Physics 112 — Solution Set #9 — Winter 2000

(1) RB Chapter 9, problem 4

We are given a hypothetical system of $N$ fermions with a single-particle density of states given by $D(\varepsilon) = \frac{\varepsilon}{\varepsilon_0^2}$. I assume that the spin degeneracy factor, $2s+1$, is included in the definition of $\varepsilon_0$.

(a) To compute the Fermi energy, we compute $N$:

$$N = \int_0^{\varepsilon_F} D(\varepsilon)\,d\varepsilon = \frac{1}{\varepsilon_0^2}\int_0^{\varepsilon_F} \varepsilon\,d\varepsilon = \frac{\varepsilon_F^2}{2\varepsilon_0^2}$$

Thus,

$$\boxed{\varepsilon_F = \varepsilon_0\sqrt{2N}}$$

(b) The result above was a computation at $T=0$, where

$$f_{FD}(\varepsilon) = \begin{cases} 1, & \varepsilon \le \varepsilon_F \\ 0, & \varepsilon > \varepsilon_F \end{cases}$$

At $T \neq 0$, we must use the result

$$N = \int_0^\infty f_{FD}(\varepsilon)\,D(\varepsilon)\,d\varepsilon$$

where

$$f_{FD}(\varepsilon) = \frac{1}{e^{(\varepsilon-\mu)/kT} + 1}$$

Thus, for the density of states given above,



$$N = \frac{1}{\varepsilon_0^2}\int_0^\infty \frac{\varepsilon\,d\varepsilon}{e^{(\varepsilon-\mu)/kT}+1}$$

Let $z \equiv e^{\mu/kT}$ and $x = \frac{\varepsilon}{kT}$. Then,

$$N = \left(\frac{kT}{\varepsilon_0}\right)^2 \int_0^\infty \frac{x\,dx}{z^{-1}e^x + 1}$$

Using the function

$$f_n(z) = \frac{1}{\Gamma(n)}\int_0^\infty \frac{x^{n-1}\,dx}{z^{-1}e^x+1}$$

defined in the mathematical notes handout,

$$N = \left(\frac{kT}{\varepsilon_0}\right)^2 f_2(z)$$

In class [or using the asymptotic expansion for $f_n(z)$ derived on p.5 of the mathematical notes], we obtained the following asymptotic expansion for $f_n(z)$:

$$f_n(z) \simeq \frac{(\ln z)^n}{\Gamma(n+1)}\left[1 + \frac{\pi^2}{6}\frac{n(n-1)}{(\ln z)^2} + O\left(\frac{1}{(\ln z)^4}\right)\right]$$

Thus, for $n=2$ and $z = e^{\mu/kT}$, we obtain

$$f_2(z) \simeq \frac{1}{2}\left(\frac{\mu}{kT}\right)^2 + \frac{\pi^2}{6}$$

and so,

$$N \simeq \frac{1}{2}\left(\frac{\mu}{\varepsilon_0}\right)^2 + \frac{\pi^2}{6}\left(\frac{kT}{\varepsilon_0}\right)^2$$

But from part (a), $N = \frac{\varepsilon_F^2}{2\varepsilon_0^2}$. Thus, solving for $\mu$:

$$\mu^2 = \varepsilon_F^2 - \frac{\pi^2}{3}(kT)^2$$

Since we have employed an asymptotic expansion to derive this result, it is valid only for $kT \ll \varepsilon_F$. Thus, we can write

$$\mu^2 = \varepsilon_F^2\left(1 - \frac{\pi^2}{3}\left(\frac{kT}{\varepsilon_F}\right)^2\right)$$

and take the square root using that fact that $(1+x)^{1/2} \simeq 1 + \frac{1}{2}x$ if $|x| \ll 1$. Thus,

$$\boxed{\mu = \varepsilon_F\left(1 - \frac{\pi^2}{6}\left(\frac{kT}{\varepsilon_F}\right)^2\right) \qquad 0 \le T \ll T_F}$$

where $\varepsilon_F \equiv kT_F$. Putting $\varepsilon_F = \varepsilon_0\sqrt{2N}$, we can write: $\mu = \varepsilon_0\sqrt{2N}\left[1 - \frac{\pi^2}{12N}\left(\frac{kT}{\varepsilon_0}\right)^2\right]$.

Note that RB eq (9.15) on p188 quotes the result

$$\mu = \varepsilon_F - \frac{\pi^2}{6}\frac{D'(\varepsilon_F)}{D(\varepsilon_F)}(kT)^2$$

This is quoted without proof. A proof would require the asymptotic expansion given in the mathematical notes. For $D(\varepsilon) = \varepsilon/\varepsilon_0^2$,

$$\frac{D'(\varepsilon_F)}{D(\varepsilon_F)} = \frac{1}{\varepsilon_F}$$

and we recover the result obtained above.



(c) Next we compute $\langle E \rangle$ by the same method

$$\langle E \rangle = \int_0^\infty \varepsilon\, f_{FD}(\varepsilon) D(\varepsilon)\, d\varepsilon$$

$$= \frac{1}{\varepsilon_0^2}\int_0^\infty \frac{\varepsilon^2 d\varepsilon}{e^{(\varepsilon-\mu)/kT}+1}$$

$$= \frac{(kT)^3}{\varepsilon_0^2}\int_0^\infty \frac{x^2 dx}{z^{-1}e^x + 1} \qquad z = e^{\mu/kT}$$

$$= \frac{2(kT)^3}{\varepsilon_0^2} f_3(z)$$

since $\Gamma(3) = 2! = 2$. Employing the asymptotic expansion

$$f_3(z) \simeq \frac{(\ln z)^3}{6}\left[1 + \frac{\pi^2}{(\ln z)^2}\right]$$

$$= \frac{1}{6}\left(\frac{\mu}{kT}\right)^3\left[1 + \pi^2\left(\frac{kT}{\mu}\right)^2\right]$$

$$\langle E \rangle = \frac{\mu^3}{3\varepsilon_0^2}\left[1 + \pi^2\left(\frac{kT}{\mu}\right)^2\right]$$

For $\mu$, we insert $\mu = \varepsilon_F\left[1 - \frac{\pi^2}{6}\left(\frac{kT}{\varepsilon_F}\right)^2\right]$ obtained in part (b).

Working consistently up to an including terms of $O\left((kT)^2\right)$, it is clear that we need only put $\mu = \varepsilon_F$ in the second term in the bracketed expression for $\langle E \rangle$. Thus, 5

$$\langle E\rangle = \frac{1}{3\varepsilon_0^2}\,\varepsilon_F^3\left[1-\frac{\pi^2}{6}\left(\frac{kT}{\varepsilon_F}\right)^2\right]^3\left[1+\pi^2\left(\frac{kT}{\varepsilon_F}\right)^2\right]$$

More details: Had we written

$$1+\frac{\pi^2(kT)^2}{\varepsilon_F^2\left[1-\frac{\pi^2}{6}\left(\frac{kT}{\varepsilon_F}\right)^2\right]}$$

then for small $kT \ll \varepsilon_F$, this becomes

$$1+\frac{\pi^2(kT)^2}{\varepsilon_F^2}\left[1+\frac{\pi^2}{6}\left(\frac{kT}{\varepsilon_F}\right)^2\right]$$

But we are dropping terms of $O((kT)^4)$.

Finally,

$$\langle E\rangle \simeq \frac{\varepsilon_F^3}{3\varepsilon_0^2}\left[1-\frac{\pi^2}{2}\left(\frac{kT}{\varepsilon_F}\right)^2\right]\left[1+\pi^2\left(\frac{kT}{\varepsilon_F}\right)^2\right]$$

$$\simeq \frac{\varepsilon_F^3}{3\varepsilon_0^2}\left[1+\frac{\pi^2}{2}\left(\frac{kT}{\varepsilon_F}\right)^2\right]$$

where again we work consistently up to $O((kT)^2)$. For example, $(1+x)^3 \simeq 1+3x$ for $|x|\ll 1$, etc.

Thus, we have shown that

$$\boxed{\langle E\rangle \simeq \frac{\varepsilon_F^3}{3\varepsilon_0^2}\left[1+\frac{\pi^2}{2}\left(\frac{kT}{\varepsilon_F}\right)^2\right]}\;.$$

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As a check, we can compute the ground state energy $E_{g.s.}$ at $T=0$ via

$$E_{g.s.} = \int_0^{\varepsilon_F}\varepsilon D(\varepsilon)\,d\varepsilon = \frac{1}{\varepsilon_0^2}\int_0^{\varepsilon_F}\varepsilon^2\,d\varepsilon = \frac{\varepsilon_F^3}{3\varepsilon_0^2}$$

and then use RB eq (9.18)

$$\langle E\rangle = E_{g.s.} + \frac{\pi^2}{6}D(\varepsilon_F)(kT)^2$$

$$= \frac{\varepsilon_F^3}{3\varepsilon_0^2} + \frac{\pi^2}{6}\frac{\varepsilon_F}{\varepsilon_0^2}(kT)^2$$

$$= \frac{\varepsilon_F^3}{3\varepsilon_0^2}\left[1+\frac{\pi^2}{2}\left(\frac{kT}{\varepsilon_F}\right)^2\right]$$

which again confirms our result.

Finally, the heat capacity is

$$C_V = \left(\frac{\partial\langle E\rangle}{\partial T}\right)$$

Using the result for $\langle E\rangle$ above, we end up with:

$$\boxed{C_V = \frac{\pi^2}{3}k^2T\frac{\varepsilon_F}{\varepsilon_0^2}}$$

In terms of $N$, $\varepsilon_0$ $k$ and $T$, use part (a) $[\varepsilon_F = \varepsilon_0\sqrt{2N}]$ to obtain:

$$\langle E\rangle = \frac{\varepsilon_0}{3}(2N)^{3/2}\left[1+\frac{\pi^2}{4N}\left(\frac{kT}{\varepsilon_0}\right)^2\right]$$

$$C_V = \frac{\pi^2 k}{3}\left(\frac{kT}{\varepsilon_0}\right)\sqrt{2N}$$

② RB Chapter 9, problem 5.

(a) The single particle partition function is

$$Z_1 = \sum_r e^{-\varepsilon_r/kT}$$

Approximating the sum by an integral using the density of states $D(\varepsilon) = \varepsilon/\varepsilon_0^2$ from the previous problem,

$$Z_1 = \int_0^\infty e^{-\varepsilon/kT} D(\varepsilon)\, d\varepsilon$$

$$= \frac{1}{\varepsilon_0^2}\int_0^\infty \varepsilon e^{-\varepsilon/kT}\, d\varepsilon$$

Put $x = \varepsilon/kT$

$$Z_1 = \left(\frac{kT}{\varepsilon_0}\right)^2 \int_0^\infty x e^{-x}\, dx$$

$$= \left(\frac{kT}{\varepsilon_0}\right)^2$$

(b) In the semi-classical limit [RB eq. (5.31) on p. 103]

$$Z_N = \frac{Z_1^N}{N!} = \frac{1}{N!}\left(\frac{kT}{\varepsilon_0}\right)^{2N}$$



(c) The total energy is given by [RB eq. (5.16) on p. 97]

$$\langle E \rangle = kT^2 \frac{\partial}{\partial T} \ln Z_N$$

$$= 2NkT^2 \frac{\partial}{\partial T} \ln T$$

$$= 2NkT$$

(d) The chemical potential is given by

$$\mu = \left(\frac{\partial F}{\partial N}\right)_{T,V}$$

Since $F = -kT \ln Z_N$

$$\mu = -kT \frac{\partial}{\partial N} \ln Z_N$$

Note that in this case $Z_N = \frac{Z_1^N}{N!}$

$$\ln \frac{Z_1^N}{N!} = \ln Z_1^N - \ln N!$$

$$= N \ln Z_1 - N \ln N + N$$

where we have used Stirling's approximation: $\ln N! \simeq N \ln N - N$.

Thus, $\frac{\partial}{\partial N} \ln Z_N = \ln Z_1 - \ln N - 1 + 1$

$$= \ln\left(\frac{Z_1}{N}\right).$$

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Thus:
$$\mu = -kT \ln\left(\frac{Z_1}{N}\right)$$

or
$$\boxed{\mu = -kT \ln\left[\frac{1}{N}\left(\frac{kT}{\varepsilon_0}\right)^2\right]}$$

<u>Remark:</u> one can sidestep Stirling's approximation by writing

$$\mu = F(N) - F(N-1)$$
$$= -kT \ln\left[\frac{Z_1^N}{N!} - \ln \frac{Z_1^{N-1}}{(N-1)!}\right]$$
$$= -kT\left[\ln Z_1 - \ln \frac{N!}{(N-1)!}\right] \qquad \text{Note: } N! = N(N-1$$
$$= -kT \ln\left(\frac{Z_1}{N}\right).$$

See RB section 7.3 on pp 156-157. The lemma for computing $\mu$ works <u>only</u> if $Z_N = Z_1^N/N!$ which is true only for some systems.

(e) <u>Method 1:</u>
$$S = -\left(\frac{\partial F}{\partial T}\right)_{N,V}$$
$$= +\frac{\partial}{\partial T}\left[kT \ln Z_N\right]$$
$$= +k \ln Z_N + kT \frac{\partial}{\partial T} \ln Z_N$$

Using the results of part (b),

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$$S = +k \ln\left[\frac{1}{N!}\left(\frac{kT}{\varepsilon_0}\right)^{2N}\right] + kT \cdot 2N \frac{\partial}{\partial T} \ln T$$

Using Stirling's approximation,

$$S = -k\left[N \ln N - N\right] + 2Nk \ln\left(\frac{kT}{\varepsilon_0}\right) + 2Nk$$
$$\boxed{S = Nk\left[\ln\left(\frac{k^2T^2}{N\varepsilon_0^2}\right) + 3\right]}$$

<u>Method 2:</u> Use RB eq (5.25) on p101
$$S = \frac{\langle E\rangle}{T} + k \ln Z_N$$
$$= 2Nk + k \ln\left[\frac{1}{N!}\left(\frac{kT}{\varepsilon_0}\right)^{2N}\right]$$

which clearly leads to the same result.

<u>Remark:</u> The result for $S$ does not appear extensive. This is because a volume factor is secretly hiding in $\varepsilon_0$. For most systems, we expect $D(\varepsilon) \propto V$. Thus, $\varepsilon_0^2$ would be proportional to $1/V$ in which case $S$ would be extensive.

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3. RB Chapter 9, problem 15

Using eq. (9.29) of RB on p196 as an estimate for the central pressure of the star,

$$P \simeq \bar{\rho} R \cdot \frac{1}{2}\frac{GM}{R^2}$$

where $\bar{\rho} = \dfrac{M}{\frac{4}{3}\pi R^3}$ is the average mass density of the star.

This pressure is equated with the pressure of a degenerate electron gas at $T=0$. In this problem, we are asked to assume that the electron gas is ultrarelativistic. Thus we cannot use the formula for $P$ given in problem 5 of this problem set, since that result applies to a non-relativistic fermi gas.

For a relativistic fermi gas, let us use RB eq 9.38 on p.199

$$P = \frac{1}{3}\left(\frac{3}{8\pi}\right)^{1/3} hc \left(\frac{N}{V}\right)^{4/3}$$

For $V$ we put the volume of the star $V = \frac{4\pi R^3}{3}$. For $N$, we put the number of electrons in the star. Since the star is electrically neutral, $N_e$ must also be the number of protons in the star. As discussed in the caption to Table 9.3 [RB p195], the star is dominated by nuclei which have as many neutrons as protons. Since the mass of the star is roughly

$$M \simeq (N_n + N_p)\, m_p$$

(since $m_p \simeq m_n \gg m_e$) and $N_n \simeq N_p \simeq N_e$, we have

$$N \equiv N_e \simeq \frac{M}{2m_p}$$

We conclude that

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$$\frac{M}{\frac{4}{3}\pi R^3}\,\frac{GM}{2R} \simeq \frac{1}{3}\left(\frac{3}{8\pi}\right)^{1/3} \hbar c \left(\frac{M}{2m_p}\right)^{4/3}\left(\frac{3}{4\pi R^3}\right)^{4/3}$$

The factors of $R$ cancel! Thus, solving for $M$

$$M^{2/3} \simeq \frac{\hbar c}{G}\,\frac{1}{(2m_p)^{4/3}}\,\frac{8\pi}{9}\left(\frac{3}{8\pi}\right)^{1/3}\left(\frac{3}{4\pi}\right)^{4/3}$$

or

$$\boxed{M \simeq \frac{1}{8\pi\sqrt{3}}\,\frac{1}{m_p^2}\left(\frac{\hbar c}{G}\right)^{3/2}}$$

(b) The electron mass does not enter, because in the ultra-relativistic limit, $cp_F \gg m_e c^2$. Specifically, for ultrarelativistic electrons, $v \simeq c$ independent of the electron mass $m_e$. Since the pressure is given by $P = \frac{1}{3}\langle \vec{p}\cdot\vec{v}\rangle (N/V)$ [RB eq (9.33) on p198], in the ultra-relativistic limit we can simply neglect $m_e$ entirely.

(c) Using the results of part (a)

$$M \simeq \frac{1}{8\pi\sqrt{3}}\,\frac{1}{(1.673\times 10^{-27}\,kg)^2}\left(\frac{(6.626\times 10^{-34}\,J\cdot s)(2.998\times 10^{8}\,m\,s^{-1})}{6.673\times 10^{-11}\,m^3\,kg^{-1}\,s^{-2}}\right)^{3/2}$$

$$= 1.33\times 10^{30}\,kg$$

Using the mass of the sun, $M_\odot = 1.989\times 10^{30}\,kg$

$$\boxed{M = 0.67\,M_\odot}$$

which is a very reasonable value.

(4) RB Chapter 9, problem 16

(a) In relativity,

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$E = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}$$

Thus, $\vec{p} = \frac{E}{c^2}\vec{v}$ and solving for $\vec{v}$

$$\vec{v} = \frac{c^2}{E}\vec{p}$$

The magnitude of $\vec{v}$ is given by $v = (\vec{v}\cdot\vec{v})^{1/2}$ and likewise $p = (\vec{p}\cdot\vec{p})^{1/2}$. Thus,

$$v = \frac{c^2 p}{E}$$

Moreover, we can combine the formulae above to obtain

$$E^2 = \vec{p}^{\,2}c^2 + m^2c^4$$

Taking the derivative with respect to $p$, and noting that $p^2 = \vec{p}^{\,2}$,

$$E\,dE = c^2 p\,dp$$

or

$$\frac{dE}{dp} = \frac{pc^2}{E} = v$$



(b) According to eq (8.18) of RB (on p174), in the classical limit,

$$\langle n_\alpha \rangle = \frac{1}{e^{(\epsilon_\alpha-\mu)/kT}} \ll 1$$

since we can drop the $+1$ in the denominator. The average value of $pv$ is

$$\langle pv \rangle = \frac{1}{N}\sum_\alpha (pv)_\alpha \langle n \rangle_\alpha$$

Following the instructions of eq (13.6) of RB, the sum in the classical limit is interpreted as an integral over

$$\int \frac{d^3x\,d^3p}{h^3} = g\frac{V}{h^3}\int d^3p$$

where we've inserted $g$ to account for the $g = 2s+1$ spin states of the particles. In the case above, the integrand depends only on the magnitude $|\vec{p}| \equiv p$, so,

$$\int d^3p = \int_0^\infty p^2\,dp\,d\Omega = 4\pi\int_0^\infty p^2\,dp$$

Thus,

$$\langle pv \rangle = g\frac{V}{Nh^3}4\pi\int_0^\infty p^2\,dp\;p\,v\,e^{-(\epsilon-\mu)/kT}$$

or

$$\langle pv \rangle = g\frac{V}{Nh^3}4\pi e^{\mu/kT}\int_0^\infty v p^3 e^{-\epsilon/kT}\,dp$$

where $\epsilon = \sqrt{p^2c^2+m^2c^4}$.

$$v = \frac{pc^2}{\epsilon}$$

To make further progress, we shall eliminate $\mu$ by computing $N$:

$$N = \sum \langle n_\alpha \rangle = g\frac{V}{h^3} 4\pi e^{\mu/kT} \int_0^\infty p^2 e^{-\varepsilon/kT} dp$$

Integrate by parts:

$$\int_0^\infty p^2 e^{-\varepsilon/kT} dp \qquad \left( \begin{array}{ll} u = e^{-\varepsilon/kT} & dv = p^2 dp \\ du = -\dfrac{e^{-\varepsilon/kT}}{kT}\dfrac{d\varepsilon}{dp} & v = \dfrac{p^3}{3} \end{array} \right)$$

$$= \frac{p^3}{3} e^{-\varepsilon/kT}\Big|_0^\infty + \frac{1}{3}\frac{1}{kT}\int_0^\infty p^3 e^{-\varepsilon/kT}\frac{d\varepsilon}{dp}\, dp$$

Note that $\lim_{p\to\infty} e^{-\varepsilon/kT} = \lim_{p\to\infty} e^{-\sqrt{p^2c^2+m^2c^4}/kT} = 0$.

Moreover, use part (a) to write $\dfrac{d\varepsilon}{dp} = v$.

Thus,

$$\int_0^\infty p^2 e^{-\varepsilon/kT} dp = \frac{1}{3kT}\int_0^\infty v p^3 e^{-\varepsilon/kT} dp$$

and so

$$N = \frac{gV}{h^3} 4\pi e^{\mu/kT} \frac{1}{3kT}\int_0^\infty v p^3 e^{-\varepsilon/kT} dp$$

Comparing this with the expression for $\langle pv \rangle$,

$$\boxed{\langle pv \rangle = 3kT}$$



(c) Using RB eq (9.33) on p198

$$P = \frac{1}{3}\langle \vec{p}\cdot\vec{v} \rangle \frac{N}{V}$$

But in part (a), we saw that $\vec{v} = \frac{c^2}{E}\vec{p}$. That is $\vec{v}$ is parallel to $\vec{p}$ and so $\vec{p}\cdot\vec{v} = pv$. Thus,

$$P = \frac{1}{3}\langle pv \rangle \frac{N}{V}$$

Using the result of part (b): $\langle pv \rangle = 3kT$, we end up with

$$\boxed{P = \frac{N}{V}kT}$$

which is the ideal gas law. Thus, in the classical limit, the ideal gas law applies to any non-interacting gas, even if it is relativistic.

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(5) (a) To compute the pressure of a non-relativistic Fermi gas at $T=0$, we can use RB eq. (1.6) on p.6 to conclude that

$$P = \frac{2}{3}\frac{\langle E\rangle}{V}$$

Note that this relation is true only for a non-interacting (i.e. ideal) gas. In class, we derived

$$\langle E\rangle = \frac{3}{5} N\epsilon_F$$

where

$$\epsilon_F = \frac{\hbar^2}{2m}\left(\frac{6\pi^2 N}{gV}\right)^{2/3}$$

and $g = 2s+1$. For electrons, $s=\frac{1}{2}$ and thus $g=2$. Then,

$$\langle E\rangle = \frac{3\hbar^2 N}{10m}\left(\frac{3\pi^2 N}{V}\right)^{2/3}$$

and so,

$$\boxed{P = \frac{\hbar^2}{5m}(3\pi^2)^{2/3}\left(\frac{N}{V}\right)^{5/3}}$$

The same result was obtained in eq (9.30) of RB on p.196.

For completeness, I give two other methods which do not depend on the result $P = \frac{2}{3}\langle E\rangle/V$ derived from kinetic theory.

Alternate Method #1

From $dE = TdS - pdV + \mu dN$, it follows that

$$p = -\left(\frac{\partial E}{\partial V}\right)_{S,N}$$

Now, in the ground state (at $T=0$), the fermions assemble into the



lowest possible set of single-particle energy levels consistent with Pauli's principle. This is a state of multiplicity 1, so $S=0$ (since $S$ is proportional to the logarithm of the multiplicity) in agreement with the third law of thermodynamics.

Thus, we can take

$$E = \frac{3\hbar^2 N}{10m}\left(\frac{3\pi^2 N}{V}\right)^{2/3}$$

and compute

$$P = -\left(\frac{\partial E}{\partial V}\right) = \frac{\hbar^2}{5m}(3\pi^2)^{2/3}\left(\frac{N}{V}\right)^{5/3}$$

Alternate Method #2

In class, I showed that for a Fermi gas at arbitrary temperature $T$,

$$PV = kTgV\lambda_{th}^{-3} f_{5/2}(z) \qquad z = e^{\mu/kT}$$

where

$$\lambda_{th} = \sqrt{\frac{2\pi\hbar^2}{mkT}}$$

and

$$f_n(z) = \frac{1}{\Gamma(n)}\int_0^\infty \frac{x^{n-1}}{z^{-1}e^x+1}\,dx$$

In the limit of $kT \ll \epsilon_F$, the leading term in the low-temperature expansion is:

$$f_n(z) \simeq \frac{(\ln z)^n}{\Gamma(n+1)}$$

The proof is simple, since we know that at $T=0$,

$$\frac{1}{z^{-1}e^{x}+1} = \begin{cases} 1 & 0 \le x \le \ln z \\ 0 & x > \ln z \end{cases}$$

Recall that $x \equiv \varepsilon/kT$ and at $T=0$, $\mu=\varepsilon_F$. Thus integrating up to $\varepsilon=\varepsilon_F$ is equivalent to integrating up to $x=\varepsilon_F/kT=\mu/kT=\ln z$. Thus,

$$f_n(z) \simeq \frac{1}{\Gamma(n)}\int_0^{\ln z} x^{n-1}\,dx = \frac{(\ln z)^n}{\Gamma(n+1)}$$

where I have used $n\Gamma(n)=\Gamma(n+1)$.

In particular,

$$f_{5/2}(z) = \frac{8}{15\sqrt{\pi}}(\ln z)^{5/2}$$

with

$$\ln z = \frac{\mu}{kT} = \frac{\varepsilon_F}{kT} = \frac{\hbar^2}{2mkT}\left(\frac{3\pi^2 N}{V}\right)^{2/3}$$

(after putting $g=2$ for the electron gas). Hence,

$$P = 2kT\left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2}\frac{8}{15\sqrt{\pi}}\left(\frac{\hbar^2}{2mkT}\right)^{5/2}\left(\frac{3\pi^2 N}{V}\right)^{5/3}$$

$$= \frac{\hbar^2}{5m}(3\pi^2)^{2/3}\left(\frac{N}{V}\right)^{5/3}$$



(b) There are a number of different ways to compute the entropy.

Using RB eq. (14.27) on p340

$$S(T) = \int_0^T \frac{C_V(T')}{T'}\,dT'$$

Note that this expression automatically embodies the third law of thermodynamics namely $S(T)\to 0$ as $T\to 0$.

For the Fermi electron gas at $T \ll T_F$, use RB eq. (9.19) on p189.

$$C_V = \frac{\pi^2 Nk}{2}\left(\frac{T}{T_F}\right)$$

Then,

$$S = \frac{\pi^2 Nk}{2T_F}\int_0^T dT'$$

or

$$\boxed{S = \frac{\pi^2 NkT}{2T_F} \qquad T \ll T_F}$$

Alternate method

Use the Euler relation: $E = TS - PV + \mu N$.

In part (a), we noted that

$$PV = kTgV\lambda_{th}^{-3} f_{5/2}(z)$$

where $z=e^{\mu/kT}$. Note that we can write $\mu = kT\ln z$ so

$$\mu N = NkT\ln z$$

Finally, in class we obtained

$$E = \frac{3}{2} kT g \lambda_{th}^{-3} V f_{5/2}(z)$$

$$N = g \lambda_{th}^{-3} V f_{3/2}(z)$$

Note that $P = \frac{2}{3}\frac{E}{V}$ as expected.

Thus,

$$S = \frac{1}{T}\left(E + pV - \mu N\right)$$

$$= \frac{1}{T}\left[\frac{5}{2} kT g \lambda_{th}^{-3} V f_{5/2}(z) - kT g \lambda_{th}^{-3} V f_{3/2}(z) \ln z\right]$$

$$= k g \lambda_{th}^{-3} V \left[\frac{5}{2} f_{5/2}(z) - f_{3/2}(z) \ln z\right]$$

Inserting the leading asymptotic behavior of $f_{5/2}(z)$ and $f_{3/2}(z)$ appropriate for the low-temperature expansions:

$$f_{5/2}(z) \simeq \frac{8}{15\sqrt{\pi}} (\ln z)^{5/2} \left[1 + \frac{5\pi^2}{8(\ln z)^2}\right]$$

$$f_{3/2}(z) \simeq \frac{4}{3\sqrt{\pi}} (\ln z)^{3/2} \left[1 + \frac{\pi^2}{8(\ln z)^2}\right]$$

we get:

$$S = \frac{4}{3\sqrt{\pi}} k g \lambda_{th}^{-3} V (\ln z)^{5/2} \left[\left(1 + \frac{5\pi^2}{8(\ln z)^2}\right) - \left(1 + \frac{\pi^2}{8(\ln z)^2}\right)\right]$$

$$= \frac{2\pi^{3/2}}{3} k g \lambda_{th}^{-3} V (\ln z)^{1/2}$$

Putting $\ln z = \frac{\mu}{kT}$ and $\lambda_{th} = \sqrt{\frac{2\pi\hbar^2}{mkT}}$ and $g=2$ for spin-1/2 electrons,



$$S = \frac{4\pi^{3/2} k}{3} \left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2} V \left(\frac{\mu}{kT}\right)^{1/2}$$

$$= \frac{4}{3} k V \left(\frac{m}{2\hbar^2}\right)^{3/2} \mu^{1/2} k T$$

To see that this is equivalent to our previous result, we note that at $T \ll T_F$ [RB eq. (9.16) on p188]

$$\mu \simeq \varepsilon_F \left(1 - \frac{\pi^2 T^2}{12 T_F^2}\right)$$

Since we are after the leading behavior for $T \ll T_F$, it is sufficient to use $\mu = \varepsilon_F$ in the expression for $S$. If we write

$$\varepsilon_F^{1/2} = \frac{\varepsilon_F^{3/2}}{\varepsilon_F} = \frac{1}{\varepsilon_F}\left(\frac{\hbar^2}{2m}\right)^{3/2}\left(\frac{3\pi^2 N}{V}\right)$$

where we have used $\varepsilon_F = \frac{\hbar^2}{2m}\left(\frac{6\pi^2 N}{gV}\right)^{2/3}$ with $g=2$ in the numerator,

$$S = \frac{4}{3} k V \left(\frac{m}{2\hbar^2}\right)^{3/2} \frac{1}{\varepsilon_F}\left(\frac{\hbar^2}{2m}\right)^{3/2}\left(\frac{3\pi^2 N}{V}\right) k T$$

$$= \frac{\pi^2 N k T}{2 T_F}$$

after putting $\varepsilon_F = k T_F$. Thus we recover our previous result. Note that in this method, we find that $S \to 0$ as $T \to 0$ automatically. This feature is an output of our computation.

6) RB Chapter 9, problem 20

The Bose-Einstein condensation temperature is

$$T_B = \frac{2\pi\hbar^2}{km}\left(\frac{N}{\zeta(\frac{3}{2})V}\right)^{2/3}$$

where $\zeta(\frac{3}{2}) \simeq 2.612$

(a) Plugging in the numbers, with $\frac{N}{V} = 10^{20}\,m^{-3}$ and $m = 3.82\times10^{-26}\,kg$

$$T_B = \frac{2\pi(1.0546\times10^{-34}\,J\cdot s)^2}{(1.381\times10^{-23}\,J/K)(3.82\times10^{-26}\,kg)}\left(\frac{10^{20}\,m^{-3}}{2.612}\right)^{2/3}$$

$$= 1.50\times10^{-6}\,K$$

(b) The number of atoms in the ground state is [RB eq (9.52) on p202]

$$N_0 = N\left[1 - \left(\frac{T}{T_B}\right)^{3/2}\right]$$

Thus, for $\frac{N_0}{N} = 0.9$, we need $\left(\frac{T}{T_B}\right)^{3/2} = 0.1$ or

$$T = (0.1)^{2/3}\,T_B$$

$$= (0.2154)(1.50\times10^{-6}\,K)$$

$$= 3.24\times10^{-7}\,K$$



(c) In the case where half the atoms are $^{23}Na$ and half the atoms are $^{21}Na$, we have a situation where the system consists of a mixture of (approximately) non-interacting ideal gases. Thus, the two gases can be treated independently. In this case, since both gases occupy the same volume, we have

$$\frac{N}{V} = 0.5\times10^{20}\,m^{-3}$$

for each one. Let us compute the corresponding Bose-Einstein condensation temperatures. If we take

$$m(^{23}Na) = 3.82\times10^{-26}\,kg$$

then

$$m(^{21}Na) \simeq \frac{21}{23}\,m(^{23}Na) = 3.49\times10^{-26}\,kg$$

Thus, for $^{23}Na$,

$$T_B = \frac{2\pi(1.0546\times10^{-34}\,J\cdot s)^2}{(1.381\times10^{-23}\,J/K)(3.82\times10^{-26}\,kg)}\left(\frac{0.5\times10^{20}\,m^{-3}}{2.612}\right)^{2/3}$$

$$= 9.48\times10^{-7}\,K$$

For $^{21}Na$,

$$T_B = \frac{2\pi(1.0546\times10^{-34}\,J\cdot s)^2}{(1.381\times10^{-23}\,J/K)(3.49\times10^{-26}\,kg)}\left(\frac{0.5\times10^{20}\,m^{-3}}{2.612}\right)^{2/3}$$

$= \frac{23}{21}\,T_B(^{23}Na)$ due to the slight difference in mass

$$= 1.04\times10^{-6}\,K$$

In the case of the system made up entirely of $^{23}Na$, we have from part (a) that $T_B = 1.50\times 10^{-6}\,K$.

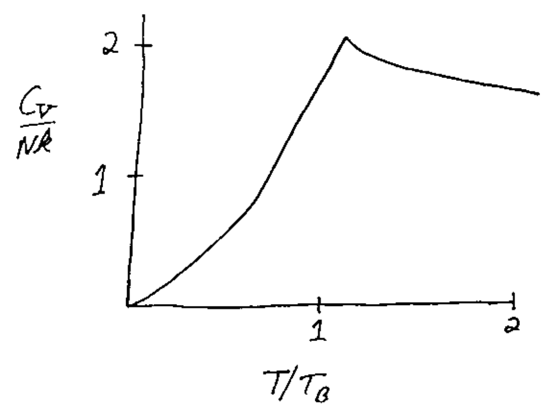
Thus, if we study both systems at $T = 1.3\times 10^{-6}\,K$, we see that:

(i) If all atoms are $^{23}Na$ atoms then $T < T_B$.

(ii) If half are $^{23}Na$ and half are $^{21}Na$ atoms, then $T > T_B$ for <u>both</u> the gas of $^{23}Na$ atoms and the gas of $^{21}Na$ atoms.

According to RB Figure 9.8 on p204, $C_V/Nk$ is a function of temperature that reaches a maximum at $T = T_B$ (for temperatures in the range $0 < T < 2T_B$)

Thus, in case (i), since $T < T_B$ we conclude that $C_V$ <u>increases</u> with increasing temperature (until $T_B$ is reached)




In case (ii), $C_V$ <u>decreases</u> with increasing temperature since in this case $T$ is above $T_B$.

This is a remarkable effect. It is a consequence of the fact that although individual $^{21}Na$ (or $^{23}Na$) atoms are indistinguishable, the $^{21}Na$ atoms are distinguishable from the $^{23}Na$ atoms.

Thus, the seemingly inconsequential mass difference between $^{21}Na$ and $^{23}Na$ has a dramatic effect in differentiating the thermodynamic behavior of the two cases.

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(7) RB Chapter 14, problem 3

(a) Using the result of the Maxwell relation derived in RB Chapter 10, problem 4 [see the second problem on Problem Set #7]

$$\left(\frac{\partial P}{\partial T}\right)_{V,N} = \left(\frac{\partial S}{\partial V}\right)_{T,N}$$

Because the entropy change in an isothermal process goes to zero as $T \to 0$ as a consequence of the third law of thermodynamics, it follows that

$$\left(\frac{\partial P}{\partial T}\right)_{V,N} \longrightarrow 0 \qquad \text{as} \quad T \to 0.$$

(b) As a check of this relation, let's look at the non-relativistic ideal Fermi gas.

In problem 5, we noted that

$$P = \frac{2}{3}\frac{\langle E\rangle}{V}$$

Thus,

$$\left(\frac{\partial P}{\partial T}\right)_V = \frac{2}{3V}\left(\frac{\partial\langle E\rangle}{\partial T}\right)_V = \frac{2C_V}{3V}$$

At $T \ll T_F$, we can use RB eq (9.19) on p 189

$$C_V = \frac{\pi^2 Nk}{2}\frac{T}{T_F}$$

Thus,

$$\left(\frac{\partial P}{\partial T}\right)_V = \frac{\pi^2 N k}{3V}\frac{T}{T_F} \qquad T \ll T_F$$

and clearly,

$$\left(\frac{\partial P}{\partial T}\right)_V \longrightarrow 0 \qquad \text{as } T \longrightarrow 0.$$

We can also verify the above behavior for a non-relativistic ideal Bose gas. Again $P = \frac{2}{3}\frac{\langle E\rangle}{V}$ applies and so

$$\left(\frac{\partial P}{\partial T}\right)_V = \frac{2C_V}{3V}$$

In class, we showed that for $T \le T_B$,

$$C_V = \frac{15}{4}\zeta\left(\tfrac{5}{2}\right)\lambda_{th}^{-3}V$$

$$= \frac{15}{4}\zeta\left(\tfrac{5}{2}\right)\left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2}V$$

Thus, as $T \to 0$,

$$\left(\frac{\partial P}{\partial T}\right)_V \propto T^{3/2} \longrightarrow 0 .$$

It is interesting to note that the classical ideal gas does <u>not</u> satisfy the third law of thermodynamics. Since $PV = NkT$,

$$\left(\frac{\partial P}{\partial T}\right)_V = \frac{Nk}{V}$$



and clearly this does not vanish as $T \to 0$. The reason is easy to comprehend. The classical ideal gas is a valid approximation in the limit of

$$\lambda_{th} \ll \left(\frac{V}{N}\right)^{1/3}$$

(see RB eq (5.47) on p106). Since $\lambda_{th} = \left(\frac{2\pi\hbar^2}{mkT}\right)^{1/2}$, this inequality implies that

$$T \gg \frac{2\pi\hbar}{mk}\left(\frac{N}{V}\right)^{2/3}$$

No matter how dilute the gas is, this inequality necessarily breaks down as $T \to 0$. In this regime, we must employ one of the quantum ideal gases in order to get correct results.