INSTRUCTIONS: You have one hour and forty-five minutes to complete this exam. During the exam, you may refer to the textbook (Boas), the class handouts (including solution sets to the homeworks, the practice problems and the practice midterm) or your own personal notes. Collaboration with your neighbor is strictly forbidden.

In answering the questions, it is not sufficient to simply give a final result. You must provide the intermediate steps needed to arrive at the final solution in order to get full credit. *However, if you are employing any result already obtained in class or in the textbook, you do not need to re-derive it.*

1. Consider the differential equation

$$(1-x^2)y'' + p(p-1)y = 0\,, \tag{1}$$

defined on the interval $-1 \leq x \leq 1$, where p is a real number. The points $x = \pm 1$ are regular singular points of the differential equation.

(a) Find the series solution of eq. (1) for arbitrary p. Your answer should consist of a linear combination of two series with arbitrary coefficients.

(b) If we demand that a solution obtained in part (a) converges to a finite value at $x = \pm 1$, then this solution *must* be a polynomial of finite degree. This requirement imposes a condition on p. Determined all possible values of p that satisfy this requirement.

(c) Cast eq. (1) in Sturm-Liouville form and determine the weight function.

(d) Denote by $\{C_n(x)\}$ the set of nth-order polynomial solutions to eq. (1), where $n = 0, 1, 2, 3, \ldots$. Write down the orthogonality relations satisfied by these polynomials.

NOTE: In part (d), you may use any general theorems on such matters that were proved in class. You do not need to prove orthogonality from scratch.

2. The associated Laguerre polynomial is denoted by $L_n^k(x)$, where n and k are non-negative integers. Evaluate this polynomial at $x = 0$. That is, determine $L_n^k(0)$ as a function of n and k.

HINT: You may find useful one of the relevant recursion relations given in Boas.

3. Consider the differential equation,

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dy}{d\theta}\right)+\ell(\ell+1)y=0\,, \tag{2}$$

whose solution $y(\theta)$ is a function of θ.

(a) Define $x \equiv \cos\theta$. Using the chain rule, show that $y(x)$ satisfies the Legendre differential equation, $(1-x^2)y''-2xy'+\ell(\ell+1)y,$.

(b) Consider the limit in which $\ell\to\infty$ and $\theta\to 0$ in such a way that the product $\ell\theta\equiv z$ is held to a fixed finite value. In this limit, we can approximate $\sin\theta\simeq\theta$ and $\ell(\ell+1)\simeq\ell^2$. Employing these approximations in eq. (2), obtain a new differential equation for $y(\theta)$. You should recognize this differential equation—then write the most general solution of this new differential equation by inspection.

(c) Using the result of part (b), determine

$$\lim_{\substack{\ell\to\infty\\ \theta\to 0\\ \ell\theta=z}} P_\ell(\cos\theta)\,,$$

where $P_\ell(\cos\theta)$ is a Legendre polynomial in the variable $\cos\theta$.

HINT: The answer is some function of z. The only subtlety here is that the general solution obtained in part (b) is an arbitrary linear combination of two linearly independent solutions. To determine the proper combination of the two solutions for part (c), consider the limit of $\theta\to 0$ with ℓ very very large but finite (in which case $\ell\theta=z=0$). After evaluating $P_\ell(\cos\theta)$ in this limit, you will be able to determine the answer to part (c) uniquely.

(d) [*EXTRA CREDIT*] Determine $P_\ell^m(\cos\theta)$ in the limit of $\ell\to\infty$ and $\theta\to 0$ with $z=\ell\theta$ held to a fixed finite value, where $P_\ell^m(\cos\theta)$ is an associated Legendre function, which is a bivariate polynomial in the two variables $\sin\theta$ and $\cos\theta$.

4. A string of length ℓ is fixed at $x=0$ and $x=\ell$. At time $t=0$, the string has an initial displacement $y(x,t)\big|_{t=0}=x(\ell-x)$, whereas the velocity of all points on the string at $t=0$ is zero.

(a) Find the displacement $y(x,t)$ as a function of x and t.

(b) Determine the fundamental frequency of the vibrations and the frequencies of the higher harmonics that appear in the motion of the string.

NOTE: In part (a), it is sufficient to express $y(x,t)$ as a series (in summation notation, please). In part (b), I am asking for the frequencies f_n and *not* the angular frequencies ω_n.