The Generating Function of the Legendre Polynomials

The Legendre polynomials can be defined via the generating function,

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n\,, \qquad\qquad \text{for } |t|<1\,, \tag{1}$$

where the positive square root is employed in eq. (1). One can verify eq. (1) by explicitly evaluating the coefficients of the power series expansion of $(1-2xt+t^2)^{-1/2}$.

1. Proof that eq. (1) yields the polynomial solutions to the Legendre differential equation

First, we recall the binomial theorem,

$$(1+z)^p = \sum_{n=0}^{\infty} \binom{p}{n} z^n\,, \tag{2}$$

where

$$\binom{p}{0} \equiv 1\,, \qquad \binom{p}{n} = \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!}\,, \quad \text{for } n=1,2,3,\ldots.$$

It follows that

$$\begin{aligned}\binom{-\frac{1}{2}}{n} &= \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\cdots(-\frac{1}{2}-n+1)}{n!} = \frac{(-1)^n\, 1\cdot 3\cdot 5\cdots(2n-1)}{2^n\, n!} \\ &= \frac{(-1)^n (2n)!}{2^n\, n!\, 2\cdot 4\cdot 6\cdots(2n)} = \frac{(-1)^n(2n)!}{2^{2n}\,[n!]^2}\,. \end{aligned} \tag{3}$$

Applying eqs. (2) and (3) to eq. (1) with $z \equiv t^2 - 2xt$, it follows that

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} \frac{(-1)^n(2n)!}{2^{2n}\,[n!]^2}\, t^n (t-2x)^n\,. \tag{4}$$

We again employ the binomial theorem,

$$(t-2x)^n = \sum_{k=0}^{n} \binom{n}{k} t^k (-2x)^{n-k}\,,$$

where for integers n and k, we can write

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}\,.$$

Inserting these results into eq. (4) yields

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^k(2n)!}{2^{2n}\,n!\,k!\,(n-k)!}\,t^{n+k}(2x)^{n-k}\,. \tag{5}$$

This is a double series. Our aim is to rearrange the double sum in order to isolate the coefficient of t^n, which we shall identify as the Legendre polynomial $P_n(x)$ [cf. eq. (1)]. This can be accomplished with the help of the following general formula, which we shall prove in the Appendix,

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} C_{k,n}t^{n+k} = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]} C_{k\,,\,n-k}t^n\,, \tag{6}$$

where $[n/2]$ is the greatest integer less than or equal to $n/2$; that is,

$$[n/2] = \begin{cases} n/2\,, & \text{if } n \text{ is even}\,, \\ (n-1)/2\,, & \text{if } n \text{ is odd}\,. \end{cases}$$

Applying eq. (6) to eq. (5), it follows that

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]} \frac{(-1)^k(2n-2k)!}{2^n\,k!\,(n-k)!\,(n-2k)!}\,t^n x^{n-2k}\,. \tag{7}$$

Comparing with eq. (1), we conclude that

$$P_n(x) = \frac{1}{2^n}\sum_{k=0}^{[n/2]} \frac{(-1)^k(2n-2k)!}{k!\,(n-k)!\,(n-2k)!}\,x^{n-2k}\,. \tag{8}$$

Eq. (8) coincides with the expression for $P_n(x)$ obtained in class via the series solution to the Legendre differential equation, with the normalization convention such that $P_n(1) = 1$ for all non-negative integers n. Hence, the proof of eq. (1) is now complete.

2. Applications of the generating function of the Legendre polynomials

Using eq. (1), one can easily check that $P_n(1) = 1$. Substituting $x = 1$ into eq. (1) yields

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} P_n(1)\,t^n\,, \qquad \text{for } |t| < 1\,.$$

Comparing the above result with

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n\,,$$

immediately implies that $P_n(1) = 1$ for all non-negative integers n.

One can also employ eq. (1) to evaluate

$$\int_{-1}^{1} [P_n(x)]^2\,dx\,.$$

If we square eq. (1) and then integrate from $x = -1$ to $x = 1$, we obtain:

$$\int_{-1}^{1} \frac{dx}{1-2xt+t^2} = \int_{-1}^{1} dx \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} P_n(x)P_m(x)t^{m+n}\,. \tag{9}$$

The integral on the left hand side of eq. (9) is elementary,

$$\begin{aligned}\int_{-1}^{1} \frac{dx}{1-2xt+t^2} &= -\frac{1}{2t}\ln(1-2xt+t^2)\bigg|_{-1}^{1} = -\frac{1}{2t}\ln\frac{(1-t)^2}{(1+t)^2} \\ &= \frac{1}{t}\left[\ln(1+t) - \ln(1-t)\right] = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}\,t^n + \sum_{n=0}^{\infty}\frac{t^n}{n+1} \\ &= 2\sum_{n=0}^{\infty}\frac{t^{2n}}{2n+1}\,. \end{aligned} \tag{10}$$

In the penultimate line above, only terms with even n survive when the two sums are added. Hence, we can relabel the summation index $n \to 2n$ to obtain the final result exhibited in eq. (10).

Next, one can show that it is permissible to interchange the order of summation and integration in eq. (9). Using the orthogonality relation,

$$\int_{-1}^{1} P_n(x)\,P_m(x)\,dx = 0\,, \qquad \text{for } m \neq n\,,$$

it then follows that

$$\int_{-1}^{1} dx \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} P_n(x)P_m(x)t^{m+n} = \sum_{n=0}^{\infty} t^{2n}\int_{-1}^{1}[P_n(x)]^2\,dx\,. \tag{11}$$

Since eq. (9) implies that eqs. (10) and (11) are equal, the coefficients of t^{2n} in the latter two equations must coincide and it immediately follows that

$$\int_{-1}^{1}[P_n(x)]^2\,dx = \frac{2}{2n+1}\,.$$

APPENDIX: Rearrangement of a double sum

A key step in the analysis presented in these notes was the general formula

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} C_{k,n}t^{n+k} = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]} C_{k\,,\,n-k}t^{n}\,, \tag{12}$$

for any quantity $C_{k,n}$ that depends on the integer indices k and n. The symbol $[n/2]$ refers to the greatest integer less than or equal to $n/2$. Eq. (12) is simply a rearrangement of the double sum. For example,

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} C_{k,n}t^{n+k} = C_{0,0} + C_{0,1}t + C_{1,1}t^2 + C_{0,2}t^2 + C_{1,2}t^3 + C_{2,2}t^4 + C_{0,3}t^3 + C_{1,3}t^4 + \cdots\,,$$

and

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]} C_{k\,,\,n-k} = C_{0,0} + C_{0,1}t + [C_{0,2} + C_{1,1}]t^2 + [C_{0,3} + C_{1,2}]t^3 + [C_{0,4} + C_{1,3} + C_{2,2}]t^4 + \cdots\,.$$

It is straightforward to check that the second sum is simply a rearrangement of the first sum, and hence both sums coincide (under the assumption that the sums in question are absolutely convergent).

We can also provide a more formal proof. Consider the sum,

$$S \equiv \sum_{n=0}^{\infty}\sum_{k=0}^{n} C_{k,n}t^{n+k}\,.$$

Define new index variables, $j = k$ and $m = n + k$. Then $0 \leq k \leq n$ implies that $0 \leq j \leq m - j$. Hence, it follows that $m \geq 2j$, which means that $0 \leq j \leq [m/2]$, since j must be an integer. Consequently,

$$S = \sum_{n=0}^{\infty}\sum_{k=0}^{n} C_{k,n}t^{n+k} = \sum_{m=0}^{\infty}\sum_{j=0}^{[m/2]} C_{j,m-j}t^{m}\,. \tag{13}$$

Since all non-negative integer powers of z appear in the definition of S, we know that the sum over m must run from 0 to infinity, as indicated above. Finally, relabeling the last sum in eq. (13) by replacing $m \to n$ and $j \to k$, we end up with eq. (12) as required.

References:

1. Earl D. Rainville, *Special Functions* (Chelsea Publishing Company, Bronx, NY, 1960).

2. George B. Arfken and Hans J. Weber, *Mathematical Methods for Physicists*, 6th edition (Elsevier Academic Press, Burlington, MA, 2005).