

$$T_f = 7T_0$$

Using $E = \frac{3}{2} NkT$,

$$\Delta E = \frac{3}{2} Nk(T_f - T_0) = \frac{3}{2} Nk(7T_0 - T_0) = 9NkT_0$$

Hence, by the first law of thermodynamics

$$Q = \Delta E + W = 9NkT_0 + 6NkT_0$$

$$= 15NkT_0$$

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(c) adiabatic expansion

Following the derivation given in class, we use the fact that $PV^\gamma = \text{constant}$ in an adiabatic process involving an ideal gas.

Thus,

$$W = \int_{V_0}^{V_f} P dV = \int_{V_0}^{V_f} P V^\gamma V^{-\gamma} dV$$

Since $PV^\gamma = P_0 V_0^\gamma$ is a constant, we can take it outside the integral, obtaining

$$\begin{aligned}
W &= P_0 V_0^\gamma \int_{V_0}^{V_f} V^{-\gamma} dV \\
&= \frac{P_0 V_0^\gamma}{1-\gamma} \left[V_f^{1-\gamma} - V_0^{1-\gamma} \right] \\
&= \frac{P_0}{1-\gamma} \left[V_f \left(\frac{V_0}{V_f} \right)^\gamma - V_0 \right] \\
&= \frac{P_0 V_0}{1-\gamma} \left[\left(\frac{V_0}{V_f} \right)^{\gamma-1} - 1 \right] \\
&= \frac{NkT_0}{1-\gamma} \left[\left(\frac{V_0}{V_f} \right)^{\gamma-1} - 1 \right]
\end{aligned}$$

Putting $\gamma = \frac{C_p}{C_v} = \frac{5}{3}$ for an ideal monatomic ideal gas

and $V_f/V_0 = 7$

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$$\begin{aligned}
W &= \frac{3}{2} NkT_0 \left[1 - \left(\frac{1}{7} \right)^{2/3} \right] \\
&= 1.09 NkT_0
\end{aligned}$$

In RB, Chapter 1, problem 4 (the third problem on the homework set), we saw that

$$TV^{\gamma-1} = \text{constant}$$

$$\text{Thus, } T_f V_f^{\gamma-1} = T_0 V_0^{\gamma-1}$$

$$T_f = T_0 \left(\frac{V_0}{V_f} \right)^{\gamma-1}$$

$$= T_0 \left(\frac{1}{7} \right)^{2/3}$$

$$= 0.27 T_0$$

and so,

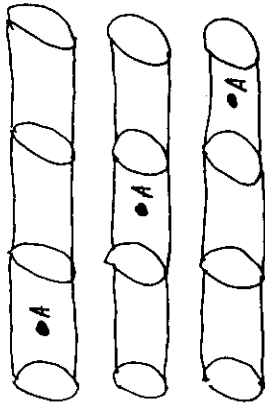
$$\Delta E = \frac{3}{2} Nk(T_f - T_0) = -1.09 NkT_0$$

As expected, $Q = \Delta E + W = 0$ since $Q=0$ is the definition of an adiabatic process.

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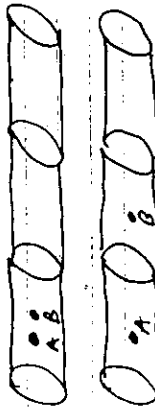
6 RB, Chapter 2, problem 3

(a) One molecule can appear in any of the three special locations. Each corresponds to a different microstate.



So there are 3 microstates in all

(b) With two molecules A and B, A can appear in any of the three special locations, and likewise for B. There are $3 \times 3 = 9$ possible microstates, e.g.



and seven other configurations

(c) For three molecules A, B, and C, there are $3 \times 3 \times 3 = 27$ microstates.

(d) For N molecules, there are 3^N microstates.

(e) If the cylinder were enlarged to contain four special locations, then the corresponding answers in parts (a)-(d) could be 4, 16, 64 and 4^N respectively.

(f) In general, one expects the number of microstates to be proportional to V^N .

7 RB, Chapter 2, problem 7

(a) Using the ideal gas law $PV = NkT$

$$N = \frac{PV}{kT} = \frac{(1.013 \times 10^5 \text{ N m}^{-2})(1.4 \times 10^{-3} \text{ m}^3)}{(1.381 \times 10^{-23} \text{ J/K})(300 \text{ K})}$$

$$= 3.42 \times 10^{22} \text{ atoms}$$

Here, I used the fact that

$$1 \text{ atm pressure} = 1.013 \times 10^5 \text{ N m}^{-2}$$

and of course, $1 \text{ J} = 1 \text{ N} \cdot \text{m}$. Thus, the above ratio is dimensionless, as expected.

(b) In sudden expansion, no work is done. Since the walls are thermally insulating, no heat is added or subtracted. Hence, by the first law of thermodynamics, $q = \Delta E + w$ which implies that $\Delta E = 0$ (since $q = w = 0$). For an ideal gas, E depends only on temperature T . Thus, $\Delta E = 0$ implies that $\Delta T = 0$.

Since the internal energy is unchanged, the increase in multiplicity is due solely to the increase in volume. The multiplicity is proportional to V^N . Hence,

$$\frac{(\text{multiplicity})_{\text{final}}}{(\text{multiplicity})_{\text{initial}}} = \left(\frac{V_f}{V_i}\right)^N = (1.2)^N$$

Using the results of part (a), this is equal to

$$(1.2)^{3.42 \times 10^{22}} = 10^{2.71 \times 10^{21}}$$

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The multiplicity has increased because of the increased volume that can be occupied by the gas atoms. Using $S = k \ln g$ where g is the multiplicity,

$$\begin{aligned} \Delta S &= k \ln \left(\frac{g_f}{g_i} \right) = k \ln \left(\frac{V_f}{V_i} \right)^N \\ &= Nk \ln \frac{V_f}{V_i} \\ &= Nk \ln 1.2 \end{aligned}$$

Plugging in the numbers,

$$\begin{aligned} \Delta S &= (3.42 \times 10^{22}) (1.381 \times 10^{-23} \text{ J/K}) \ln 1.2 \\ &= 8.61 \times 10^{-2} \text{ J/K} \end{aligned}$$

(c) Instead of a sudden expansion, we could increase the entropy by raising its temperature. If we add heat to the system quasi-statically, then

$$\Delta S = \frac{q}{T}$$

But at constant V , $C_V = \frac{(f) \text{ constant } V \cdot \Delta T}{\Delta T}$. Thus,

$$\Delta S = \frac{C_V \Delta T}{T}$$

We can integrate this equation

$$S_f - S_i = \int_{T_i}^{T_f} C_V \frac{dT}{T}$$

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For an ideal gas, C_V is a constant, independent of temperature. Gaseous helium is monatomic, so

$$C_V = \frac{3}{2} Nk.$$

$$\text{Thus, } S_f - S_i = \frac{3}{2} Nk \int_{T_i}^{T_f} \frac{dT}{T} = \frac{3}{2} Nk \ln \left(\frac{T_f}{T_i} \right)$$

To obtain the same entropy change as in part (b), we set $S_f - S_i = Nk \ln 1.2$.

$$\text{Then, } \ln \left(\frac{T_f}{T_i} \right) = \frac{2}{3} \ln 1.2$$

$$= 0.12$$

or

$$\begin{aligned} T_f &= 1.13 T_i \\ &= (1.13)(300 \text{ K}) \\ &= 339 \text{ K} \end{aligned}$$

Thus, the required input heat is:

$$\begin{aligned} q &= C_V \Delta T = \frac{3}{2} Nk (T_f - T_i) \\ &= \frac{3}{2} (3.42 \times 10^{22}) (1.381 \times 10^{-23} \text{ J/K}) (39 \text{ K}) \\ &= 27.5 \text{ J} \end{aligned}$$