Physics 112 Solution Set #3 Winter 2000

1. The easiest way to solve this problem is to construct the sample space and count the various possibilities.

Suppose the sample population is 10,000. Since 1% of the population has cancer, this means that there are 100 cancer victims. Of the people who have cancer, 99% will test positive. That is, of the 100 cancer victims, there will be 99 positive test results.

Of the people who do <u>not</u> have cancer, 2% will test positive. That is, of the 10,000 − 100 = 9900 people without cancer, there will be

$$9900 \times 0.02 = 198$$

"false" positives.

The total number of positive test results is $99+198=297$. Of this sample, 99 have cancer. Thus, the probability of having cancer if you test positive is

$$\frac{99}{297} = \frac{1}{3}.$$

We can derive the same result in a more complicated way using Bayes' theorem. For completeness, I provide the derivation here. Let:

$P(A)$ = probability of testing positive
$P(C)$ = probability of having cancer

We are given the following facts:

(i) $P(C) = 1\%$, i.e. 1% of the population has cancer
(ii) $P_C(A) = 99\%$, i.e. of the people who have cancer, 99% will test positive.
(iii) $P_{\bar{C}}(A) = 2\%$, i.e. of the people who do <u>not</u> have cancer (denoted by $\bar{C}$), 2% will test positive.

We are asked to compute $P_A(C)$ = probability of having cancer if you test positive

Using Bayes' theorem,

$$P(AC) = P_A(C)P(A) = P_C(A)P(C)$$

Hence,

$$P_A(C) = \frac{P(C)P_C(A)}{P(A)}$$

Finally, we note that

$$P(A) = P(AC) + P(A\bar{C})$$

since C and $\bar{C}$ are independent, while $C+\bar{C}$ yields the entire sample space. Again using Bayes' theorem,

$$P(A\bar{C}) = P_{\bar{C}}(A)P(\bar{C}) = P_{\bar{C}}(A)[1-P(C)]$$

since $P(C)+P(\bar{C})=1$. Thus, we can write

$$P(A) = P_C(A)P(C) + P_{\bar{C}}(A)[1-P(C)]$$

Our final formula then reads:

$$P_A(C) = \frac{P(C)P_C(A)}{P_C(A)P(C) + P_{\bar{C}}(A)[1-P(C)]}$$

Plugging in $P(C)=1\%$, $P_C(A)=99\%$ and $P_{\bar{C}}(A)=2\%$,

$$P_A(C) = \frac{(0.01)(0.99)}{(0.01)(0.99) + (0.02)(0.99)}$$

or

$$\boxed{P_A(C) = \frac{1}{3}}.$$

(2) Suppose a year has N days ($N=365$, since for simplicity we ignore birthdays on February 29 which can only occur in leap years), and there are n people selected at random. Let us count the number of possible birthday combinations. In total, there are N^n possibilities.

Among these possibilities, let us count the number of such choices which contain __no__ matching birthdays. The answer is:

$$\underbrace{N(N-1)(N-2)\cdots(N-n+1)}_{n \text{ factors}}.$$

The reason is that there are N choices for the first birthday, $N-1$ for the second (since it does not match the first birthday), etc.

Let p_n be the probability that n people selected at random all have different birthdays. Clearly,

$$p_n = \frac{N(N-1)(N-2)\cdots(N-n+1)}{N^n}$$

(a) For $n=2$,

$$p_2 = \frac{(365)(364)}{365^2} = \frac{364}{365} = 0.99726$$

(b) For $n=3$,

$$p_3 = \frac{(365)(364)(363)}{365^3} = \left(\frac{364}{365}\right)\left(\frac{363}{365}\right) = 0.99180$$

(c) Assuming $n \ll N$, we can approximate $\ln p_n$ as follows. From the general formula,

$$p_n = \left(1-\frac{1}{N}\right)\left(1-\frac{2}{N}\right)\cdots\left(1-\frac{n-1}{N}\right)$$

Thus, using $\ln(1+x) \simeq x$ for $|x| \ll 1$,

$$\ln p_n \simeq -\frac{1}{N}\left[1+2+3+\ldots+(n-1)\right]$$

But,

$$1+2+3+\ldots+(n-1) = \frac{n(n-1)}{2}$$

Thus,

$$\boxed{\ln p_n \simeq \frac{-n(n-1)}{2N}}$$

(d) Noting that $\ln\left(\frac{1}{2}\right) = -0.69315$, we must solve for n:

$$n(n-1) = 506 \qquad \text{(after putting } N=365\text{)}$$

$$n = 23$$

Alternatively, we check that:

n	$p_n \simeq \exp\left(\frac{-n(n-1)}{730}\right)$	p_n (exact)
22	0.53106	0.524305
23	0.49998	0.492703
24	0.46946	0.461656

Since we expect the approximation to be rather good, we conclude that $n=23$ is the smallest value of n such that $p_n < \frac{1}{2}$. In other words, for $n=23$, there is a 50-50 chance that at least two birthdays will match.

I have included in the table above the exact value of p_n, which I computed using a calculator.

(3) In class, I showed that for N balls (distinguishable) distributed in k containers, there were k^N possible arrangements. As for the number of possible configurations with N_1 balls in container 1, N_2 balls in container 2, …, N_k balls in container k, the result obtained in class was

$$\frac{N!}{N_1!\,N_2!\cdots N_k!}$$

In this problem, $N=2$ and $k=3$.

(a) For $N=2$ and $k=3$ there should be $3^2=9$ possible arrangements when the balls are distinguishable. Label the two balls ① and ②. The nine possibilities are:

				configuration	
1.	①②			$\{2,0,0\}$	$1=\frac{2!}{2!0!0!}$
2.		①②		$\{0,2,0\}$	$1=\frac{2!}{0!2!0!}$
3.			①②	$\{0,0,2\}$	$1=\frac{2!}{0!0!2!}$
4.	①	②		$\{1,1,0\}$	$2=\frac{2!}{1!1!0!}$
5.	②	①		$\{1,1,0\}$	
6.	①		②	$\{1,0,1\}$	$2=\frac{2!}{1!0!1!}$
7.	②		①	$\{1,0,1\}$	
8.		①	②	$\{0,1,1\}$	$2=\frac{2!}{0!1!1!}$
9.		②	①	$\{0,1,1\}$	

(b) If the balls are indistinguishable, then arrangements 4 and 5 are identical. Likewise for arrangements 6 and 7. Likewise again for arrangements 8 and 9. Thus, there are in total 6 possible distinct arrangements.

In class, the number of distinct arrangements for N indistinguishable balls in k containers was found to be

$$\frac{(N+k-1)!}{N!\,(k-1)!}$$

For $N=2$ and $k=3$, we obtain $\frac{4!}{2!\,2!}=6$ possible arrangements.

(c) If the balls are indistinguishable and [redacted] at most one ball can be placed in a given container, then arrangements 1, 2 and 3 are disallowed. We are left with 3 possible distinct arrangements.

In class, the number of distinct arrangements for N distinguishable balls in k containers, with no more than one ball per container, was found to be:

$$\frac{k!}{N!\,(N-k)!}$$

For $N=2$ and $k=3$, we obtain $\frac{3!}{2!\,1!}=3$ possible arrangements.

---

(4) For parts (a) and (b), there are three methods. Each one is worth knowing.

<u>Method 1</u>: probabilistic technique

Consider a random variable x_i defined by

$$x_i = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } q = 1-p \end{cases}$$

Then, it is easy to see that the probability that

$$x_1 + x_2 + \ldots + x_n = r$$

where r is an integer between 0 and n is given by the binomial distribution

$$f(r) = C(n,r)p^r q^{n-r}$$

since there are $C(n,r)$ different ways for r of the x_i to be equal to 1 and $n-r$ of the x_i to be equal to zero.

In preparation for our calculation, let us note that the mean of x_i is given by

$$\langle x_i \rangle = p$$

where we have used the general definition $\langle f \rangle = \sum_s f(s)\,p(s)$.

The mean of the binomial distribution is then given by:

$$\langle x_1 + x_2 + \ldots + x_n \rangle = n\langle x_i \rangle = np$$

where we have used the fact that in general $\langle f+g \rangle = \langle f \rangle + \langle g \rangle$ which follows directly from the definition of the mean.

The variance of the binomial distribution is given by

$$\langle (x_1 + x_2 + \ldots + x_n)^2 \rangle - \langle x_1 + x_2 + \ldots + x_n \rangle^2$$

Expanding out $(x_1 + x_2 + \ldots + x_n)^2$, two distinct terms emerge: x_i^2 and $x_i x_j$ $(i \neq j)$.

For $i \neq j$,

$$x_i x_j = \begin{cases} 1 & \text{with probability } p^2 \\ 0 & \text{with probability } 1-p^2 \end{cases}$$

In contrast,

$$x_i^2 = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1-p \end{cases}$$

Thus, the means of these two quantities are given by

$$\langle x_i x_j \rangle = p^2 \quad i \neq j$$

$$\langle x_i^2 \rangle = p$$

Therefore, noting that in the expansion $(x_1 + x_2 + \ldots + x_n)^2$, the total number of terms is n^2, of which n are of the form x_i^2 and $n^2 - n$ are of the form $x_i x_j$ $(i \neq j)$, we conclude that

$$\langle (x_1 + x_2 + \ldots + x_n)^2 \rangle = n\langle x_i^2 \rangle + (n^2 - n)\langle x_i x_j \rangle$$

$$= np + (n^2 - n)p^2$$

Thus, the variance is given by:

$$np + (n^2 - n)p^2 - n^2 p^2 = np(1-p)$$

$$= npq$$

where we have used $\langle x_1 + x_2 + \ldots + x_n \rangle^2 = n^2 p^2$ from our previous calculation

Note that we can shorten the proof somewhat by noting that if two random variables are <u>independent</u>, then

$$\mathrm{Var}(x_i+x_j) = \mathrm{var}(x_i) + \mathrm{var}(x_j)$$

(Unlike in the case of the mean, the above result fails if the two random variables are not independent. In this problem, separate "trials" are independent events, so we may use the above result.)

Now, for the binomial distribution,

$$\begin{aligned}\mathrm{Var}(x_i) &= \langle x_i^2\rangle - \langle x_i\rangle^2\\ &= p-p^2\\ &= p(1-p)\\ &= pq\end{aligned}$$

and so,

$$\mathrm{Var}(x_1+x_2+\ldots+x_n) = n\,\mathrm{Var}(x_i) = npq$$

as before.

<u>Method 2</u>: The direct approach

The mean of the binomial distribution is given by

$$\sum_{r=0}^{n} r f(r)$$

with $f(r) = C(n,r)p^r q^{n-r} = \dfrac{n!}{r!(n-r)!}\,p^r q^{n-r}$

and $p+q=1$. Thus,

$$\sum_{r=0}^{n} r\,\frac{n!}{r!(n-r)!}\,p^r q^{n-r}$$

$$= \sum_{r=1}^{n} \frac{n!}{(r-1)!(n-r)!}\,p^r q^{n-r}$$

since the $r=0$ term does not contribute in the first line above, and we used $r! = r(r-1)!$ to cancel out the factor of r in the second line above. Now, let $s=r-1$. We then get

$$= \sum_{s=0}^{n-1} \frac{n!}{s!(n-s-1)!}\,p^{s+1} q^{n-s-1}$$

$$= np\sum_{s=0}^{n-1} \frac{(n-1)!}{s!(n-s-1)!}\,p^{s} q^{n-s-1}$$

By the binomial theorem,

$$(p+q)^{n-1} = \sum_{s=0}^{n-1} \frac{(n-1)!}{s!(n-s-1)!}\,p^{s} q^{n-s-1}$$

Since $p+q=1$, we have $(p+q)^{n-1}=1$, and we are left with:

$$\sum_{r=0}^{n} r f(r) = np.$$

The variance of the binomial distribution is

$$\sum_{r=0}^{n} r^2 f(r) - \left(\sum_{r=0}^{n} r f(r)\right)^2$$

$$= \sum_{r=0}^{n} r^2 f(r) - n^2 p^2$$

To evaluate the sum, it is convenient to write: $r^2 = r(r-1)+r$. Then, we obtain

$$= \sum_{r=0}^{n} r(r-1) f(r) + \sum_{r=0}^{n} r f(r) - n^2 p^2$$

$$= \sum_{r=2}^{n} r(r-1) f(r) + np - n^2 p^2$$

where we have used the previous result for the mean. Also, because of the factor $r(r-1)$, the sum starts at $r=2$. The motivation for the above manipulation becomes clear when you notice that

$$r(r-1) f(r) = r(r-1) \frac{n!}{r!(n-r)!} p^r q^{n-r}$$

$$= \frac{n!}{(r-2)!(n-r)!} p^r q^{n-r}$$

Since $r! = r(r-1)(r-2)!$ This time, we let $s = r-2$ to obtain:

$$\sum_{s=0}^{n-2} \frac{n!}{s!(n-s-2)!} p^{s+2} q^{n-s-2} + np - n^2 p^2$$

$$= p^2 n(n-1) \sum_{s=0}^{n-2} \frac{(n-2)!}{s!(n-s-2)!} p^s q^{n-s-2} + np - n^2 p^2$$

$$= p^2 n(n-1) + np - n^2 p^2$$

$$= np(1-p)$$

$$= npq$$

where we used the binomial theorem again to conclude that

$$(p+q)^{n-2} = \sum_{s=0}^{n-2} \frac{(n-2)!}{s!(n-s-2)!} p^s q^{n-s-2}$$

$$= 1.$$

<u>Method 3:</u> The slick approach

Start from the binomial expansion

$$(p+q)^n = \sum_{r=0}^{n} \frac{n!}{r!(n-r)!} p^r q^{n-r}$$

Take $p\frac{\partial}{\partial p}$ of both sides:

$$p\frac{\partial}{\partial p}(p+q)^n = np(p+q)^{n-1}$$

$$p\frac{\partial}{\partial p}\sum_{r=0}^{n} \frac{n!}{r!(n-r)!} p^r q^{n-r} = \sum_{r=0}^{n} r \frac{n!}{r!(n-r)!} p^r q^{n-r}$$

We recognize the right hand side above as the mean of the binomial distribution. Thus,

$$\sum_{r=0}^{n} r f(r) = np(p+q)^{n-1} = np$$

since $p+q=1$.

To get the variance, take $p\frac{\partial}{\partial p} p\frac{\partial}{\partial p}$ of both sides:

$$p\frac{\partial}{\partial p} p\frac{\partial}{\partial p}(p+q)^n = p\frac{\partial}{\partial p} np(p+q)^{n-1}$$

$$= np\left[(p+q)^{n-1} + (n-1)p(p+q)^{n-2}\right]$$

and

$$p\frac{\partial}{\partial p} p\frac{\partial}{\partial p}\sum_{r=0}^{n} \frac{n!}{r!(n-r)!} p^r q^{n-r} = \sum_{r=0}^{n} r^2 \frac{n!}{r!(n-r)!} p^r q^{n-r}$$

$$= \sum_{r=0}^{n} r^2 f(r)$$

Thus,

$$\sum_{r=0}^{n} r^2 f(r) = np\left[(p+q)^{n-1} + (n-1)p(p+q)^{n-2}\right]$$

$$= np\left[1 + (n-1)p\right]$$

again using $p+q=1$. The variance is given by

$$\sum_{r=0}^{n} r^2 f(r) - \left(\sum_{r=0}^{n} r f(r)\right)^2$$

$$= np\left[1+(n-1)p\right] - n^2p^2$$

$$= np(1-p)$$

$$= npq.$$

(c) Using eq. (8.1) on p723 of Boas ("Mathematical Methods in the Physical Sciences", 2nd edition), the binomial distribution is well approximated at large n by:

$$f(r) = C(n,r)p^r q^{n-r} \simeq \frac{1}{\sqrt{2\pi npq}}\, e^{-(r-np)^2/(2npq)}$$

The mean and variance of this distribution are computed as follows:

<u>Mean</u>:

$$\int_{-\infty}^{\infty} r f(r)\,dr = \frac{1}{\sqrt{2\pi npq}} \int_{-\infty}^{\infty} r e^{-(r-np)^2/(2npq)}\,dr$$

Let $x = r - np$. Noting that

$$\int_{-\infty}^{\infty} f(r)\,dr = \frac{1}{\sqrt{2\pi npq}} \int_{-\infty}^{\infty} e^{-(r-np)^2/(2npq)}\,dr = 1$$

as required for a properly normalized probability distribution, we have:

$$\int_{-\infty}^{\infty} r f(r)\,dr = \frac{1}{\sqrt{2\pi npq}} \int_{-\infty}^{\infty} (x+np)\, e^{-x^2/(2npq)}\,dx$$

$$= np$$

since the integrand $xe^{-x^2/(2npq)}$ is odd under $x \to -x$ and therefore integrates to zero.

<u>The variance</u>

$$\int_{-\infty}^{\infty} r^2 f(r)\,dr - \left(\int_{-\infty}^{\infty} r f(r)\,dr\right)^2$$

$$= \int_{-\infty}^{\infty} r^2 f(r)\,dr - n^2p^2$$

$$= \int_{-\infty}^{\infty} (r^2 - n^2p^2) f(r)\,dr$$

since $f(r)$ is a properly normalized probability distribution which integrates to unity.

$$= \frac{1}{\sqrt{2\pi npq}} \int_{-\infty}^{\infty} (r^2 - n^2p^2)\, e^{-(r-np)^2/(2npq)}\,dr$$

Again, we let $x = r - np$. Then $r^2 - n^2p^2 = (x+np)^2 - n^2p^2 = x^2 + 2xnp$.

$$= \frac{1}{\sqrt{2\pi npq}} \int_{-\infty}^{\infty} (x^2 + 2xnp)\, e^{-x^2/(2npq)}\,dx$$

$$= \frac{1}{\sqrt{2\pi npq}} \int_{-\infty}^{\infty} x^2 e^{-x^2/(2npq)}\,dx$$

$$= \frac{1}{2}(2npq)$$

$$= npq$$

<u>Addendum</u>: Proof of eq. (8.1) on p 723 of Boas.

To compute the large n behavior of $f(r) = C(n,r)p^r q^{n-r}$ we use Stirling's approximation:

$$\log n! = (n+\tfrac{1}{2})\log n - n + \tfrac{1}{2}\log 2\pi$$

Then,

$$\log f(r) = \log \frac{n!}{r!(n-r)!} p^r q^{n-r}$$

$$= \log n! - \log r! - \log (n-r)! + r\log p + (n-r)\log q$$

$$= (n+\tfrac{1}{2})\log n - (r+\tfrac{1}{2})\log r - (n-r+\tfrac{1}{2})\log (n-r)$$
$$-\tfrac{1}{2}\log 2\pi + r\log p + (n-r)\log q$$

It is convenient in the first term to write $n = r + (n-r)$ and then collect all terms proportional to r and $n-r$, respectively

$$= \tfrac{1}{2}\left[\log n - \log r - \log (n-r) - \log 2\pi\right]$$
$$+ r\left[\log n - \log r + \log p\right]$$
$$+ (n-r)\left[\log n - \log (n-r) + \log q\right]$$

Thus,

$$\log f(r) = \tfrac{1}{2}\log\left(\frac{n}{2\pi r(n-r)}\right) + r\log\left(\frac{np}{r}\right) + (n-r)\log\left(\frac{nq}{n-r}\right).$$

Now, the maximum of the distribution occurs at the mean np and the width of the distribution is characterized by the square root of the variance, $\sqrt{npq}$.

Thus, we expect that the function $f(r)$ is appreciable for

$$r = np + \sqrt{npq}\,x$$
$$n - r = nq - \sqrt{npq}\,x \qquad (p+q=1)$$

where $x \ll \sqrt{n}$. That is, we note that $f(r)$ approaches zero quickly after we deviate a few standard deviations from the mean.

Then,

$$\log\left(\frac{n}{2\pi r(n-r)}\right) = \log\left(\frac{n}{2\pi n^2 pq\left[1 + O\left(\frac{x}{\sqrt{n}}\right)\right]}\right)$$

$$= -\log(2\pi npq) - \log\left(1 + O\left(\frac{x}{\sqrt{n}}\right)\right)$$

$$= -\log(2\pi npq) + O\left(\frac{x}{\sqrt{n}}\right)$$

In the large n limit, we can drop the term of $O\left(\frac{x}{\sqrt{n}}\right)$.

Similarly,

$$r\log\left(\frac{np}{r}\right) + (n-r)\log\left(\frac{nq}{n-r}\right)$$

$$= -\left(np + \sqrt{npq}\,x\right)\log\left(1 + \frac{x\sqrt{npq}}{np}\right) - \left(nq - \sqrt{npq}\,x\right)\log\left(1 - \frac{x\sqrt{npq}}{nq}\right)$$

Expanding out the logarithms,

$$= -\left(np + x\sqrt{npq}\right)\left(\frac{x\sqrt{npq}}{np} - \frac{x^2 npq}{2n^2p^2}\right)$$

$$+ \left(nq - x\sqrt{npq}\right)\left(\frac{x\sqrt{npq}}{nq} + \frac{x^2 npq}{2n^2q^2}\right)$$

$$= -x\sqrt{npq} + \frac{x^2 q}{2} - x^2 q + o\left(\frac{x^3}{\sqrt{n}}\right)$$

$$+ x\sqrt{npq} + \frac{x^2 p}{2} - x^2 p + o\left(\frac{x^3}{\sqrt{n}}\right)$$

$$= -\frac{x^2}{2}(q+p) + o\left(\frac{x^3}{\sqrt{n}}\right)$$

$$= -\frac{x^2}{2} + o\left(\frac{x^3}{\sqrt{n}}\right)$$

Thus,

$$\log f(r) = \frac{-1}{2}\log(2\pi npq) - \frac{x^2}{2} + o\left(\frac{x}{\sqrt{n}}\right)$$

so as $n \to \infty$,

$$f(r) = \frac{1}{\sqrt{2\pi npq}}\, e^{-x^2/2}$$

By definition,

$$x = \frac{r - np}{\sqrt{npq}}$$

and so,

$$f(r) = \frac{1}{\sqrt{2\pi npq}}\, e^{-(r-np)^2/(2npq)}$$

which is the desired result.

5 The multiplicity of the spin system is given approximately by

$$g(N,s) \simeq g(N,0)\, e^{-2s^2/N}$$

while the total energy is given by

$$E = -2smB$$

for a spin excess of $2s$.

(a) The entropy is given by

$$S = k \ln g$$
$$= k\left[\ln g(N,0) - \frac{2s^2}{N}\right]$$

Using $s = \dfrac{-E}{2mB}$, it follows that

$$\boxed{S = k\left[\ln g(N,0) - \frac{E^2}{2m^2B^2N}\right]}$$

(b) Using eq. (4.21) of RB,

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_{\text{fixed external parameters}}$$

or

$$\boxed{\frac{1}{T} = \frac{-kE}{m^2B^2N}}$$

Note: in such spin systems, the temperature can be either positive or negative depending on the sign of the spin excess $2s$.

(c) The equilibrium value of the fractional magnetization,

$$\frac{\langle M\rangle}{Nm} = \frac{2\langle s\rangle}{N}$$

is obtained by noting that $E = -2smB$, so

$$\langle E\rangle = -2\langle s\rangle mB.$$

But, $E \simeq \langle E\rangle$ since fluctuations from the mean value are negligible if $N \gg 1$. Thus,

$$2\langle s\rangle \simeq \frac{-E}{mB}$$

and so

$$\frac{\langle M\rangle}{Nm} = \frac{-E}{mBN}$$

Using the result of part (b),

$$\boxed{\frac{\langle M\rangle}{Nm} = \frac{mB}{kT}}$$

6. RB Chapter 5, problem 3

In this problem, we use the fact that [RB eq. (5.6)]

$$P(\varepsilon_n) \propto e^{-\varepsilon_n/kT}$$

For atomic hydrogen,

$$\varepsilon_n = \frac{-13.6}{n^2}\ \text{eV}$$

It is convenient to work in eV energy units. Since $1\,\text{eV} = 1.602\times10^{-19}$ joule it follows that

$$k = 0.8617\times10^{-4}\ \text{eV/K}$$

(see p420 of RB). Thus, for $T = 7000\,\text{K}$,

$$\frac{\varepsilon_n}{kT} = \frac{-(13.6\,\text{eV})/n^2}{(0.8617\times10^{-4}\,\text{eV/K})(7000\,\text{K})}$$

$$= \frac{-22.55}{n^2}$$

When computing the relative probabilities $P(\varepsilon_n)$, we must take into account the degeneracies of energy levels. If the degeneracy of the n^{th} level is g_n, then it is more accurate to write:

$$P(\varepsilon_n) \propto g_n e^{-\varepsilon_n/kT}$$

For the $n=1$ level of hydrogen, $g=2$ corresponding to two distinct spin states (up and down) for the electron.

For the $n=2$ level of hydrogen, possible values of the orbital angular momentum quantum number are $\ell=0$ and $\ell=1$. For each of these states, there is a degeneracy of $2(2\ell+1)$. The factor of 2 is for electron spin as before. The factor of $2\ell+1$ is for the possible values of the quantum number m which takes on integer values $-\ell \le m \le \ell$.

The m quantum number reflects the possible values of the z-component of the orbital angular momentum (in units of $\hbar$), which are quantized.

Finally, we note that for atomic hydrogen, $\ell \le n-1$. With these results in mind, we easily work out the corresponding degeneracy factors:

n	ℓ	g
1	0	2
2	0	2
2	1	$2\times3=6$

(a) $$\frac{P(n=2 \text{ and } \ell=0)}{P(n=1)} = \frac{2e^{22.55/4}}{2e^{22.55}} = 4.53\times10^{-8}$$

(b) $$\frac{P(n=2 \text{ and } \ell=0 \text{ or } 1)}{P(n=1)} = \frac{8e^{22.55/4}}{2e^{22.55}} = 1.81\times10^{-7}$$

In part (b), we noted that the total degeneracy of the $n=2$ state of hydrogen (where $\ell=0$ or $\ell=1$) is $2+6=8$.

(25)

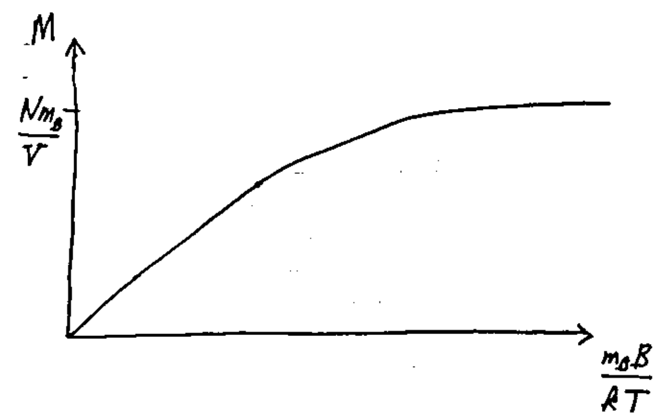
(7) RB Chapter 5, problem 4

(a) According to RB eq. (5.13), the average magnetic moment along $\vec{B}$ is given by

$$m_B \tanh\left(\frac{m_B B}{kT}\right)$$

This is the average magnetic moment of a single spin. If the spin system contains N spins and occupies a volume V, then the total magnetic moment per unit volume (which is called the magnetization, M) is given by

$$M = \frac{N m_B}{V} \tanh\left(\frac{m_B B}{kT}\right)$$



(b) The magnetic susceptibility is given by

$$\chi = \left(\frac{\partial M}{\partial B}\right)_T = \frac{N m_B^2}{VkT} \frac{1}{\cosh^2(m_B B/kT)}$$

(26)

(c) When $m_B B \ll kT$,

$$\cosh\left(\frac{m_B B}{kT}\right) \simeq 1$$

and we find from part (b),

$$\chi = \left(\frac{\partial M}{\partial B}\right)_T \simeq \frac{N m_B^2}{VkT}$$

The result

$$\chi = \frac{\text{constant}}{T}$$

is known as Curie's law.

(8) The Monty Hall Problem

Clearly, before Monty Hall reveals the goat, the relevant probabilities are (for the three door problem):

probability that the car is behind the door you chose $= \frac{1}{3}$

probability that the car is behind one of the two doors you did not choose $= \frac{2}{3}$

Now, Monty Hall opens one of the doors you didn't choose to reveal the goat. The key observation is that none of the above probabilities change. Since there always exists a door you didn't choose, behind which is a goat, no new information has been gained when Monty Hall reveals the goat. This conclusion, however, is based on the assumption that if Monty Hall has a choice of two doors to open (which happens when the car is behind the door you chose), then he makes his choice of which door to open at random. If this is not true, then when Monty Hall opens one of the doors, new information is gained and the probabilities do change. This latter possibility is analyzed in the handout ("The Car and the Goats", by Leonard Gillman, in The American Mathematical Monthly 99 (1992) pp 3–7).

In the problem I gave you, Monty Hall opens a door at random if there is a choice to be made. Hence, the probabilities quoted above remain unchanged. Since there is now only one unopened door (apart from the one you originally chose), it follows that:

probability that the car is behind the unopen door and not behind the door you chose $= \frac{2}{3}$

probability that the car is behind the door you chose $= \frac{1}{3}$

If given the opportunity to switch, clearly you should.

The trick in this problem centers around the following false argument that states that after Monty Hall reveals the goat, the car can only be behind two doors. With no further information, the probability should be $\frac{1}{2}$ that the car is behind either door.

To see this is false, consider all the possible arrangements of cars, goats, and Monty Hall's choice of which door to open. Before Monty Hall opens a door, there are three possible arrangements

1 2 3	probability
C G G	1/3
G C G	1/3
G G C	1/3

where by definition, door 1 is the door you chose. Let me circle the goat Monty Hall reveals:

1 2 3	probability
C G G	$1/6 = 1/3 \times 1/2$
C G G	$1/6 = 1/3 \times 1/2$
G C G	$1/3 = 1/3 \times 1$
G G C	$1/3 = 1/3 \times 1$

In all cases, I multiplied the probability that you chose the car times the probability that Monty Hall opened either door 2 or 3. Monty Hall has a choice in the first two cases, but in the last two cases, Monty Hall can only open one door.

Now, if [redacted] Monty Hall opened door 2, then only two of the four cases survive with relative probabilities 1/6 and 1/3. Thus,

$$\text{probability that you have chosen the car originally} = \frac{1/6}{1/6 + 1/3} = \frac{1}{3}$$

$$\text{probability that you will get the car if you switch} = \frac{1/3}{1/3 + 1/6} = \frac{2}{3}$$

If Monty Hall opened door 3, you would reach the same conclusion.

Finally, let us consider the n-door version. You make a choice of doors (call it door 1). The probability that it has a car behind it is $\frac{1}{n}$. If $n \gg 1$, the probability is essentially zero. Now, of the remaining $n-1$ doors, Monty Hall opens $n-2$ of them, revealing $n-2$ goats. One door remains unopened. What is the probability that the car is behind this door. Clearly, it is $1 - \frac{1}{n} = \frac{n-1}{n}$. For $n \gg 1$, it is almost a sure bet that the goat is behind the unopened door you didn't choose. Think about it. Originally, there was virtually no chance of you choosing the car, which means that [redacted] it was virtually a sure bet that the [redacted] car was behind one of the other unopened $n-1$ doors. Monty Hall can always find $n-2$ doors to open to reveal $n-2$ goats. Having done so, do you think that suddenly you would have a so-so chance to have originally chosen the car? No way! For the large n version of the problem, switching gets you the car with almost complete certainty!

So, how did those math and statistic professors screw up? It's a mystery to me!