DUE: THURSDAY November 15, 2012

To receive full credit, you must exhibit the intermediate steps that lead you to your final results.

1. In class, the addition theorem for spherical harmonics was presented,*

$$P_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell} Y_\ell^m(\theta,\phi)Y_\ell^m(\theta',\phi')^*\,,$$

where γ is the angle between the vectors $\vec{\boldsymbol{r}}$ and $\vec{\boldsymbol{r}}'$ with polar and azimuthal angles (θ,ϕ) and (θ',ϕ'), respectively, and the spherical harmonics are defined by

$$Y_\ell^m(\theta,\phi) = (-1)^m\sqrt{\frac{(2\ell+1)}{4\pi}\,\frac{(\ell-m)!}{(\ell+m)!}}\;P_\ell^m(\cos\theta)\,e^{im\phi}\,,\qquad \text{for}\;\begin{cases}\ell = 0,1,2,3,\ldots\,,\\ m=-\ell,-\ell+1,\ldots,\ell-1,\ell\,.\end{cases}$$

The normalization of the spherical harmonics has been chosen so that they are orthonormal. That is,†

$$\int Y_\ell^m(\theta,\phi)\,Y_{\ell'}^{m'}(\theta,\phi)^*\,d\Omega = \delta_{\ell\ell'}\,\delta_{mm'}\,,$$

where $d\Omega = \sin\theta\,d\theta\,d\phi$ is the differential solid angle in spherical coordinates. We then used the addition theorem for spherical harmonics in class to derive:

$$\frac{1}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|} = 4\pi\sum_{\ell=0}^{\infty}\frac{1}{2\ell+1}\,\frac{r'^{\,\ell}}{r^{\ell+1}}\sum_{m=-\ell}^{\ell}Y_\ell^m(\theta,\phi)Y_\ell^m(\theta',\phi')^*\,,\qquad \text{for } r>r'\,,\tag{1}$$

where $r\equiv|\vec{\boldsymbol{r}}|$ and $r'\equiv|\vec{\boldsymbol{r}}'|$. In the case of $r<r'$, the result above holds if we interchange r and r' on the right hand side of eq. (1).

With the help of eq. (1), evaluate the integral†

$$\int\frac{\cos\theta}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\,d\Omega\,.$$

Consider separately the cases of $r>r'$ and $r<r'$.

HINT: Write $\cos\theta$ as a Legendre polynomial and show it is proportional to $Y_1^0(\theta,\phi)$. [You will need to find the constant of proportionality.] Then, make use of the orthonormality properties of the spherical harmonics, and perform the summations. Evaluate explicitly any remaining spherical harmonics.

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*A proof of this theorem is given in the class handout entitled *The Spherical Harmonics*.

†The notation $\int d\Omega$ instructs you to integrate over the full solid angle, i.e. $0\le\theta\le\pi$ and $0\le\phi\le2\pi$.

2. Boas, p. 643, problem 13.5–3, part (a)

3. Boas, p. 643, problem 13.5–7

4. Boas, p. 644, problem 13.5–15

5. Boas, p. 650, problem 13.7–9

6. Boas, p. 650, problem 13.7–11

7. Boas, p. 651, problem 13.7–14

8. Boas, p. 658, problem 13.7–16

9. Boas, p. 658, problem 13.8–1

10. Boas, p. 658, problem 13.8–3. Note that Example 1 refers to the example worked out on pp. 655–657 of Boas.

11. Boas, p. 664, problem 13.10–20