Physics 112 — Solution Set #5 — Winter 2000

(1) RB Chapter 6, problem 1

(a) The pressure of a classical ideal gas is

$$P = \frac{NkT}{V} = (10^{23}\,m^{-3})(1.381\times10^{-23}\,J/K)(5800\,K)$$
$$= 8.001\times10^{3}\,N\,m^{-2}$$

(since $1J = 1Nm$).

The pressure of a photon gas is:

$$P = \frac{1}{3}\frac{\langle E\rangle}{V} = \frac{4\sigma_B}{3c}T^4$$

Using eqs. 6.15 and 6.18 of RB and noting that $\sigma_B \equiv \frac{2\pi^5 k^4}{15c^2h^3}$ (see eq 6.29), which implies that $\frac{\langle E\rangle}{V} = \frac{4\sigma_B}{c}T^4$.

Thus,

$$P = \frac{4(5.67\times10^{-8}\,W\,m^{-2}K^{-4})(5800\,K)^4}{3\times10^{8}\,m\,s^{-1}}$$

$$= 2.85\times10^{-1}\,N\,m^{-2}$$

(since $1W = 1Js^{-1} = 1Nms^{-1}$).

We conclude that:

$$\frac{P_{radiation}}{P_{particles}} = 3.56\times10^{-5}$$

(b) We repeat the computation for the center of the sun, where $T = 1.5\times10^7 K$ and $N/V = 5\times10^{31}\,m^{-3}$.

$$P_{particles} = \frac{NkT}{V} = (5\times10^{31}m^{-3})(1.381\times10^{-23}J/K)(1.5\times10^{7}K)$$
$$= 1.036\times10^{16}\,N\,m^{-2}$$

$$P_{radiation} = \frac{4\sigma_B}{3c}T^4 = \frac{\frac{4}{3}(5.67\times10^{-8}\,W\,m^{-2}K^{-4})(1.5\times10^{7}K)^4}{3\times10^{8}\,ms^{-1}}$$
$$= 1.28\times10^{13}\,N\,m^{-2}$$

and so,

$$\frac{P_{radiation}}{P_{particles}} = 1.23\times10^{-3}$$

(2) RB Chapter 6, problem 5

(a) Since $P = \frac{1}{3}\frac{\langle E \rangle}{V} = \frac{4\sigma_B}{3c} T^4$ is constant in an isothermal process,

$$w = \int_{V_1}^{V_2} P\,dV = \frac{4\sigma_B}{3c} T^4 \int_{V_1}^{V_2} dV$$

or

$$\boxed{w = \frac{4\sigma_B T^4}{3c}(V_2 - V_1)}$$

From the first law of thermodynamics,

$$q = \Delta E + w$$

Since $E \approx \langle E \rangle = \frac{4\sigma_B}{c} V T^4$,

$$\Delta E = \frac{4\sigma_B T^4}{c}(V_2 - V_1)$$

Hence,

$$\boxed{q = \frac{16\sigma_B T^4}{3c}(V_2 - V_1)}$$

(b) In class, we showed that a photon gas has entropy

$$S = \frac{16\sigma_B V T^3}{3c}$$

i.e. VT^3 is constant.

Consider now the four steps of the Carnot cycle.

1. Isothermal expansion from V_1 to V_2 at temperature T_h.

In part (a), we computed

$$\boxed{w_1 = \frac{4\sigma_B T_h^4}{3c}(V_2 - V_1)}$$

2. Adiabatic expansion from V_2 to V_3. Initial temperature is T_h and final temperature is T_c.

The easiest method for computing w is to note that

$$q = \Delta E + w_2 = 0$$

for an adiabatic transition. Thus, $w = -\Delta E$.
But,

$$\Delta E = \frac{4\sigma_B}{c}(T_c^4 V_3 - T_h^4 V_2)$$

So, we need to evaluate V_3. For a quasi-static reversible abiabatic process, the entropy does not change. For a photon gas, this means that VT^3 is constant. Hence,

$$V_2 T_h^3 = V_3 T_c^3$$

Inserting this into the expression for ΔE, we can rewrite this as

$$\Delta E = \frac{4\sigma_B}{c} T_h^3 V_2 (T_c - T_h).$$

Thus,

$$\boxed{w_2 = -\Delta E = \frac{4\sigma_B}{c} V_2 T_h^3 (T_h - T_c)}$$

3. Isothermal compression from V_3 to V_4 at temperature T_c.

We again make use of part (a), and obtain

$$\boxed{w_3 = \frac{4\sigma_B T_c^4}{3c}(V_4 - V_3)}$$

4. Adiabatic compression from V_4 to V_1. The initial temperature is T_c and the final temperature is T_h.

Again, as in step 2, $q=0$ so

$$w_4 = -\Delta E = -\frac{4\sigma_B}{c}(T_h^4 V_1 - T_c^4 V_4)$$

We can evaluate V_4 since this is an adiabatic transition:

$$V_4 T_c^3 = V_1 T_h^3$$

Thus,

$$\boxed{w_4 = -\frac{4\sigma_B}{c} V_1 T_h^3 (T_h - T_c)}$$

The total work done by the gas of photons during the cycle is:

$$W = w_1 + w_2 + w_3 + w_4$$

First, note that

$$w_2 + w_4 = \frac{4\sigma_B}{c} V_2 T_h^3 (T_h - T_c) - \frac{4\sigma_B}{c} V_1 T_h^3 (T_h - T_c)$$

$$= \frac{4\sigma_B}{c} T_h^3 (V_2 - V_1)(T_h - T_c)$$

Next, we compute:

$$w_1 + w_3 = \frac{4\sigma_B T_h^4}{3c}(V_2 - V_1) + \frac{4\sigma_B T_c^4}{3c}(V_4 - V_3)$$

Using $V_4 = V_1\left(\frac{T_h^3}{T_c^3}\right)$ and $V_3 = V_2\left(\frac{T_h^3}{T_c^3}\right)$, we substitute for $V_4 - V_3$ in the above expression. The final result is

$$w_1 + w_3 = \frac{4\sigma_B}{3c} T_h^4 (V_2 - V_1) + \frac{4\sigma_B}{3c} T_h^3 T_c (V_1 - V_2)$$

$$= \frac{4\sigma_B T_h^3}{3c}(T_h - T_c)(V_2 - V_1)$$

Therefore,

$$W = w_1 + w_2 + w_3 + w_4$$

or

$$\boxed{w = \frac{16\sigma_B T_h^3}{3}(T_h - T_c)(V_2 - V_1)}$$

Note that $T_h > T_c$ and $V_2 > V_1$, so that $w > 0$ as expected.

(c) The efficiency of the Carnot engine is defined by $\eta = \frac{w}{q_h}$.

In part (a), we found that

$$q_h = \frac{16\sigma_B T_h^4}{3c}(V_2 - V_1)$$

Using the result for w obtained above,

$$\eta = \frac{T_h - T_c}{T_h} = 1 - \frac{T_c}{T_h}$$

which is precisely the Carnot efficiency derived in RB Chapter 3.

3. RB Chapter 6, problem 8

(a) According to the problem, 4.2 J of energy is required to raise the temperature of $1\,cm^3$ of water by 1 K. Using

$$E = \frac{4\sigma_B V T^4}{c}$$

for the energy of a photon gas, if the temperature of the radiation is lowered from 450 K to 440 K, then the energy lost by the gas is

$$\frac{4\sigma_B V}{c}\left(T_i^4 - T_f^4\right) \qquad T_i = 450\,K,\quad T_f = 440\,K$$

Assume that this energy is equal to the required 4.2 J.

Thus,

$$4.2\,J = \frac{4\sigma_B V}{c}\left[(450)^4 - (440)^4\right]$$

Solving for V,

$$V = \frac{(4.2\,J)(3\times 10^8\,m\,s^{-1})}{4(5.67\times 10^{-8}\,W\,m^{-2}K^{-4})\left[(450\,K)^4 - (440\,K)^4\right]}$$

$$= 1.58\times 10^6\ m^3$$

If this were a cube, it would have a length of $L = 116\ m$.

(b) Part (a) implies that the energy stored in the radiation is very small if $V = 0.1\,m^3$ (a typical size of the oven). The energy to heat the water is due to the heating coils. The power delivered by the coils is voltage × current or

$$\text{Power} = (240\text{ volts})(20\text{ amps}) = 4800\text{ watts}$$

$$= 4800\ J\,s^{-1}$$

Thus, to obtain the 4.2 J needed to raise the temperature of the water, we need a time of

$$\frac{4.2\,J}{4800\,J\,s^{-1}} = 8.75\times 10^{-4}\,s$$

This is the minimal time required. Of course, in practice not all the power delivered from the coils is used to heat up the water. Moreover, the coils themselves require time and energy to warm up.

④ RB Chapter 6, problem 11

(a) The energy flux of a perfect blackbody is equal to $\sigma_B T^4$. Flux means per unit area per unit time. Thus the units of $\sigma_B T^4$ is $W m^{-2}$ or $J m^{-2} s^{-1}$.

The total power emitted by the sun is

$$(\sigma_B T_\odot^4)\, 4\pi R_\odot^2 \qquad (T_\odot = 5800\,K)$$

where $R_\odot$ is the radius of the sun. This is because $4\pi R_\odot^2$ is the surface area of the sun, and radiation is emitted from the surface.

Let d = distance from the sun to the earth. The radiation is emitted uniformly in all directions. Thus, the energy flux at the earth's orbital distance is equal to

$$\frac{\text{total power emitted by sun}}{4\pi d^2}$$

since $4\pi d^2$ is the total surface area at a distance d from the sun receiving the sun's energy. That is,

$$\text{energy flux at earth's orbital distance} = \sigma_B T_\odot^4 \frac{R_\odot^2}{d^2}$$

Using $R_\odot = 6.960 \times 10^8\,m$ (see p420 of RB)
$d = 1.496 \times 10^{11}\,m$

we find:

$$\text{energy flux at earth's orbital distance} = \frac{(5.67\times10^{-8}\,W m^{-2} K^{-4})(5800K)^4 (6.96\times10^8 m)^2}{(1.496\times10^{11} m)^2}$$

$$= 1389\ W m^{-2}$$

(b) Suppose the earth radiates as a perfect black body at temperature T_E. Then, the earth radiates an energy [redacted] per unit time of

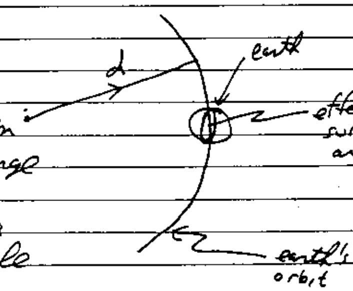
$$\sigma_B T_E^4 (4\pi R_E^2)$$

If the earth is in a thermal steady state, this radiated flux must be exactly balanced by the incoming solar flux of radiation. In part (a), we computed the solar energy flux at the earth's orbital distance to be

$$\sigma_B T_\odot^4 \frac{R_\odot^2}{d^2}$$

To get the solar energy per unit time absorbed by the earth, we must multiply by the effective surface area. This is a little tricky.

In the diagram at the right, I show part of the earth's orbit and (not to scale) the sphere of the earth. The effective surface area is the projection of the sphere of the earth onto the large surface of radius d which contains the earth's orbit. This effective area is just πR_E^2, the area of the circle the makes up the earth's equator. (Imagine shining a flashlight on a sphere in front of the wall. The shadow on the wall will be disk of radius R and area πR^2, where R is the radius of the sphere.)



Thus, the solar energy per unit time absorbed by the earth is

$$\sigma_B T_\odot^4 \frac{R_\odot^2}{d^2}\, \pi R_E^2$$

Setting this equal to $\sigma_B T_E^4 (4\pi R_E^2)$, and solving for T_E,

$$\sigma_B T_E^4 (4\pi R_E^2) = \sigma_B T_\odot^4 \frac{R_\odot^2}{d^2} \pi R_E^2$$

$$= (1389\ \mathrm{W m^{-2}})\, \pi R_E^2$$

using the results of part (a). Thus,

$$T_E = \left(\frac{1389\ \mathrm{W m^{-2}}}{4(5.67\times 10^{-8}\ \mathrm{W m^{-2} K^{-4}})} \right)^{1/4}$$

or

$$\boxed{T_E = 280\ K}$$

Assumptions made:

(i) The sun and earth radiate as perfect black bodies
(ii) The surface of the earth can be regarded as a perfect sphere characterized by a single temperature T_E
(iii) All solar energy is absorbed by the earth; for example reflection by the atmosphere is negligible
(iv) There are no other sources of energy in the earth's core (this is not strictly true, since radioactive nuclei inside the earth do provide a small source of energy)

Nevertheless, the result obtained is pretty good.

(5) RB Chapter 6, problem 18

In class, we obtained:

$$E = \frac{3k^4 T^4 V}{2\pi^2 \hbar^3 v_s^3} \int_0^{x_D} dx\, \frac{x^3}{e^x - 1}$$

where $x_D \equiv \frac{\Theta_D}{T}$. Here, we consider the high temperature limit, $T \gg \Theta_D$, which corresponds to $x_D \ll 1$. This means that over the entire integration range, $x \ll 1$, and we can expand the integrand:

$$\frac{x^3}{e^x - 1} = \frac{x^3}{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) - 1}$$

$$= \frac{x^3}{x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots}$$

$$= \frac{x^2}{1 + \frac{x}{2!} + \frac{x^2}{3!} + \dots}$$

The leading term, obtained in class, corresponds to keeping just 1 in the denominator. Let's keep two more terms

$$\frac{x^3}{e^x - 1} \simeq \frac{x^2}{1 + \frac{x}{2} + \frac{x^2}{6}}$$

$$\simeq x^2\left(1 - \frac{x}{2} - \frac{x^2}{6} + \frac{x^2}{4}\right)$$

$$\simeq x^2\left(1 - \frac{x}{2} + \frac{x^2}{12}\right)$$

In obtaining this result, I used the expansion

$$\frac{1}{1+z} = 1 - z + z^2 - \dots$$

where $z = \frac{x}{2} + \frac{x^2}{6}$. Note that I did not have to keep the term of $O(x^4)$ obtained when squaring z, since I am only interested in terms up to and including $O(x^2)$.

Thus,

$$\int_0^{x_D} dx\, \frac{x^3}{e^x - 1} \simeq \int_0^{x_D} x^2\left(1 - \frac{x}{2} + \frac{x^2}{12}\right) dx$$

$$= \frac{x_D^3}{3} - \frac{x_D^4}{8} + \frac{x_D^5}{60}$$

$$= \frac{x_D^3}{3}\left(1 - \frac{3x_D}{8} + \frac{x_D^2}{20}\right)$$

Therefore,

$$E = \frac{3k^4T^4V}{2\pi^2\hbar^3 v_s^3}\,\frac{x_D^3}{3}\left(1 - \frac{3x_D}{8} + \frac{x_D^2}{20}\right)$$

Using the definition

$$x_D = \frac{\hbar v_s}{kT}\left(\frac{6\pi^2 N}{V}\right)^{1/3}$$

we can write:

$$\frac{3N}{x_D^3} = \frac{k^3T^3V}{2\pi^2\hbar^3 v_s^3}$$

The expression for E simplifies to

$$\boxed{E = 3NkT\left[1 - \frac{3}{8}\frac{\theta_D}{T} + \frac{1}{20}\frac{\theta_D^2}{T^2}\right]}$$

where we have put $x_D = \theta_D/T$.

The heat capacity is

$$\boxed{C_V = \frac{\partial E}{\partial T} = 3Nk\left[1 - \frac{1}{20}\frac{\theta_D^2}{T^2}\right]}$$

Note that the second term in the expansion of E is independent of T, so its derivative with respect to T vanishes. Thus, we needed to compute E up to order θ_D^2/T^2 with respect to the lowest order approximation in order to obtain the first non-trivial correction term in the expression for C_V.

(b) For $\theta_D/T = 0.5$

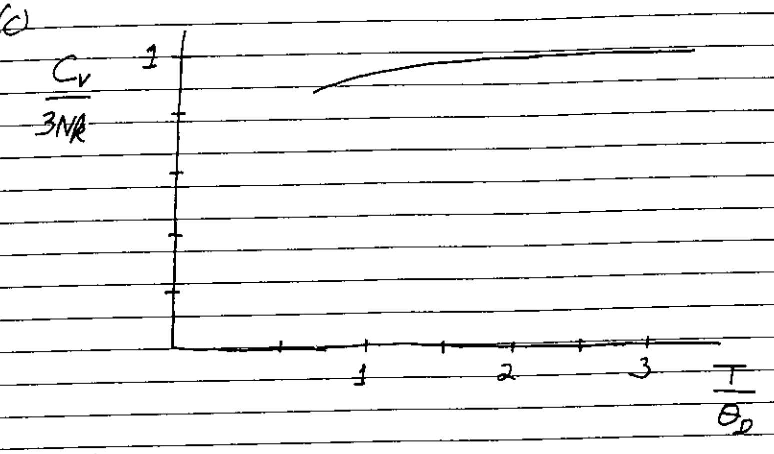
$$\frac{E - 3NkT}{3NkT} \simeq -\frac{3}{8}\frac{\theta_D}{T} = -\frac{3}{16}$$

which is a correction of about -19%, and

$$\frac{C_V - 3Nk}{3Nk} = -\frac{1}{20}\frac{\theta_D^2}{T^2} = -\frac{1}{80}$$

which is a correction of -1.25%.

(c)



6 RB Chapter 6, problem 19

(a) The entropy of an ideal gas is given by

$$S = kN\left[\ln\left(\frac{V}{N}\left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2}\right) + \frac{5}{2}\right]$$

Thus,
$$S_f - S_i = kN\left[\ln\left(V_f T_f^{3/2}\right) - \ln\left(V_i T_i^{3/2}\right)\right]$$
$$= kN \ln\left(\frac{V_f}{V_i}\left(\frac{T_f}{T_i}\right)^{3/2}\right)$$

Given $V_f = 3V_i$ and $T_f = \frac{4}{3}T_i$, with $N = 10^{20}$ atoms,

$$S_f - S_i = (1.381\times10^{-23}\,J/K)(10^{20})\ln\left(3\cdot\left(\frac{4}{3}\right)^{3/2}\right)$$
$$= 2.11\times10^{-3}\,J/K.$$

(b) According to eq 3.9 of RB on p.62

$$S_f - S_i = \int_{T_i}^{T_f} \frac{C(T)}{T}\,dT$$

According to table 6.2 on p137, the Debye temperature of diamonds is $\theta_D = 2230\,K$. In this problem, $T_i = 4.2\,K$ and $T_f = 77K$, so $T \ll \theta_D$ throughout the integration range.

Thus, we can use RB eq 6.57 for the heat capacity:

$$C_V = \frac{12\pi^4}{5} Nk \left(\frac{T}{\Theta_D}\right)^3$$

For diamonds in helium, we are given the value of

$$C_V(T=4.2K) = 10^{-6}\, J/K$$

Thus,

$$C_V(T) = 10^{-6}\, J/K \left(\frac{T}{4.2K}\right)^3$$

So,

$$S_f - S_i = \frac{10^{-6}\, J/K}{(4.2K)^3} \int_{4.2}^{77} T^2 dT$$

$$= \frac{1}{3} \frac{10^{-6}}{(4.2)^3} \left[(77)^3 - (4.2)^3 \right]$$

$$= 2.05 \times 10^{-3}\, J/K.$$

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(7) RB, Chapter 6 problem 21

(a) The energy loss per unit time is given by

$$\frac{dE}{dt} = -\sigma_B T^4 (4\pi r_0^2)$$

where $4\pi r_0^2$ is the surface area of the silver sphere. By the chain rule,

$$\frac{dE}{dt} = \frac{dE}{dT}\frac{dT}{dt} = C(T)\frac{dT}{dt}$$

where $C(T)$ is the heat capacity. Thus,

$$\frac{dT}{dt} = \frac{1}{C(T)}\frac{dE}{dt}$$

$$= \frac{-1}{C(T)} \sigma_B T^4 4\pi r_0^2$$

Since $\Theta_D = 225K$ for silver, in the range of temperatures between 250K and 300K, we will approximate

$$C(T) = 3Nk$$

(see eg. Figures 6.8 and 6.9 on p136 and 139 of RB).

So,

$$\int_{T_i}^{T_f} \frac{dT}{T^4} = -\frac{1}{3Nk}\sigma_B 4\pi r_0^2 \int_{t_i}^{t_f} dt$$

The cooling time required, $\Delta t \equiv t_f - t_i$ is then given by:

$$\Delta t = \frac{Nk}{4\pi r_0^2 \sigma_B}\left(\frac{1}{T_f^3} - \frac{1}{T_i^3}\right)$$

For $N = 2.45\times10^{23}$
$r_0 = 1\,cm = 0.01\,m$
$T_i = 300\,K$
$T_f = 250\,K$

$$\Delta t = \frac{(2.45\times10^{23})(1.381\times10^{-23}\,J/K)}{4\pi(0.01\,m)^2(5.67\times10^{-8}\,Wm^{-2}K^{-4})}\left[\frac{1}{(250K)^3} - \frac{1}{(300K)^3}\right]$$

$= 1280\,s$ (note: $1\,W = 1\,Js^{-1}$)

$= 21.3$ minutes

(b) In cooling from 20K to 10K, the temperature range is now much lower than θ_D. Thus, we must use eq 6.57 of RB

$$C(T) = \frac{12\pi^4}{5}Nk\left(\frac{T}{\theta_D}\right)^3$$

Thus,

$$\frac{dT}{dt} = -\frac{5\theta_D^3}{12\pi^4 Nk}4\pi r_0^2\sigma_B T$$

So,

$$\int_{T_i}^{T_f}\frac{dT}{T} = -\frac{5\theta_D^3\sigma_B 4\pi r_0^2}{12\pi^4 Nk}\Delta t$$

Thus,

$$\Delta t = \frac{3\pi^3 Nk}{5\theta_D^3\sigma_B r_0^2}\ln\left(\frac{T_i}{T_f}\right)$$

$$= \frac{3\pi^3(2.45\times10^{23})(1.381\times10^{-23}\,J/K)}{5(225\,K)^3(0.01m)^2(5.67\times10^{-8}\,Wm^{-2}K^{-4})}\ln\left(\frac{20}{10}\right)$$

$= 6.76\times10^5$ seconds

$= 7.82$ days

(c) The asymptotic value of the sphere's temperature, set adrift in outer space, would be 2.7K, the temperature of the cosmic microwave background radiation.