

2.4 BRANCHES OF FUNCTIONS

In Section 2.2 we defined the principal square root function and investigated some of its properties. We left unanswered some questions concerning the choices of square roots. We now look at these questions because they are similar to situations involving other elementary functions.

In our definition of a function in Section 2.1, we specified that each value of the independent variable in the domain is mapped onto one and *only one* value in the range. As a result, we often talk about a single-valued function, which emphasizes the “only one” part of the definition and allows us to distinguish such functions from multiple-valued functions, which we now introduce.

Let $w = f(z)$ denote a function whose domain is the set D and whose range is the set R . If w is a value in the range, then there is an associated inverse relation $z = g(w)$ that assigns to each value w the value (or values) of z in D for which the equation $f(z) = w$ holds. But unless f takes on the value w at most once in D , then the inverse relation g is necessarily many valued, and we say that g is a multivalued function. For example, the inverse of the function $w = f(z) = z^2$ is the square root function $z = g(w) = w^{\frac{1}{2}}$. For each value z other than $z = 0$, then, the two points z and $-z$ are mapped onto the same point $w = f(z)$; hence g is, in general, a two-valued function.

The study of limits, continuity, and derivatives loses all meaning if an arbitrary or ambiguous assignment of function values is made. For this reason we did not allow multivalued functions to be considered when we defined these concepts. When working with inverse functions, you have to specify carefully one of the many possible inverse values when constructing an inverse function, as when you determine implicit functions in calculus. If the values of a function f are determined by an equation that they satisfy rather than by an explicit formula, then we say that the function is defined implicitly or that f is an implicit function. In the theory of complex variables we present a similar concept.

We now let $w = f(z)$ be a multiple-valued function. A branch of f is any single-valued function f_0 that is continuous in some domain (except, perhaps, on the boundary). At each point z in the domain, it assigns one of the values of $f(z)$.

■ **EXAMPLE 2.20** We consider some branches of the two-valued square root function $f(z) = z^{\frac{1}{2}}$ ($z \neq 0$). Recall that the principal square root function is

$$f_1(z) = |z|^{\frac{1}{2}} e^{i\frac{\text{Arg}(z)}{2}} = r^{\frac{1}{2}} e^{i\frac{\theta}{2}} = r^{\frac{1}{2}} \cos \frac{\theta}{2} + ir^{\frac{1}{2}} \sin \frac{\theta}{2}, \quad (2-28)$$

where $r = |z|$ and $\theta = \text{Arg}(z)$ so that $-\pi < \theta \leq \pi$. The function f_1 is a branch of f . Using the same notation, we can find other branches of the square root function. For example, if we let

$$f_2(z) = |z|^{\frac{1}{2}} e^{i\frac{\text{Arg}(z)+2\pi}{2}} = r^{\frac{1}{2}} e^{i\frac{\theta+2\pi}{2}} = r^{\frac{1}{2}} \cos \left(\frac{\theta+2\pi}{2} \right) + ir^{\frac{1}{2}} \sin \left(\frac{\theta+2\pi}{2} \right), \quad (2-29)$$

then

$$f_2(z) = r^{\frac{1}{2}} e^{i\frac{\theta+2\pi}{2}} = r^{\frac{1}{2}} e^{i\frac{\theta}{2}} e^{i\pi} = -r^{\frac{1}{2}} e^{i\frac{\theta}{2}} = -f_1(z),$$

so f_1 and f_2 can be thought of as “plus” and “minus” square root functions. The negative real axis is called a branch cut for the functions f_1 and f_2 . Each point on the branch cut is a point of discontinuity for both functions f_1 and f_2 .

■ **EXAMPLE 2.21** Show that the function f_1 is discontinuous along the negative real axis.

Solution Let $z_0 = r_0 e^{i\pi}$ denote a negative real number. We compute the limit as z approaches z_0 through the upper half-plane $\{z : \text{Im}(z) > 0\}$ and the limit as z approaches z_0 through the lower half-plane $\{z : \text{Im}(z) < 0\}$. In polar coordinates these limits are given by

$$\begin{aligned} \lim_{(r,\theta) \rightarrow (r_0,\pi)} f_1(re^{i\theta}) &= \lim_{(r,\theta) \rightarrow (r_0,\pi)} r^{\frac{1}{2}} \left(\cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \right) = ir_0^{\frac{1}{2}}, \quad \text{and} \\ \lim_{(r,\theta) \rightarrow (r_0,-\pi)} f_1(re^{i\theta}) &= \lim_{(r,\theta) \rightarrow (r_0,-\pi)} r^{\frac{1}{2}} \left(\cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \right) = -ir_0^{\frac{1}{2}}. \end{aligned}$$

The two limits are distinct, so the function f_1 is discontinuous at z_0 .

Remark 2.4 Likewise, f_2 is discontinuous at z_0 . The mappings $w = f_1(z)$, $w = f_2(z)$, and the branch cut are illustrated in Figure 2.18. ■

We can construct other branches of the square root function by specifying that an argument of z given by $\theta = \arg z$ is to lie in the interval $\alpha < \theta \leq \alpha + 2\pi$.

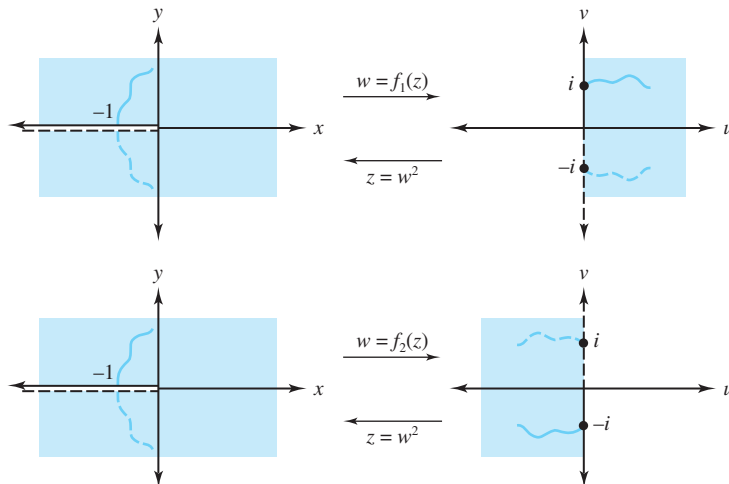


Figure 2.18 The branches f_1 and f_2 of $f(z) = z^{\frac{1}{2}}$.

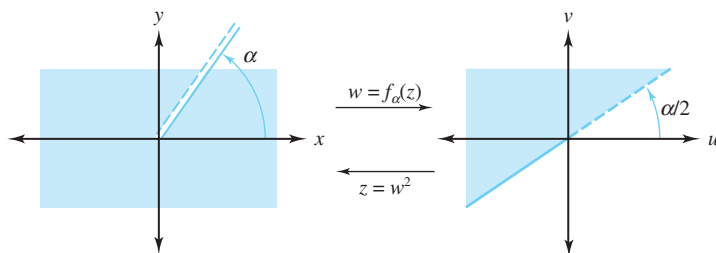


Figure 2.19 The branch f_α of $f(z) = z^{\frac{1}{2}}$.

The corresponding branch, denoted f_α , is

$$f_\alpha(z) = r^{\frac{1}{2}} \cos \frac{\theta}{2} + ir^{\frac{1}{2}} \sin \frac{\theta}{2}, \quad (2-30)$$

where $z = re^{i\theta} \neq 0$ and $\alpha < \theta \leq \alpha + 2\pi$.

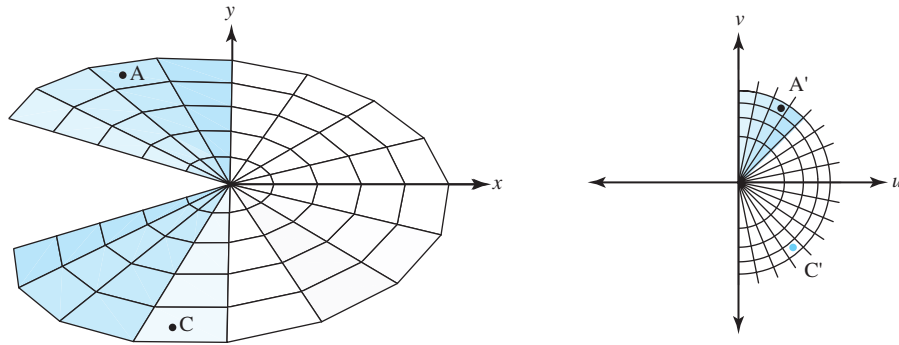
The branch cut for f_α is the ray $r \geq 0$, $\theta = \alpha$, which includes the origin. The point $z = 0$, common to all branch cuts for the multivalued square root function, is called a branch point. The mapping $w = f_\alpha(z)$ and its branch cut are illustrated in Figure 2.19.

2.4.1 The Riemann Surface for $w = z^{\frac{1}{2}}$

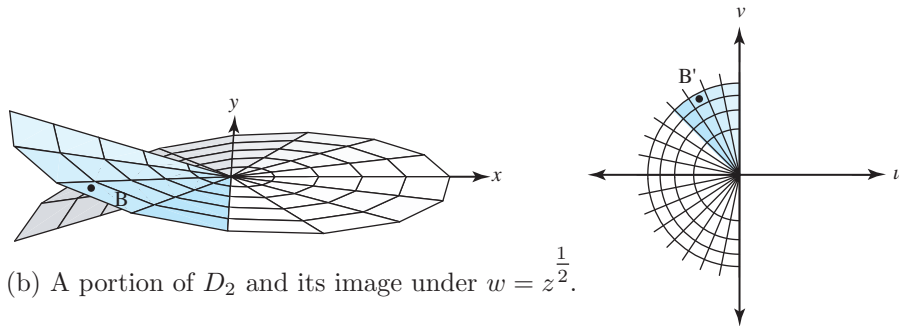
A Riemann surface is a construct useful for visualizing a multivalued function. It was introduced by G. F. B. Riemann (1826–1866) in 1851. The idea is ingenious—a geometric construction that permits surfaces to be the domain or range of a multivalued function. Riemann surfaces depend on the function being investigated. We now give a nontechnical formulation of the Riemann surface for the multivalued square root function.

Consider $w = f(z) = z^{\frac{1}{2}}$, which has two values for any $z \neq 0$. Each function f_1 and f_2 in Figure 2.18 is single-valued on the domain formed by cutting the z plane along the negative x -axis. Let D_1 and D_2 be the domains of f_1 and f_2 , respectively. The range set for f_1 is the set H_1 consisting of the right half-plane, and the positive v -axis; the range set for f_2 is the set H_2 consisting of the left half-plane and the negative v -axis. The sets H_1 and H_2 are “glued together” along the positive v -axis and the negative v -axis to form the w plane with the origin deleted.

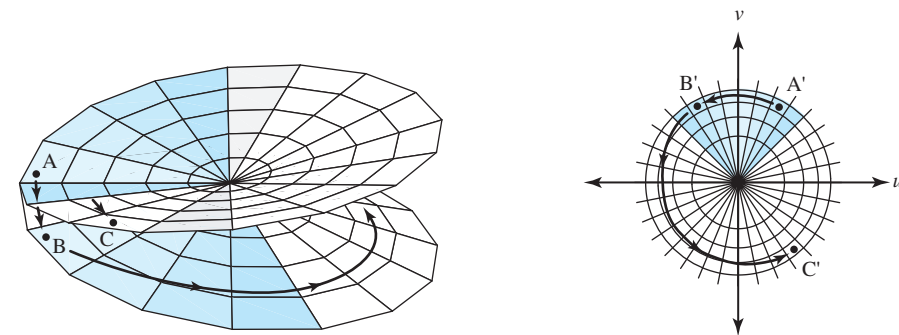
We stack D_1 directly above D_2 . The edge of D_1 in the upper half-plane is joined to the edge of D_2 in the lower half-plane, and the edge of D_1 in the lower half-plane is joined to the edge of D_2 in the upper half-plane. When these domains are glued together in this manner, they form R , which is a Riemann surface domain for the mapping $w = f(z) = z^{\frac{1}{2}}$. The portions of D_1 , D_2 , and R that lie in $\{z : |z| < 1\}$ are shown in Figure 2.20.



(a) A portion of D_1 and its image under $w = z^{\frac{1}{2}}$.



(b) A portion of D_2 and its image under $w = z^{\frac{1}{2}}$.



(c) A portion of R and its image under $w = z^{\frac{1}{2}}$.

Figure 2.20 Formation of the Riemann surface for $w = z^{\frac{1}{2}}$: (a) a portion of D_1 and its image under $w = z^{\frac{1}{2}}$; (b) a portion of D_2 and its image under $w = z^{\frac{1}{2}}$; (c) a portion of R and its image under $w = z^{\frac{1}{2}}$.

The beauty of this structure is that it makes this “full square root function” continuous for all $z \neq 0$. Normally, the principal square root function would be discontinuous along the negative real axis, as points near -1 but above that axis would get mapped to points close to i , and points near -1 but below the axis would get mapped to points close to $-i$. As Figure 2.20(c) indicates, however, between the point A and the point B , the domain switches from the edge of D_1 in the upper half-plane to the edge of D_2 in the lower half-plane. The corresponding mapped points A' and B' are exactly where they should be. The surface works in such a way that going directly between the edges of D_1 in the upper and lower half-planes is impossible (likewise for D_2). Going counterclockwise, the only way to get from the point A to the point C , for example, is to follow the path indicated by the arrows in Figure 2.20(c).