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D) RB Chapter 2, problem 8

(a) For a classical ideal gas, the multiplicity g depends on the volume V and the temperature T as:

$$g \propto V^N T^{3N/2}$$

where N is the number of molecules in the gas. Since $S = k \ln g$, the change in the entropy of the gas when the volume V is reduced at constant T is:

$$\Delta S_{\text{gas}} = S_f - S_i = k \ln \left(\frac{g_f}{g_i} \right) = Nk \ln \left(\frac{V_f}{V_i} \right)$$

where i refers to the initial state and f refers to the final state. Note that the temperature and the constant of proportionality in $g \propto V^N T^{3N/2}$ drop out in the ratio of multiplicities.

In this problem, $V_i = V_0 = 10^{-3} \text{ m}^3$, $V_f = 0.98 V_0$ and $N = 3 \times 10^{22}$ atoms. Thus,

$$\begin{aligned} \Delta S_{\text{gas}} &= (1.381 \times 10^{-23} \text{ J/K}) (3 \times 10^{22}) \ln 0.98 \\ &= -8.37 \times 10^{-3} \text{ J/K}. \end{aligned}$$

(b) The total work done by external forces in compressing the gas is

$$w = - \int_{V_i}^{V_f} P dV$$

Note the minus sign. In class, we showed that $P \Delta V$ is the work done by the gas, which is the negative of the work done on the gas.

②

Using the ideal gas law, $PV = NkT$. Thus, at fixed temperature T ,

$$w = -NkT \int_{V_i}^{V_f} \frac{dV}{V} = -NkT \ln \left(\frac{V_f}{V_i} \right)$$

As in part (a), $V_f/V_i = 0.98$. Using $T = 300 \text{ K}$ and $N = 3 \times 10^{22}$ atoms,

$$\begin{aligned} w &= -(3 \times 10^{22}) (1.38 \times 10^{-23} \text{ J/K}) (300 \text{ K}) \ln 0.98 \\ &= 2.51 \text{ J} \end{aligned}$$

(c) The energy of an ideal gas depends only on the temperature T . Since T is fixed in this problem, there is no change in the gas energy, i.e. $\Delta E = 0$. By the first law of thermodynamics,

$$q = \Delta E - w$$

where w is the work done on the gas, as in part (b), and q is the heat energy absorbed by the gas, which is the negative of the heat energy, $q_{\text{env}} = -q$ absorbed by the environment. Since $\Delta E = 0$, it follows that $q_{\text{env}} = w = 2.51 \text{ J}$.

(d) The entropy change in the environment (for a quasi-static process at constant temperature) is given by

$$\Delta S_{\text{env}} = \frac{q_{\text{env}}}{T} = \frac{2.51 \text{ J}}{300 \text{ K}} = 8.37 \times 10^{-3} \text{ J/K}.$$

Note that

$$\Delta S_{\text{gas}} + \Delta S_{\text{env}} = 0$$

as expected for a quasi-static non-dissipative process.

② RB, Chapter 2, problem 10

(a) The heat capacity of water is $C_v = (4.2 \text{ J/gram}\cdot\text{K}) (18 \text{ grams})$
 $= 75.6 \text{ J/K}$

Then,

$$\Delta S = \int_{T_i}^{T_f} \frac{C_v}{T} dT$$

For $T_i = 0^\circ\text{C} = 273 \text{ K}$
 $T_f = 1^\circ\text{C} = 274 \text{ K}$

$$\Delta S = C_v \ln\left(\frac{T_f}{T_i}\right) = (75.6 \text{ J/K}) \ln\left(\frac{274}{273}\right)$$

$$= 0.28 \text{ J/K.}$$

Note: in this case, $T_f = T_i + \Delta T$ where ΔT is small, so
 $\ln\left(\frac{T_f}{T_i}\right) = \ln\left(1 + \frac{\Delta T}{T_i}\right) \approx \frac{\Delta T}{T_i}$

Noting that $q = C_v \Delta T = 75.6 \text{ J}$, we could have computed

$$\Delta S = \frac{q}{T_i} = \frac{75.6 \text{ J}}{273 \text{ K}} = 0.28 \text{ J/K}$$

with the same result.

The change in multiplicity is

$$\frac{g_f}{g_i} = \exp \frac{\Delta S}{k} = \exp \frac{0.28 \text{ J/K}}{1.38 \times 10^{-23} \text{ J/K}} = e^{2.03 \times 10^{22}}$$

We can rewrite this as

$$\frac{g_f}{g_i} = 10^{(0.4343)(2.03 \times 10^{22})} = 10^{8.8 \times 10^{21}}$$

(b) On p. 37 of RB, he computes the increase in the multiplicity by a factor of $10^{6.9 \times 10^{23}}$ in the melting of 18 grams of ice to liquid water. The reason that this change in multiplicity is larger than the result found in part (a) is due to the fact ice has a periodic crystal structure, while water is more irregular in the placement of atoms. Thus, there is a larger change in multiplicity when changing from a more ordered to less ordered state as compared with the increase in multiplicity due simply to the increase in the allowed energies of the individual molecules when the temperature of water is raised by 1°C .

(c) If we raise the temperature from $T = 0^\circ\text{C}$ to $T = 70^\circ\text{C}$, the change of entropy, using the results of part (a), is given by:

$$\Delta S = C_v \ln\left(\frac{T_f}{T_i}\right) = (75.6 \text{ J/K}) \ln\left(\frac{343}{273}\right)$$

$$= 17.3 \text{ J/K}$$

where we have set $T_i = 0^\circ\text{C} = 273 \text{ K}$ and $T_f = 70^\circ\text{C} = 343 \text{ K}$.

Compared to $\Delta S = 0.28 \text{ J/K}$, this is an increase by a factor of 62.

Note that $q = C_v \Delta T = 5.3 \times 10^3 \text{ J}$, an increase in the heat absorbed by a factor of 70 as compared with part (a). However, ΔS increases by a somewhat smaller factor since the temperature in the denominator of $\Delta S = q/T$ increases during the heating interval.

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③ RB Chapter 3, problem 1

(a) Given $T_h = 550^\circ\text{C} = 823\text{ K}$
 $T_c = 30^\circ\text{C} = 303\text{ K}$

the Carnot efficiency is

$$\eta_c = 1 - \frac{T_c}{T_h} = 1 - \frac{303}{823} = 0.632$$

(b) Given $W = 1500\text{ J}$, we are asked to compute Q_h and Q_c .

By definition,

$$\eta_c = \frac{W}{Q_h}$$

$$\text{so } Q_h = \frac{W}{\eta_c} = \frac{1500\text{ J}}{0.632} = 2374\text{ J}$$

Moreover, the first law of thermodynamics implies that

$$W = Q_h - Q_c$$

Hence,

$$Q_c = Q_h - W = 874\text{ J}$$

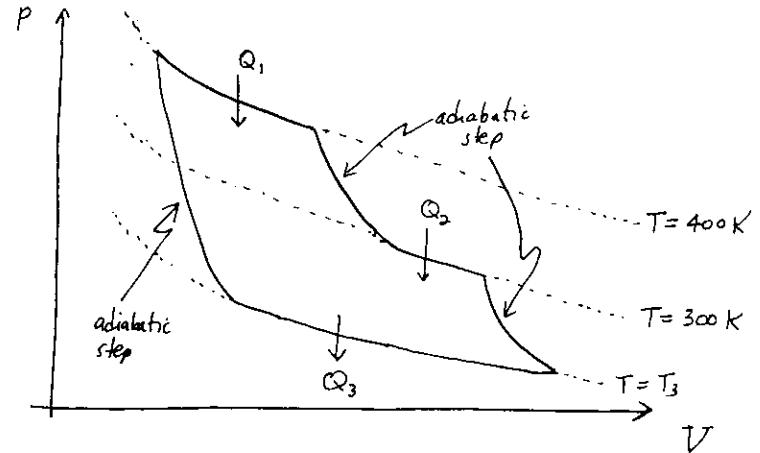
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④ RB Chapter 3, problem 4

(a) Over the complete cycle, $\Delta S_{\text{system}} = 0$ because the system returns to its original macrostate and hence its original multiplicity.

(b) Following the idea of the Carnot cycle, we consider isothermal followed by adiabatic steps. The adiabatic steps (performed quasi-statically) are used to change the temperature of the system so it is ready for the next isothermal process.

Here is a possible cycle in the P - V plane



Three isotherms are shown above, plus the adiabatic steps required to complete the cycle.

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(c) During the isothermal steps, $\Delta S = \frac{Q}{T}$, where Q is the

heat energy absorbed by the system. During the adiabatic steps, $\Delta S = 0$. Both these results are obtained under the assumption that the cycle is reversible, i.e. each step in the cycle is quasi-static and non-dissipative.

Thus, using part (a),

$$\Delta S_{\text{system}} = \frac{Q_1}{T_1} + \frac{Q_2}{T_2} - \frac{Q_3}{T_3} = 0.$$

Note that the minus sign in the third term appears since the heat Q_3 leaves the system.

Moreover, using the first law of thermodynamics, the total work done by the system is

$$W = Q_1 + Q_2 - Q_3$$

Putting in the numbers: $W = 100 \text{ J}$, $Q_1 = 200 \text{ J}$, $Q_2 = 300 \text{ J}$ gives $Q_3 = 400 \text{ J}$. Then, from $\Delta S_{\text{system}} = 0$,

$$\begin{aligned} \frac{Q_3}{T_3} &= \frac{Q_1}{T_1} + \frac{Q_2}{T_2} = \frac{200 \text{ J}}{400 \text{ K}} + \frac{300 \text{ J}}{300 \text{ K}} \\ &= 1.5 \text{ J/K} \end{aligned}$$

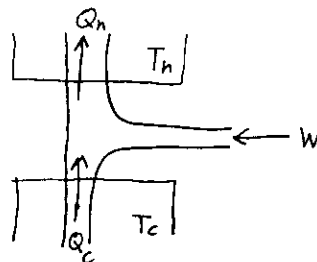
Thus,

$$T_3 = \frac{Q_3}{1.5 \text{ J/K}} = \frac{400 \text{ J}}{1.5 \text{ J/K}} = 267 \text{ K}.$$

(8)

(5) RB Chapter 3, problem 8

(a) The picture of the Carnot refrigerator is:



$$\Delta S_{\text{system}} = 0 \Rightarrow \frac{Q_c}{T_c} - \frac{Q_h}{T_h} = 0$$

The first law of thermodynamics implies that

$$Q_h = Q_c + W$$

We are given that

$$T_c = 25^\circ\text{C} = 298 \text{ K}$$

$$T_h = 32^\circ\text{C} = 305 \text{ K}$$

$$Q_c = 3000 \text{ J/s}$$

Then,

$$\begin{aligned} Q_h &= Q_c \left(\frac{T_h}{T_c} \right) = 3000 \text{ J/s} \left(\frac{305 \text{ K}}{298 \text{ K}} \right) \\ &= 3070.5 \text{ J/s} \end{aligned}$$

and therefore

$$W = Q_h - Q_c = 70.5 \text{ J/s}$$

which is the electrical power needed to run the air conditioner

(9)

(b) Suppose $T_h = 39^\circ\text{C}$, i.e. $\Delta T = T_h - T_c$ doubles relative to part (a). We assume that Q_h also doubles.

$$\begin{aligned} W &= Q_h - Q_c \\ &= Q_c \left(\frac{T_h}{T_c} \right) - Q_c \\ &= Q_c \left(\frac{T_h - T_c}{T_c} \right) \\ &= \frac{Q_c \Delta T}{T_c} \end{aligned}$$

If Q_c is also proportional to ΔT , then W is proportional to $(\Delta T)^2$. Thus, the required electrical power is quadrupled if ΔT is doubled.

With the numbers of part (a), if $T_h = 39^\circ\text{C}$ and $Q_c = 6000 \text{ J/s}$, then $W = 282 \text{ J/s}$.

(10)

⑥ Chapter 3, problem 9

(a) We use eq (3.9) of RB

$$S_f - S_i = \int_{T_i}^{T_f} \frac{C(T)}{T} dT$$

In this problem, we assume that $C(T) \equiv C$ is a constant. Thus, the change in entropy of the penny from initial temperature T_i to the final temperature $T_f = T_{\text{res}}$ of the heat reservoir is

$$\Delta S_{\text{penny}} = \int_{T_i}^{T_{\text{res}}} \frac{C}{T} dT = C \ln \left(\frac{T_{\text{res}}}{T_i} \right)$$

To compute the entropy change of the reservoir, we compute

$$\Delta S_{\text{res}} = \frac{q_{\text{res}}}{T_{\text{res}}}$$

where q_{res} is the heat energy absorbed by the reservoir. But

$$q_{\text{res}} = -q$$

where q is the heat energy absorbed by the penny. By definition of the heat capacity C ,

$$q = C \Delta T = C(T_{\text{res}} - T_i)$$

Hence, we conclude that

$$\Delta S_{\text{res}} = \frac{-C(T_{\text{res}} - T_i)}{T_{\text{res}}} = -C \left(1 - \frac{T_i}{T_{\text{res}}} \right)$$

(11)

(b) The total entropy change is

$$\begin{aligned}\Delta S &= \Delta S_{\text{penny}} + \Delta S_{\text{res}} \\ &= C \left[\ln\left(\frac{T_{\text{res}}}{T_i}\right) - 1 + \frac{T_i}{T_{\text{res}}} \right]\end{aligned}$$

I will now demonstrate that $\Delta S \geq 0$. Let $x = \frac{T_i}{T_{\text{res}}}$ and consider the function:

$$f(x) = x - 1 - \ln x \quad (\text{defined for all } x \geq 0)$$

Then, taking derivatives,

$$f'(x) = 1 - \frac{1}{x}$$

$$f''(x) = \frac{1}{x^2}$$

Thus, $f'(x) = 0 \Rightarrow x = 1$, and since $f''(1) > 0$ we conclude that the point $x = 1$ is a minimum of the function. Since $f(1) = 0$, we conclude that $f(x) \geq 0$ (for all $x \geq 0$).

Since $\Delta S = C f\left(\frac{T_i}{T_{\text{res}}}\right)$, and $C > 0$, we conclude that $\Delta S \geq 0$.

(c) The conclusion of part (b) does not depend on whether $T_i < T_{\text{res}}$ or $T_i > T_{\text{res}}$.

(12)

⑦ RB Chapter 4, problem 1

From Appendix A of RB on p. 420, we see that

$$\hbar = \frac{h}{2\pi} = 1.055 \times 10^{-34} \text{ J}\cdot\text{s}$$

$$m_{\text{He}^4} = 6.649 \times 10^{-27} \text{ kg}$$

$$1 \text{ eV} = 1.602 \times 10^{-19} \text{ J}$$

In class, we derived the density of states

$$D(\epsilon) = \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \epsilon^{1/2}$$

(a) The number of single-particle energy states with $0 \leq \epsilon \leq \epsilon_{\text{max}}$ is given by

$$\begin{aligned}\int_0^{\epsilon_{\text{max}}} D(\epsilon) d\epsilon &= \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \int_0^{\epsilon_{\text{max}}} \epsilon^{1/2} d\epsilon \\ &= \frac{V}{6\pi^2} \left(\frac{2m\epsilon_{\text{max}}}{\hbar^2} \right)^{3/2}\end{aligned}$$

Putting in $V = 10^{-3} \text{ m}^3$

$$\epsilon_{\text{max}} = \frac{1}{40} \text{ eV} = 4 \times 10^{-21} \text{ J}$$

we obtain

$$\begin{aligned}\text{number of energy states} &= \frac{10^{-3} \text{ m}^3}{6\pi^2} \left[\frac{(2)(6.649 \times 10^{-27})(4 \times 10^{-21})}{(1.055 \times 10^{-34})^2} \right]^{3/2} \\ \text{with } \epsilon \leq 0.025 \text{ eV} & \\ &= 5.59 \times 10^{27} \text{ states}\end{aligned}$$

(13)

(b) In the range $0.025 \text{ eV} \leq \epsilon \leq 0.026 \text{ eV}$, we could repeat the computation of part (a) for $\epsilon_{\text{max}} = 0.026 \text{ eV}$ and subtract the result of part (a). That is,

$$\begin{aligned} \text{number of energy states} &= \frac{10^{-3}}{6\pi^2} \left[\frac{(2)(6.699 \times 10^{-27})(0.026)(1.602 \times 10^{-19})}{(1.055 \times 10^{-34})^2} \right]^{3/2} \\ \text{with } \epsilon \leq 0.026 \text{ eV} & \\ &= 5.93 \times 10^{27} \end{aligned}$$

Using part (a), we end up with

$$\begin{aligned} \text{number of energy states} &= 5.93 \times 10^{27} - 5.59 \times 10^{27} \\ \text{with } 0.025 \text{ eV} \leq \epsilon \leq 0.026 \text{ eV} & \\ &= 3.4 \times 10^{26} \text{ states} \end{aligned}$$

However, since the energy range in question

$$\Delta \epsilon = 0.026 \text{ eV} - 0.025 \text{ eV} = 0.001 \text{ eV}$$

is much less than 0.025 eV , then to a very good approximation,

$$\begin{aligned} \text{number of energy states} &\approx \mathcal{D}(\epsilon) \Delta \epsilon \\ \text{with } 0.025 \text{ eV} \leq \epsilon \leq 0.026 \text{ eV} & \end{aligned}$$

where we evaluate the product at $\epsilon = 0.025 \text{ eV} = 4 \times 10^{-21} \text{ J}$

Then, noting that $\Delta \epsilon = 10^{-3} \text{ eV} = 1.6 \times 10^{-22} \text{ J}$

$$\begin{aligned} \mathcal{D}(\epsilon) \Delta \epsilon &= \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \epsilon^{1/2} \Delta \epsilon \\ &= \frac{10^{-3}}{4\pi^2} \left[\frac{(2)(6.699 \times 10^{-27})}{(1.055 \times 10^{-34})^2} \right]^{3/2} (4 \times 10^{-21})^{1/2} (1.6 \times 10^{-22}) \\ &= 3.35 \times 10^{26} \text{ states} \end{aligned}$$

(14)

Although an approximation is involved, our final result is quite close to the result obtained by a more accurate computation.