1. Evaluate the following limits:

(a) $\displaystyle\lim_{x\to 0}\left(\frac{1+x}{x}-\frac{1}{\sin x}\right)$

Using the Taylor expansion for $\sin x$ about $x=0$,

$$\lim_{x\to 0}\left(\frac{1+x}{x}-\frac{1}{\sin x}\right)=\lim_{x\to 0}\left(1+\frac{1}{x}-\frac{1}{\sin x}\right)=\lim_{x\to 0}\left(1+\frac{1}{x}-\frac{1}{x-\frac{1}{6}x^3+\mathcal{O}(x^5)}\right)$$
$$=\lim_{x\to 0}\left[1+\frac{1}{x}-\frac{1}{x}\left(\frac{1}{1+\frac{1}{6}x^2+\mathcal{O}(x^4)}\right)\right].$$

We can omit the $\mathcal{O}(x^4)$ term in the denominator. Using the expansion of the geometric series

$$\frac{1}{1-y}=\sum_{n=0}^{\infty}y^n=1+y+\mathcal{O}(y^2)\,,$$

we may take $y\equiv -\frac{1}{6}x^2$ to obtain:

$$\frac{1}{1+\frac{1}{6}x^2}=1-\tfrac{1}{6}x^2+\mathcal{O}(x^4)\,.$$

Hence,

$$\lim_{x\to 0}\left(\frac{1+x}{x}-\frac{1}{\sin x}\right)=\lim_{x\to 0}\left(1+\frac{1}{x}-\frac{1}{x}\left[1-\tfrac{1}{6}x^2+\mathcal{O}(x^4)\right]\right)=\lim_{x\to 0}\left(1+\tfrac{1}{6}x+\mathcal{O}(x^3)\right)=1\,.$$

Note that our calculation above also provides the behavior as $x\to 0$, as well as the order of the neglected terms.

(b) $\displaystyle\lim_{n\to\infty}\sqrt{n^2+3n}-n\,,$

Using the expansion $\sqrt{1+x}=1+\frac{1}{2}x+\mathcal{O}(x^2)$, where $x\equiv 3/n$ in the computation below,

$$\lim_{n\to\infty}\sqrt{n^2+3n}-n=\lim_{n\to\infty}\left(n\sqrt{1+\frac{3}{n}}-n\right)=\lim_{n\to\infty}n\left(\sqrt{1+\frac{3}{n}}-1\right)$$
$$=\lim_{n\to\infty}\left(n\left[1+\frac{3}{2n}-1+\mathcal{O}(n^{-2})\right]\right)=\lim_{n\to\infty}\left[\tfrac{3}{2}+\mathcal{O}(n^{-1})\right]=\tfrac{3}{2}\,.$$

(c) $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$ .

By L'Hospital's rule,

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{1/x}{\frac{1}{2}x^{-1/2}} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} \to 0\,.$$

The moral of the story: $x^p$ (for any $p > 0$) grows faster than $\ln x$ as $x \to \infty$.

2. Find the radius of convergence of the following three series:

$$\text{(a)} \sum_{n=1}^{\infty} \frac{x^n}{\ln(n+1)}$$

Using the ratio test,

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{\ln(n+2)} \frac{\ln(n+1)}{x^n} \right| = |x| \lim_{n\to\infty} \frac{\ln(n+2)}{\ln(n+1)} = |x| < 1\,.$$

By ratio test we know that the series converges for $-1 < x < 1$. Thus, the radius of convergence is 1.

$$\text{(b)} \sum_{n=0}^{\infty} \frac{(n!)^2 x^n}{(2n)!}$$

Using the ratio test,

$$\lim_{n\to\infty} \left| \frac{(n+1)!^2 x^{n+1}}{(2n+2)!} \frac{(2n)!}{(n!)^2 x^n} \right| = \lim_{n\to\infty} \frac{(n+1)^2 |x|}{(2n+1)(2n+2)} = \frac{|x|}{4} < 1\,.$$

Thus, the radius of convergence is 4.

$$\text{(c)} \sum_{n=0}^{\infty} \frac{n^2 (x-5)^n}{5^n (n^2+1)}$$

Using the ratio test,

$$\lim_{n\to\infty} \left| \frac{(n+1)^2 (x-5)^{n+1}}{5^{n+1}[(n+1)^2+1]} \frac{(n^2+1)5^n}{n^2 (x-5)^n} \right| = \lim_{n\to\infty} \frac{(n+1)^2 (n^2+1)}{n^2 (n^2+2n+2)} \frac{|x-5|}{5} = \frac{|x-5|}{5} < 1\,.$$

Thus, we see that $|x-5| < 5$ or $0 < x < 10$. That is, the radius of convergence is 5.

3. Determine whether the following series are absolutely convergent, conditionally convergent or divergent.

$$\text{(a)}\ \sum_{n=1}^{\infty}(-1)^n\frac{n}{n+1}\,, \qquad \text{(b)}\ \sum_{n=1}^{\infty}\frac{(-1)^n}{2^{\,\ln n}}\,, \qquad \text{(c)}\ \sum_{n=2}^{\infty}\frac{1}{n\ln n}\,.$$

For series (a), use the preliminary test.

$$\lim_{n\to\infty}\frac{n}{n+1}=1\neq 0\,.$$

Thus, series (a) diverges.

Series (b) satisfies the conditions of the alternating series test. In particular, the terms of the series are monotonically decreasing, since since $\ln(n+1)>\ln n$ implies that

$$0<\frac{1}{2^{\ln(n+1)}}<\frac{1}{2^{\ln n}}\,.$$

Moreover, the coefficients of the series approach zero as $n\to\infty$,

$$\lim_{n\to\infty}\frac{1}{2^{\ln n}}=0\,.$$

Therefore series (b) is convergent. To show that series (b) is not absolutely convergent, we examine the convergence properties of the series

$$\sum_{n=1}^{\infty}\frac{1}{2^{\,\ln n}}\,.$$

We can show that this series is divergent using the comparison test. First, use the fact that $e>2$ to conclude that $e^{\ln n}>2^{\ln n}$. Consequently,

$$\frac{1}{2^{\ln n}}>\frac{1}{e^{\ln n}}=\frac{1}{n}\quad\Longrightarrow\quad\sum_{n=1}^{\infty}\frac{1}{2^{\ln n}}>\sum_{n=1}^{\infty}\frac{1}{n}\,.$$

Since the harmonic series is known to be divergent, it follows that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2^{\ln n}}$ diverges as well. Hence, series (b) is not absolutely convergent, in which case it must be conditionally convergent.

For series (c), we employ the integral test:

$$\int^{\infty}\frac{dn}{n\ln n}=\int^{\infty}\frac{d\ln n}{\ln n}=\ln(\ln n)\bigg|^{\infty}=\infty\,.$$

Thus, series (c) is divergent.

4. Without using your calculator, compute the cube root of 1.09, with an accuracy of four decimal places.

To solve this problem, we use the binomial expansion:

$$(1+x)^p = 1 + \sum_{n-1}^{\infty} p(p-1)(p-2)\cdots(p-n+1)\,\frac{x^n}{n!}$$

$$= 1 + px + \tfrac{1}{2}p(p-1)x^2 + \tfrac{1}{6}p(p-1)(p-2)x^3 + \cdots\,.$$

We shall use this result to approximate

$$\sqrt[3]{1.09} = (1+0.09)^{1/3}\,.$$

Put $p = 1/3$ and $x = 0.09$ into the binomial expansion [eq. (1)]. Then

$$\sqrt[3]{1.09} = (1+0.09)^{1/3} = 1 + \tfrac{1}{3}(0.09) + \left(\tfrac{1}{2}\right)\left(\tfrac{1}{3}\right)\left(-\tfrac{2}{3}\right)(0.09)^2 + \cdots$$

$$= 1 + 0.03 - 0.0009 + \cdots$$

$$\approx 1.0291\,.$$

To be sure that we have four decimal place accuracy, look at the size of the first neglected term:

$$\left(\tfrac{1}{6}\right)\left(\tfrac{1}{3}\right)\left(-\tfrac{2}{3}\right)\left(-\tfrac{5}{3}\right)(0.09)^3 = 4.5\times 10^{-5}\,,$$

which only affects the fifth decimal place. Thus, to four decimal place accuracy, $\sqrt[3]{1.09} = 1.0291$.

5. What is the *behavior* of the function:

$$f(x) = -1 + \frac{1}{x^2}\left[\frac{1}{(1+x^2)^{3/2}} - \frac{1}{(1+x^2)^{5/2}}\right]\,,$$

as $x \to 0$? (Obtaining the limit as $x \to 0$ is not sufficient.)

Using the binomial expansion,

$$(1+x^2)^p = 1 + px^2 + \tfrac{1}{2}p(p-1)x^4 + \mathcal{O}(x^6)\,,$$

twice, we compute:

$$f(x) = -1 + \frac{1}{x^2}\left[\frac{1}{(1+x^2)^{3/2}} - \frac{1}{(1+x^2)^{5/2}}\right]$$

$$= -1 + \frac{1}{x^2}\left[1 - \tfrac{3}{2}x^2 + \tfrac{15}{8}x^4 - 1 + \tfrac{5}{2}x^2 - \tfrac{35}{8}x^4 + \mathcal{O}(x^6)\right]$$

$$= -1 + 1 - \tfrac{5}{2}x^2 + \mathcal{O}(x^4)$$

$$= -\tfrac{5}{2}x^2 + \mathcal{O}(x^4)$$

6. Evaluate $f(x) = \ln\sqrt{(1+x)/(1-x)} - \tan x$ at $x = 0.0015$ without a calculator. Determine the numerical accuracy of your result. Is your calculator a useful tool for this problem? (Try it!)

The relevant Maclaurin series are given in the class handout on Taylor series.

$$\begin{aligned}
\ln(1+x) &= x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 - \tfrac{1}{6}x^6 + \tfrac{1}{7}x^7 - \tfrac{1}{8}x^8 + \mathcal{O}(x^9)\,, \\
\ln(1-x) &= -x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 - \tfrac{1}{5}x^5 - \tfrac{1}{6}x^6 - \tfrac{1}{7}x^7 - \tfrac{1}{8}x^8 + \mathcal{O}(x^9)\,, \\
\tan x &= x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + \tfrac{17}{315}x^7 + \mathcal{O}(x^9)\,.
\end{aligned}$$

Hence, we can expand $f(x)$ about $x = 0$ to obtain:

$$\begin{aligned}
f(x) &= \tfrac{1}{2}[\ln(1+x) - \ln(1-x)] - \tan x \\
&= \tfrac{1}{2}\biggl[\bigl(x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 - \tfrac{1}{6}x^6 + \tfrac{1}{7}x^7 - \tfrac{1}{8}x^8\bigr) \\
&\qquad - \bigl(-x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 - \tfrac{1}{5}x^5 - \tfrac{1}{6}x^6 - \tfrac{1}{7}x^7 - \tfrac{1}{8}x^8\bigr)\biggr] \\
&\qquad - \bigl(x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + \tfrac{17}{315}x^7\bigr) + \mathcal{O}(x^9) \\
&= \tfrac{1}{15}x^5 + \tfrac{4}{45}x^7 + \mathcal{O}(x^9)\,.
\end{aligned}$$

Using the first term of the approximation just obtained, $f(x) \simeq \frac{1}{15}x^5$, and plugging in the numbers, we obtain:

$$f(0.0015) \approx \frac{(0.0015)^5}{15} = (15^5 \times 10^{-20})/15 = 3^4 \times 5^4 \times 10^{-20} = 81 \times 625 \times 10^{-20} = 5.0625 \times 10^{-16}\,.$$

To determine the accuracy of this approximation, we compute the second term in the series approximation for $f(x)$,

$$4(0.0015)^7/45 = 1.51875 \times 10^{-21} = 0.000002 \times 10^{-16}\,,$$

which is negligible compared to the leading term of the expansion.

7. For each expression find all possible values and express your result both in the form $x + iy$ and in polar form $re^{i\theta}$, where $\theta$ is the principal value of the argument.

(a) $i^{77} + i^{202}$

Let us first determine the $x + iy$ form of the number. The key observation is that $i^4 = 1$. Thus if we can divide 77 and 202 by four, only the remainder matters. For this problem, $77 = 4(19) + 1$ and $202 = 4(50) + 1$, in which case,

$$i^{77} + i^{202} = i^{4(19)+1} + i^{4(50)+2} = i^1 + i^2 = -1 + i = \sqrt{2}\, e^{3i\pi/4}\,.$$

The last step was accomplished by writing $-1+i = re^{i\theta}$ and noting that the complex magnitude of $-1+i$ is:

$$r = |-1+i| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}\,.$$

The argument $\theta$ is determined from $\tan\theta = y/x = -1$. Since $-1+i$ corresponds to $(x,y) = (-1,1)$, which lies in the second quadrant of the complex plane, it follows that $\theta = 135^\circ = 3\pi/4$.

(b) $\dfrac{3+i}{2+i}$

To determine the $x+iy$ form, we multiply the numerator and denominator by the conjugate of the denominator.

$$\frac{3+i}{2+i} = \left(\frac{3+i}{2+i}\right)\left(\frac{2-i}{2-i}\right) = \frac{(6+1)-i}{4+1} = \tfrac{7}{5} - \tfrac{1}{5}i\,.$$

Writing $\frac{7}{5} - \frac{1}{5}i = re^{i\theta}$, we obtain:

$$r = \sqrt{\frac{7^2}{5^2} + \frac{1}{5^2}} = \sqrt{\frac{49+1}{25}} = \sqrt{2} \quad \text{and} \quad \tan\theta = -\frac{1}{7}\,.$$

Since $\frac{7}{5} - \frac{1}{5}i$ lies in the fourth quadrant of the complex plane, we must have $-\pi/2 < \theta < 0$. A numerical computation yields $\theta = -0.1419$, so that

$$\frac{3+i}{2+i} = \sqrt{2}e^{-0.1419i}\,.$$

(c) $\sqrt{-2+2i\sqrt{3}}$

First, we determine the polar form for $-2+2i\sqrt{3} \equiv r_0 e^{i\theta_0}$.

$$r_0 = \sqrt{(-2)^2 + (2\sqrt{3})^2} = \sqrt{4+12} = 4 \quad \text{and} \quad \tan\theta_0 = -\sqrt{3}\,.$$

Since $(-2, 2\sqrt{3})$ resides in the second quadrant of the complex plane, if follows that $\theta_0 = 2\pi/3$. Hence,

$$-2+2i\sqrt{3} = 4e^{2\pi i/3 + 2\pi i k}\,, \qquad k = 0, \pm 1, \pm 2, \ldots\,.$$

Taking the square root then yields:

$$\sqrt{-2+2i\sqrt{3}} = \sqrt{4e^{2\pi i/3+2\pi i k}} = 2e^{i\pi/3 + \pi i k} = \pm 2e^{i\pi/3}\,.$$

We can now determine the $x+iy$ form of the square root by writing:

$$\sqrt{-2+2i\sqrt{3}} = \pm 2e^{i\pi/3} = \pm 2\left(\cos\tfrac{1}{3}\pi + i\sin\tfrac{1}{3}\pi\right) = \pm(1+\sqrt{3}\,i)\,,$$

where we have used $\cos(\pi/3) = \frac{1}{2}$ and $\sin(\pi/3) = \sqrt{3}/2$.

(d) $\left(\dfrac{1+i}{1-i}\right)^4$

The simplest way to evaluate this expression is to separately express the numerator and denominator in polar form. Using

$$1+i=\sqrt{2}e^{i\pi/4}\,, \qquad\qquad 1-i=\sqrt{2}e^{-i\pi/4}\,,$$

it follows that:

$$\left(\frac{1+i}{1-i}\right)^4=\left(\frac{\sqrt{2}e^{i\pi/4}}{\sqrt{2}e^{-i\pi/4}}\right)^4=\left(e^{i\pi/2}\right)^4=e^{2\pi i}=1\,.$$

(e) $\sqrt[4]{16}$

Let us first write 16 in polar form, $16=16e^{2k\pi i}$, where $k=0\,,\pm1\,,\pm2\,,\ldots$. Taking the fourth root then yields:

$$\sqrt[4]{16}=(16)^{1/4}=\left(16e^{2k\pi i}\right)^{1/4}=2e^{k\pi i/2}$$

Only four distinct roots exist, which correspond to $k=-1,0,1,2$:

$$\sqrt[4]{16}=2e^{-\pi i/2},2e^{0i/2},2e^{\pi i/2},2e^{\pi i}\,.$$

Those are the polar forms for the four roots. The $x+iy$ forms are easily obtained using Euler's formula. The end result is: $\sqrt[4]{16}=-2i,2,2i,-2$.

8. Let $z=1-i$. Express each of the following in the form of $x+iy$. For any multi-valued function, you should indicate all possible values of the result.

(a) $\cos(1/z)$

We compute $1/z$ by multiplying the numerator and denominator by $1+i$,

$$\frac{1}{1-i}=\left(\frac{1}{1-i}\right)\left(\frac{1+i}{1+i}\right)=\frac{1+i}{1+1}=\tfrac{1}{2}+\tfrac{1}{2}i$$

Using the addition formula for the cosines, $\cos(z_1+z_2)=\cos z_1\cos z_2-\sin z_1\sin z_2$,

$$\cos(1/z)=\cos\left(\frac{1}{1-i}\right)=\cos(\tfrac{1}{2}+\tfrac{1}{2}i)=\cos(\tfrac{1}{2})\cos(\tfrac{1}{2}i)-\sin(\tfrac{1}{2})\sin(\tfrac{1}{2}i)$$

$$=\cos(\tfrac{1}{2})\cosh(\tfrac{1}{2})-i\sin(\tfrac{1}{2})\sinh(\tfrac{1}{2})\,,$$

after using $\cos(iz)=\cosh z$ and $\sin(iz)=i\sinh z$. Plugging in the following numbers: $\cos(\frac{1}{2})=0.877583$, $\sin(\frac{1}{2})=0.479426$, $\cosh(\frac{1}{2})=1.12763$, and $\sinh(\frac{1}{2})=0.521095$, we arrive at:

$$\cos(1/z)=0.989585-0.249826i$$

(b) $z^z$

Using $z \equiv 1 - i = \sqrt{2}e^{-i\pi/4}$ and the definition of the power function,

$$z^z = \left(\sqrt{2}e^{-i\pi/4}\right)^{1-i} = \exp\left\{(1-i)\ln(\sqrt{2}e^{-i\pi/4})\right\}$$

$$= \exp\left\{(1-i)[\tfrac{1}{2}\ln 2 - i\pi/4 + 2\pi ik]\right\}$$

$$= \exp\left\{\tfrac{1}{2}\ln 2 - \pi/4 + 2\pi k\right\}e^{-i(\frac{1}{2}\ln 2 + \pi/4)}$$

$$= \sqrt{2}e^{-\pi/4+2\pi k}\left[\cos\left(\tfrac{1}{2}\ln 2 + \tfrac{1}{4}\pi\right) - i\sin\left(\tfrac{1}{2}\ln 2 + \tfrac{1}{4}\pi\right)\right], \qquad k = 0, \pm1, \pm2, \dots .$$

There are infinitely many solutions, which is characteristic of a power of a complex number with nonzero imaginary component.

(c) $\tan(z-1)$

Using the definition of $\tan\theta$,

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{1}{i}\left(\frac{e^{i\theta} - e^{-i\theta}}{e^{i\theta} + e^{-i\theta}}\right),$$

we have for $z = 1 - i$,

$$\tan(z-1) = \tan(-i) = \frac{1}{i}\left(\frac{e^{1} - e^{-1}}{e^{1} + e^{-1}}\right) = -i\tanh(1) = -0.761594i\,.$$

(d) Ln $z$

First express $z$ in polar form so $z = \sqrt{2}e^{-i\pi/4}$. The principal value of the argument, which lies in the interval $-\pi < \operatorname{Arg} z \leq \pi$, is $\operatorname{Arg}(1-i) = -\pi/4$. Thus we have:

$$\operatorname{Ln} z = \operatorname{Ln}\left(\sqrt{2}e^{-i\pi/4}\right) = \tfrac{1}{2}\operatorname{Ln} 2 - \tfrac{1}{4}i\pi = 0.346574 - 0.785398i$$

(e) $\arg z$

In part (d), we noted that $\operatorname{Arg}(1-i) = -\pi/4$. Hence,

$$\arg z = \operatorname{Arg} z + 2\pi k = -\tfrac{1}{4}\pi + 2\pi k\,, \qquad k = 0, \pm1, \pm2, \dots .$$

9. Solve for all possible values of the real numbers $x$ and $y$ in the following equations:

(a) $x + iy = y + ix$.

Equating the real and imaginary parts, one obtains $x = y$ as the only solution.

(b) $\dfrac{x + iy}{x - iy} = -i$.

Multiplying both sides by $x - iy$ yields

$$x + iy = -y - ix\,.$$

Equating the real and imaginary parts, one obtains $x = -y$ as the only solution.

10. Find the disk of convergence of the following complex power series:

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}\, z^n$

Using the complex version of the ratio test, the computation is identical to that of problem 2(b). The series therefore converges inside the disk $|z| < 4$ in the complex plane. That is, the radius of convergence is 4.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{z^{2n}}{(2n+1)!}\,.$

Again we use the ratio test:

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{z^{2n+2}}{(2n+3)!} \frac{(2n+1)!}{z^{2n}} \right| &= |z^2| \lim_{n\to\infty} \frac{(2n+1)!}{(2n+3)(2n+2)((2n+1)!} \\
&= |z^2| \lim_{n\to\infty} \frac{1}{(2n+3)(2n+2)} = 0\,,
\end{aligned}$$

for all finite values of $z$. Therefore, the sum converges for all finite $z$ in the complex plane. The radius of convergence is infinite.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(z-i)^n}{n!}\,.$

Again we use the ratio test:

$$\lim_{n\to\infty} \left| \frac{(z-i)^{n+1}}{n+1} \frac{n}{(z-i)^n} \right| = |z-i| \lim_{n\to\infty} \frac{n}{n+1} = |z-i|\,.$$

Therefore the sum converges for $|z - i| < 1$, and the radius of convergence is 1.

11. Evaluate the integral

$$\int_0^\pi \sin 3x \cos 4x \, dx \, .$$

We shall evaluate this integral by replacing the sine and cosine by their exponential forms,

$$\sin 3x = \frac{e^{3ix} - e^{-3ix}}{2i} \, , \quad \text{and} \quad \cos 4x = \frac{e^{4ix} - e^{-4ix}}{2} \, .$$

Expanding the integral out then evaluating each term we find:

$$\begin{aligned}
\int_0^\pi \sin 3x \cos 4x \, dx &= \int_0^\pi \left(\frac{e^{3ix} - e^{-3ix}}{2i}\right)\left(\frac{e^{4ix} + e^{-4ix}}{2}\right) dx \\
&= \frac{1}{4i}\int_0^\pi \left(e^{7ix} + e^{-ix} - e^{ix} - e^{-7ix}\right) dx \\
&= \frac{1}{4i}\left[\frac{1}{7i}e^{7ix} - \frac{1}{i}e^{-ix} - \frac{1}{i}e^{ix} + \frac{1}{7i}e^{-7ix}\right]\Bigg|_0^\pi = -\frac{6}{7} \, .
\end{aligned}$$

12. Evaluate the following quantities:

(a) $(1)^\pi$

(b) $\arg(e^{x+iy})$, where $x$ and $y$ are real numbers

We evaluate the two complex numbers above as follows:

(a) Using the definition of the power function,

$$1^\pi = e^{\pi \ln 1} = e^{\pi(2\pi i n)} \, , \qquad \text{for} \;\; n = 0 \pm 1, \pm 2, \pm 3, \ldots \, ,$$

where we have used the fact that the multi-valued complex logarithm is defined by

$$\ln z = \text{Ln}|z| + i(\text{Arg}\, z + 2\pi n) \, , \qquad \text{for} \;\; n = 0 \pm 1, \pm 2, \pm 3, \ldots \, , \tag{1}$$

so that $\ln 1 = 2\pi i n$. We now use Euler's formula to express $1^\pi$ in $x + iy$ form,

$$1^\pi = e^{2\pi^2 i n} = \cos(2\pi^2 n) + i \sin(2\pi^2 n), \qquad \text{for} \;\; n = 0 \pm 1, \pm 2, \pm 3, \ldots \, .$$

(b) If we write $e^{x+iy}$ in polar form, we can write $e^{x+iy} = e^x e^{iy} = re^{i\theta}$, where $r = e^x$ and $\theta = y$. Thus $\arg(e^{x+iy}) = y + 2\pi n$, where $n$ is any integer.

13. Find all complex number solutions $z$ to the equation, $e^z = 1 - i$.

The solution of $e^z = 1-i$ is $z = \ln(1-i)$. To evaluate this, we use the polar form to write $1 - i = \sqrt{2}\, e^{-i\pi/4}$. Hence,

$$z = \ln(1-i) = \tfrac{1}{2}\text{Ln } 2 - \frac{i\pi}{4} + 2n\pi i\,,$$

where $n$ is an arbitrary integer.

14. Consider the real-valued function:

$$f(x) = \tfrac{1}{2}\ln\left(\frac{1+x}{1-x}\right)\,.$$

(a) Derive the Taylor series expansion of $f(x)$ about the point $x = 0$. Write the series using summation notation (that is, you will need to determine the general term in the series).

As a first step, we shall write:

$$\tfrac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \tfrac{1}{2}\left[\ln(1+x) - \ln(1-x)\right]\,. \tag{2}$$

We can then employ eq. (13.4) on p. 26 of Boas,

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n} = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 - \cdots\,, \qquad \text{for } -1 < x \leq 1\,. \tag{3}$$

By replacing $x$ with $-x$ in the above expansion, we obtain:

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} = -x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 - \tfrac{1}{5}x^5 - \cdots\,, \qquad \text{for } \; -1 \leq x < 1\,, \tag{4}$$

where we have used $(-1)^{n+1}(-x) = (-1)^{n+1}(-1)^n x^n = -x^n$. Inserting these two series expansions into eq. (2), we see that all even power terms cancel, which leaves us with:

$$\boxed{\tfrac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} = x + \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 + \cdots\,, \qquad \text{for } \; -1 < x < 1} \tag{5}$$

Note that this result is in fact given explicitly in Example 1 on p. 36 of Boas.

(b) Determine all possible values of $x$ for which the series obtained in part (a) converges.

If we require that $x$ satisfy the conditions specified in eqs. (3) and (4), we would conclude that $-1 < x < 1$, as indicated in eq. (5). We can check this using the ratio test, by evaluating

$$\rho \equiv \lim_{n\to\infty} \left| \frac{x^{2n+3}}{2n+3} \cdot \frac{2n+1}{x^{2n+1}} \right| = |x|^2 \left( \lim_{n\to\infty} \frac{2n+1}{2n+3} \right) = |x|^2 .$$

Thus, by the ratio test, the series given in eq. (5) converges for $\rho < 1$, which implies that $|x| < 1$, and we conclude that the radius of convergence is equal to 1. The points $x = \pm 1$ must be checked independently. But, since $\ln[(1+x)/(1-x)]$ diverges when $x = 1$ and when $x = -1$, we confirm that all possible values of $x$ for which the series given in eq. (5) converges lie within the open interval $-1 < x < 1$.

(c) Evaluate explicitly the sum

$$\sum_{n=0}^{\infty} \frac{1}{2^{2n}} \frac{1}{2n+1} .$$

Use your calculator to compute the sum of the first four terms of the series, and compare this numerical approximation with the exact result.

Plugging $x = \frac{1}{2}$ into eq. (5), we immediately obtain

$$\tfrac{1}{2} \ln \left( \frac{1+\frac{1}{2}}{1-\frac{1}{2}} \right) = \sum_{n=0}^{\infty} \frac{1}{2^{2n+1}} \frac{1}{2n+1} .$$

Multiplying both sides of the equation by 2, and evaluating the argument of the logarithm then yields

$$\boxed{\ln 3 = \sum_{n=0}^{\infty} \frac{1}{2^{2n}} \frac{1}{2n+1}}$$

Using my calculator, I then compute:

$$1 + \tfrac{1}{4} \cdot \tfrac{1}{3} + \tfrac{1}{16} \cdot \tfrac{1}{5} + \tfrac{1}{64} \cdot \tfrac{1}{7} = 1.098065 ,$$

which should be compared to $\ln 3 = 1.098612$. Thus, we have accuracy to four significant figures (not bad for a four-term approximation!). This series obviously converges much faster than the series for $\ln 2$ obtained by setting $x = 1$ in eq. (3).

15. Assume that $p$ is a real parameter such that $-1 < p < 1$.

(a) Compute the following sum:

$$\sum_{n=0}^{\infty} p^n e^{in\theta} \,. \tag{6}$$

The sum of an infinite geometric series is given by:

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}\,, \qquad\qquad \text{for } \ |z| < 1\,, \tag{7}$$

which is valid for any complex number $z$ that lies within the unit circle centered at the origin of the complex plane.

By setting $z = pe^{i\theta}$, we recognize eq. (6) as a geometric series. Hence,

$$\boxed{\sum_{n=0}^{\infty} p^n e^{in\theta} = \sum_{n=0}^{\infty} (pe^{i\theta})^n = \frac{1}{1-pe^{i\theta}}\,, \qquad \text{for } \quad -1 < p < 1} \tag{8}$$

Note that since $p$ is a real parameter, the condition $|pe^{i\theta}| < 1$ is equivalent to $|p| < 1$, or equivalently $-1 < p < 1$.

(b) Using the results of part (a), compute the sum

$$\sum_{n=0}^{\infty} p^n \cos(n\theta)\,.$$

Verify that your result for the sum in part (b) has the correct form in the $\theta \to 0$ limit.

Using the fact that $\mathrm{Re}(e^{in\theta}) = \cos n\theta$, we simply take the real part of eq. (8),

$$\sum_{n=0}^{\infty} p^n \cos(n\theta) = \mathrm{Re}\left(\sum_{n=0}^{\infty} p^n e^{in\theta}\right) = \mathrm{Re}\left(\frac{1}{1-pe^{i\theta}}\right)\,.$$

To evaluate the last expression, we write:

$$\frac{1}{1-pe^{i\theta}} = \left(\frac{1}{1-pe^{i\theta}}\right)\left(\frac{1-pe^{-i\theta}}{1-pe^{-i\theta}}\right) = \frac{1-pe^{-i\theta}}{1-p[e^{i\theta}+e^{-i\theta}]+p^2} = \frac{1-p\cos\theta+ip\sin\theta}{1-2p\cos\theta+p^2}\,, \tag{9}$$

where we have used

$$\cos\theta = \tfrac{1}{2}(e^{i\theta}+e^{-i\theta})$$

in the last step. Thus, taking the real part of eq. (9), we end up with

$$\boxed{\sum_{n=0}^{\infty} p^n \cos(n\theta) = \frac{1 - p\cos\theta}{1 - 2p\cos\theta + p^2}\,, \qquad \text{for} \quad -1 < p < 1}$$

Finally, we check the $\theta \to 0$ limit. Inserting $\theta = 0$ into the sum formula above, and using $\cos 0 = 1$,

$$\sum_{n=0}^{\infty} p^n = \frac{1-p}{1-2p+p^2} = \frac{1-p}{(1-p)^2} = \frac{1}{1-p}\,, \qquad \text{for} \quad -1 < p < 1\,,$$

which is the correct formula for the sum of an infinite geometric series [cf. eq. (7)].

As a bonus, one can also obtain the corresponding sum formula with the cosine function replaced by the sine function. Since $\mathrm{Im}(e^{in\theta}) = \sin n\theta$, we can take the imaginary part of eq. (9) and use it to obtain:

$$\sum_{n=0}^{\infty} p^n \sin(n\theta) = \frac{p\sin\theta}{1 - 2p\cos\theta + p^2}\,, \qquad \text{for} \quad -1 < p < 1\,.$$

16. Consider the following matrices:

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & -1 & 0 \\ 0 & 5 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 3 & -1 & 0 \end{pmatrix},$$

Compute $AB$, $BA$, $\det A$, $\det B$, $\det AB$ and $\det BA$. Verify that $AB \neq BA$ and $\det AB = (\det A)(\det B)$.

First we compute the two matrix products $AB$ and $BA$,

$$AB = \begin{pmatrix} 1 & 0 & 2 \\ 3 & -1 & 0 \\ 0 & 5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 3 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 7 & -1 & 0 \\ 3 & 1 & -1 \\ 3 & 9 & 5 \end{pmatrix},$$

$$BA = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 3 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 3 & -1 & 0 \\ 0 & 5 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -1 & 2 \\ 6 & 3 & 1 \\ 0 & 1 & 6 \end{pmatrix}.$$

Next, we evaluate the determinants,

$$\det A = \begin{vmatrix} 1 & 0 & 2 \\ 3 & -1 & 0 \\ 0 & 5 & 1 \end{vmatrix} = -1 + 2(15) = 29\,,$$

$$\det B = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 3 & -1 & 0 \end{vmatrix} = 1 + 3 = 4\,,$$

$$\det AB = \begin{vmatrix} 7 & -1 & 0 \\ 3 & 1 & -1 \\ 3 & 9 & 5 \end{vmatrix} = 7(5+9) + 1(15+3) = 98 + 18 = 116\,,$$

$$\det BA = \begin{vmatrix} 4 & -1 & 2 \\ 6 & 3 & 1 \\ 0 & 1 & 6 \end{vmatrix} = 4(18-1) - 6(-6-2) = 68 + 48 = 116\,.$$

Note that even though $AB \neq BA$, one finds:

$$\det(AB) = \det(BA) = (\det\ A)(\det\ B)\,,$$

i.e. $116 = (29)(4)$, as expected.

17. Let $A$ be a $3 \times 3$ matrix. The determinant of $A$ is denoted by $\det\ A$.

(a) Is the equation $\det(3A) = 3\det\ A$ true or false? Explain.

The determinant is multiplied by $k$ if you multiply *one* of the rows by $k$. Here, $3A$ is a matrix obtained from $A$ by multiplying each of the three rows of $A$ by a factor of 3. Hence, $\det(3A) = 27\det\ A$. In general, for an $n \times n$ matrix, $\det(kA) = k^n \det\ A$.

(b) Suppose that $\det\ A = 1$. Let $B$ be a matrix obtained from $A$ by permuting the order of the rows so that the first row of $A$ is the second row of $B$, the second row of $A$ is the third row of $B$ and the third row of $A$ is the first row of $B$. (This is called a *cyclic permutation*.) What is the value of $\det\ A$?

Each time you interchange a pair of rows, the determinant changes by an overall sign. In this case, one can obtain $B$ from $A$ by two pairwise interchanges. Thus, $\det\ B = 1$.

(c) Suppose that the $3 \times 3$ matrix $A \neq 0$ but $\det\ A = 0$. What can you say about the rank of $A$?

If $\det\ A = 0$, then the rank of $A$ must be less than three. Since $A \neq 0$, the rank must be greater than zero. Thus, either the rank of $A$ is one or two. No further deduction can be drawn without additional information.

18. Consider the system of equations:

$$
\begin{aligned}
5x + 2y + \phantom{2}z &= 2 \\
x + \phantom{2}y + 2z &= 1 \\
3x - 3z &= 0\,.
\end{aligned}
$$

(a) What is the augmented matrix for this system of equations?

The augmented matrix consists of adding a fourth column to the coefficient matrix. The fourth column consists of the numbers appearing on the right hand side of the above set of equations,

$$
\left(\begin{array}{rrr|r} 5 & 2 & 1 & 2 \\ 1 & 1 & 2 & 1 \\ 3 & 0 & -3 & 0 \end{array}\right).
$$

(b) Solve this system of equations using Gaussian elimination. That is, reduce the augmented matrix to reduced row echelon form with a series of elementary row operations.

The augmented matrix can be reduced to reduced row echelon form with the follow series of elementary row operations.

$$
\begin{aligned}
\left(\begin{array}{rrr|r} 5 & 2 & 1 & 2 \\ 1 & 1 & 2 & 1 \\ 3 & 0 & -3 & 0 \end{array}\right)
&\xrightarrow[R_1 \leftrightarrow R_3]{} \left(\begin{array}{rrr|r} 3 & 0 & -3 & 0 \\ 1 & 1 & 2 & 1 \\ 5 & 2 & 1 & 2 \end{array}\right) \\
&\xrightarrow[R_1 \to R_1/3]{} \left(\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 1 & 1 & 2 & 1 \\ 5 & 2 & 1 & 2 \end{array}\right) \\
&\xrightarrow[R_2 \to R_2 - R_1]{} \left(\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & 3 & 1 \\ 5 & 2 & 1 & 2 \end{array}\right) \\
&\xrightarrow[R_3 \to R_3 - 5R_1]{} \left(\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 2 & 6 & 2 \end{array}\right) \\
&\xrightarrow[R_3 \to R_3 - 2R_2]{} \left(\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right).
\end{aligned}
$$

Thus, our original set of equations is equivalent to:

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} ,$$

or equivalently,

$$x - z = 0 \,, \qquad \text{and} \qquad y + 3z = 1 \,. \tag{10}$$

Setting $z = t$, where $t$ is any number, then the infinite set of solutions to the original system of equations consists of the set

$$(x\,,\,y\,,\,z) = (t\,,\,1 - 3t\,,\,t)\,. \tag{11}$$

(c) What is the rank of the augmented matrix of part (b)?

The rank is equal to the number of non-zero rows in the reduced row echelon form. Hence, the rank is equal to two.

(d) Remove the third equation above, and solve the new system of two equations and three unknowns using Gaussian elimination. What is the rank of the corresponding augmented matrix?

The augmented matrix for the revised problem is:

$$\left(\begin{array}{ccc|c} 5 & 2 & 1 & 2 \\ 1 & 1 & 2 & 1 \end{array}\right) .$$

Again, we reduce the augmented matrix to reduced row echelon form by a series of elementary row operations:

$$\begin{aligned} \left(\begin{array}{ccc|c} 5 & 2 & 1 & 2 \\ 1 & 1 & 2 & 1 \end{array}\right) &\xrightarrow[R_1 \leftrightarrow R_2]{} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 5 & 2 & 1 & 2 \end{array}\right) \\ &\xrightarrow[R_2 \to 5R_1 - R_2]{} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 3 & 9 & 3 \end{array}\right) \\ &\xrightarrow[R_2 \to R_2/3]{} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & 3 & 1 \end{array}\right) \\ &\xrightarrow[R_1 \to R_1 - R_2]{} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 3 & 1 \end{array}\right) . \end{aligned}$$

Thus, the solutions obtained in eqs. (10) and (11) still hold. The rank of the augmented matrix is two as before.

19. Evaluate the following determinant by hand:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix}.$$

The simplest method is to apply a set of elementary row operations of the form $R_j \to R_j + kR_i$ (where $k$ is some non-zero constant), which do not change the value of the determinant, such that the final resulting matrix is in upper triangular form.

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix} \xrightarrow[R_2 \to R_2 - R_1]{} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix} \xrightarrow[R_3 \to R_3 - R_1]{} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 5 & 9 \\ 1 & 4 & 10 & 20 \end{vmatrix} \xrightarrow[R_4 \to R_4 - R_1]{} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 5 & 9 \\ 0 & 3 & 9 & 19 \end{vmatrix}.$$

The first column below the main diagonal is now filled with zeros. We proceed similarly until all elements below the main diagonal consist of zeros:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 5 & 9 \\ 0 & 3 & 9 & 19 \end{vmatrix} \xrightarrow[R_4 \to R_4 - R_1]{} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 5 & 9 \\ 0 & 3 & 9 & 19 \end{vmatrix} \xrightarrow[R_3 \to R_3 - 2R_2]{} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 3 & 9 & 19 \end{vmatrix}$$

$$\xrightarrow[R_4 \to R_4 - 3R_2]{} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 10 \end{vmatrix} \xrightarrow[R_4 \to R_4 - 3R_3]{} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1\,,$$

where in the last step, we used the fact that the determinant of an upper triangular matrix is equal to the product of the diagonal elements. Hence, we conclude that

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix} = 1\,.$$

20. Find $z$ by Cramer's rule (*NOTE*: you are not being asked to find $x$ and $y$),

$$\begin{cases} (a-b)x - (a-b)y + 3b^2 z = 3ab\,, \\ (a+2b)x - (a-2b)y - (3ab^2+3b^2)z = 3b^2\,, \\ bx + ay - (2b^2+a^2)z = 0\,, \end{cases} \tag{12}$$

where $a$ and $b$ are arbitrary real numbers.

Given a set of equations,

$$C\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \tag{13}$$

where $C$ is an $n \times n$ coefficient matrix, Cramer's rule states that the solution to this set of equations is given by

$$x_i = \frac{\det\, C^{(i)}}{\det\, C}\,, \qquad\qquad n = 1, 2, 3\ldots, n\,,$$

where $C^{(i)}$ is obtained by replacing the $i$th column of $C$ with the right hand side of eq. (13).

Applying Cramer's rule to solve the set of equations given by eq. (12), one obtains the following expression for $z$:

$$z = \frac{\begin{vmatrix} a-b & -(a-b) & 3ab \\ a+2b & -(a+2b) & 3b^2 \\ b & a & 0 \end{vmatrix}}{\begin{vmatrix} a-b & -(a-b) & 3b^2 \\ a+2b & -(a+2b) & -(3ab+3b^2) \\ b & a & -(2b^2+a^2) \end{vmatrix}}\,.$$

Performing the elementary column operation $C_1 \to C_1 + C_2$ on both determinants above (which does not modify the value of either determinant), one obtains:

$$z = \frac{\begin{vmatrix} 0 & -(a-b) & 3ab \\ 0 & -(a+2b) & 3b^2 \\ a+b & a & 0 \end{vmatrix}}{\begin{vmatrix} 0 & -(a-b) & 3b^2 \\ 0 & -(a+2b) & -(3ab+3b^2) \\ a+b & a & -(2b^2+a^2) \end{vmatrix}} = \frac{(a+b)\begin{vmatrix} -(a-b) & 3ab \\ -(a+2b) & 3b^2 \end{vmatrix}}{(a+b)\begin{vmatrix} -(a-b) & 3b^2 \\ -(a+2b) & -(3ab+3b^2) \end{vmatrix}}$$

$$= \frac{-3b^2(a-b)+3ab(a+2b)}{3b(b+a)(a-b)+3b^2(a+2b)} = \frac{a(a+2b)-b(a-b)}{a^2-b^2+b(a+2b)} = \frac{a^2+ab+b^2}{a^2+ab+b^2} = 1\,,$$

where we have expanded both determinants using the cofactor expansion. Thus, we conclude that $z = 1$.

21. A complex number $x + iy$ can be represented by the $2 \times 2$ matrix

$$Z = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}, \tag{14}$$

where $x$ and $y$ are real numbers. Verify that this is a sensible representation by answering the following questions.

(a) Show that the matrix representation of $(x + iy)(a + ib)$ is equal to

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix} \begin{pmatrix} a & -b \\ b & a \end{pmatrix}. \tag{15}$$

To show this, you should express the product $(x + iy)(a + ib)$ in the form of $X + iY$ and show that the matrix product above, when evaluated, is consistent with the form given by eq. (14).

First we note that:

$$(x + iy)(a + ib) = (xa - yb) + i(xb + ya).$$

The matrix product in eq. (15) is given by

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix} \begin{pmatrix} a & -b \\ b & b \end{pmatrix} = \begin{pmatrix} xa - yb & -(xb + ya) \\ xb + ya & xa - yb \end{pmatrix}.$$

Indeed, this is the matrix representation of $(xa - yb) + i(xb + ya)$.

(b) Show that the matrix representation of the complex number $(x + iy)^{-1}$ is correctly given by the inverse of eq. (14). Here, the inverse of $Z$ (denoted by $Z^{-1}$) satisfies the matrix equation, $ZZ^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

First, we write:

$$\frac{1}{x + iy} = \frac{1}{x + iy} \cdot \frac{x - iy}{x - iy} = \frac{x - iy}{x^2 + y^2}. \tag{16}$$

Recall that the inverse of a $2 \times 2$ matrix,

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

is given by:

$$M^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \qquad ad - bc \neq 0. \tag{17}$$

Hence, the inverse of the matrix exhibited in eq. (14) is given by:

$$Z^{-1} = \frac{1}{x^2+y^2}\begin{pmatrix} x & y \\ -y & x \end{pmatrix} = \begin{pmatrix} \dfrac{x}{x^2+y^2} & \dfrac{y}{x^2+y^2} \\ -\dfrac{y}{x^2+y^2} & \dfrac{x}{x^2+y^2} \end{pmatrix},$$

which is the matrix representation of

$$(x+iy)^{-1} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2},$$

in light of eq. (16).

(c) How is the determinant of the matrix given in eq. (14) related to the corresponding complex number, $x+iy$?

The determinant,

$$\begin{vmatrix} x & -y \\ y & x \end{vmatrix} = x^2+y^2,$$

is equal to the modulus of the corresponding complex number,

$$|x+iy|^2 = (x+iy)(x-iy) = x^2+y^2.$$

*REMARK:* The representation of complex numbers by $2\times 2$ matrices is especially useful in a number of mathematical applications. One can extend these results and show that any $n\times n$ complex matrix, $M$, can be represented by a $2n\times 2n$ real matrix, $M_R$, where every complex element is replaced by its corresponding $2\times 2$ matrix representative. One can then prove that $\det M_R = |\det M|^2$.