1. Consider the function
\[
f(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta},
\] (1)
where \(0 < x < \infty\), and \(\alpha\) and \(\beta\) are positive constants.

Define the family of integrals
\[
I_n \equiv \int_0^\infty x^n f(x) \, dx,
\]
where \(n\) is a non-negative integer.*

(a) Evaluate the integral \(I_n\) for arbitrary \(n\). Your answer should be expressed in terms of functions that depend on \(\alpha\), \(\beta\) and \(n\).

(b) For the cases of \(I_1\) and \(I_2\), simplify the corresponding result obtained in part (a). Then, compute the quantity \(I_2 - (I_1)^2\). Your final answers should be expressed in terms of simple elementary functions of \(\alpha\) and \(\beta\).

(c) Suppose that the integration variable \(x\) has dimensions of length. What are the dimension of \(\alpha\) and \(\beta\) required for the consistency of dimensional analysis? Determine the dimension of \(I_n\) as a function of \(n\). Then, check that your results for \(I_1\) and \(I_2\) in part (b) exhibit the correct dimensions of length.

2. If \(n\) is a positive integer, then the double factorial is defined as:
\[
(2n)!! \equiv 2 \cdot 4 \cdot 6 \cdots (2n), \quad (2n-1)!! \equiv 1 \cdot 3 \cdot 5 \cdots (2n-1).
\]

(a) Express \((2n)!!\) in terms of the gamma function. [HINT: Factor out a 2 from each term in the definition of \((2n)!!\)]

(b) Express \((2n-1)!!\) in terms of a ratio of two gamma functions. [HINT: First evaluate the product \((2n)!!(2n-1)!!\), and then use the result of part (a)]

(c) Find the leading behavior of the ratio
\[
r_n \equiv \frac{(2n-1)!!}{(2n)!!}, \quad \text{as} \quad n \to \infty.
\]

*In statistics, eq. (1) is known as the gamma distribution, which is sometimes employed to describe the probability that a random variable \(x\) lies within an interval on the positive real axis. In this case, \(I_1\) is the mean of the distribution and \(I_2 - (I_1)^2\) is the variance.
3. Consider the matrix

\[ M = \begin{pmatrix} x & 1 & 0 & 1 \\ 1 & x & 1 & 0 \\ 0 & 1 & x & 1 \\ 1 & 0 & 1 & x \end{pmatrix}. \]

(a) Find all values of \( x \) for which the inverse of \( M \) does not exist. [HINT: You do not have to compute \( M^{-1} \) to answer this question.]

(b) Determine the rank of \( M \) as a function of \( x \). [HINT: Consider the explicit form of \( M \) for the values of \( x \) found in part (a).]

4. In the following problem, \( c \) is some number (which may be complex). Note that parts (b) and (c) of this problem are independent of part (a).

(a) For what values of \( c \) (if any) will the following equations,

\[ x + y = cx, \]
\[ -x + y = cy, \]

have non-trivial (i.e. non-zero) solutions?

(b) For what values of \( c \) (if any) will the following equations,

\[ x + y + z = 6, \]
\[ x + cy + cz = 2, \]

have either a unique solution, an infinite number of solutions, or no solutions.

(c) Determine the values of \( x, y \) and \( z \) that solve the equations of part (b) as a function of \( c \), assuming solutions exist. If an infinite number of solutions exist, write the solution set for \((x, y, z)\) that indicates all the possible solutions.