Series solutions to a second order linear differential equation with regular singular points

Consider the second-order linear differential equation,

$$\frac{d^2y}{dx^2} + \frac{p(x)}{x}\frac{dy}{dx} + \frac{q(x)}{x^2}y = 0\,, \tag{1}$$

where the functions $p(x)$ and $q(x)$ are real analytic functions in the neighborhood of $x = 0$. The two linearly independent solutions of eq. (1) will be denoted by $y_1(x)$ and $y_2(x)$, respectively. We assume that

$$p_0 \equiv p(0) \neq 0\,, \qquad q_0 \equiv q(0) \neq 0\,.$$

In this case, the point $x = 0$ is a regular singular point of the differential equation. Every second order linear differential equation of this type possesses at least one solution of the form of the Frobenius series,

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n\,, \qquad \text{where } a_0 \neq 0\,, \tag{2}$$

for values of $x > 0$. For values of $x < 0$, if r is an integer then one can employ the solution given by eq. (2) without modification. If r is not an integer, then one can obtain the solution from eq. (2) by analytic continuation in the complex plane. In particular, for non-integer r there is a branch point at $x = 0$ and it is convenient to choose the branch cut to lie along the negative real axis. In particular, for $x > 0$ we have $y(-x) = \lim_{\epsilon\to 0} y(-x + i\epsilon)$, where ϵ is a positive infinitesimal quantity. In this way, one is able to obtain a series solution to eq. (1) that is valid in the entire complex plane excluding the branch cut that runs from $x = 0$ to $x = -\infty$.

If one is only interested in solutions to eq. (1) along the real axis, then one cannot analytically continue from the positive real axis to the negative real axis through the origin when r is a non-integer. In this case, the solutions for $x > 0$ and $x < 0$ are distinct and uncorrelated. If r is a non-integer, one possible choice is to replace x^r by $|x|^r$ in eq. (2),[1] since for $x < 0$

$$x^r = e^{r\ln x} = e^{r[\ln|x| + i\pi]} = |x|^r e^{i\pi r}\,,$$

and the phase factor $e^{i\pi r}$ can be absorbed into an overall arbitrary constant. Likewise, as shown later in these notes, when one solution, $y(x)$, is given by eq. (2), then in some cases a second linearly independent solution exists of the form $y(x)\ln x + w(x)$, where $w(x)$ is a

[1] For integer r, there is no branch point at $x = 0$, and eq. (2) is a valid solution for any real value of x.

second Frobenius series. For values of $x < 0$, it is permissible to replace $\ln x$ with $\ln|x|$, since for real $x < 0$ we have

$$\ln(x) = \ln|x| + i\pi\,,$$

in which case $y(x)\ln x + w(x) = y(x)\ln|x| + i\pi y(x) + w(x)$. Since $y(x)$ is the first solution to the differential equation, it follows that if $y(x)$ and $y(x)\ln x + w(x)$ are linearly independent then $y(x)$ and $y(x)\ln|x| + w(x)$ are also linearly independent. The advantage of these conventions is that the resulting series solutions of eq. (1) are manifestly real for any value of x. In these notes, we shall always provide the solutions to eq. (1) relevant for $x > 0$. We leave it to the reader to replace x^r by $|x|^r$ and $\ln x$ by $\ln|x|$ in the appropriate places in order that the solutions obtained are valid for $x < 0$.

As noted below eq. (1), we assume that $p(x)$ and $q(x)$ are real analytic functions in the neighborhood of $x = 0$. Thus, they possess Taylor series expansions about $x = 0$,

$$p(x) = \sum_{n=0}^{\infty} p_n x^n \quad \text{and} \quad q(x) = \sum_{n=0}^{\infty} q_n x^n\,. \tag{3}$$

In many practical problems, $p(x)$ and $q(x)$ are polynomials of finite degree. In these notes we shall be more general by employing eq. (3) as the analysis is not that more complicated.

Inserting eqs. (2) and (3) into eq. (1), we get

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + \left(\sum_{n=0}^{\infty} p_n x^n\right)\left(\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-2}\right) + \left(\sum_{n=0}^{\infty} q_n x^n\right)\left(\sum_{n=0}^{\infty} a_n x^{n+r-2}\right) = 0\,. \tag{4}$$

Since eq. (4) is an identity, all the coefficients of x^{n+r-2} (for $n = 0, 1, 2, 3, \ldots$) must be zero. The coefficient of x^{r-2} is $a_0 F(r)$, where

$$\boxed{F(r) = r(r-1) + p_0 r + q_0 = (r - r_1)(r - r_2) = 0} \tag{5}$$

This quadratic equation is called the *indicial equation*; the roots of this equation are denoted by r_1 and r_2. The coefficients of x^{n+r-2} for $n = 1, 2, 3, \ldots$ yield recurrence relations for the a_n. After some algebraic manipulation and simplification, the resulting recurrence relation is:

$$(n+r)(n+r-1)a_n + \sum_{k=0}^{n}\left[(k+r)p_{n-k} + q_{n-k}\right]a_k = 0\,, \qquad \text{for } n = 1, 2, 3, \ldots. \tag{6}$$

Note that this relation can be rewritten in the following form:

$$\boxed{F(r+n)a_n(r) = -\sum_{k=0}^{n-1}\left[(k+r)p_{n-k} + q_{n-k}\right]a_k(r)\,, \qquad \text{for } n = 1, 2, 3, \ldots} \tag{7}$$

after using the definition of $F(r) = r(r-1) + p_0 r + q_0$ [cf. eq. (5)]. In eq. (7), we have written $a_n(r)$ in place of a_n to emphasize that the solution to the recurrence relation depends on the parameter r. Normally, we will be interested only in the values of $a_n(r)$ for $r = r_1$ or $r = r_2$. However, we will see in the subsequent analysis that the r dependence of $a_n(r)$ is also a useful quantity.

In particular, it is convenient for later use to consider the following function of two variables,

$$y(x, r) = x^r \sum_{n=0}^{\infty} a_n(r) x^n \,, \qquad \text{where } a_0(r) \equiv a_0 \neq 0 \,, \tag{8}$$

where a_0 is some nonzero constant that is *independent* of r.[2] If we insert eqs. (3) and (8) into eq. (1) and make use of eq. (7), only the term proportional to x^{r-2} survives,

$$\frac{\partial^2 y}{\partial x^2} + \frac{p(x)}{x}\frac{\partial y}{\partial x} + \frac{q(x)}{x^2} y(x, r) = a_0 F(r) x^{r-2} = a_0 (r - r_1)(r - r_2) x^{r-2} \,. \tag{9}$$

Of course, $y(x, r_1)$ and $y(x, r_2)$ are solutions to eq. (1) as expected.

There are three separate cases depending on the values of the roots r_1 and r_2.

Case 1: $r_1 \neq r_2$ and $r_1 - r_2$ is not an integer. In this case, the two linearly independent solutions of eq. (1) are given by:

$$y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n(r_1) x^n \,, \qquad \text{where } a_0(r_1) \neq 0 \,, \tag{10}$$

$$y_2(x) = x^{r_2} \sum_{n=0}^{\infty} a_n(r_2) x^n \,, \qquad \text{where } a_0(r_2) \neq 0 \,. \tag{11}$$

The coefficients $a_n(r_1)$ and $a_n(r_2)$ for $n = 1, 2, 3, \ldots$ are determined from the recurrence relation and the indicial equation. That is, one first determines the general solution, $a_n(r)$, of eq. (7), and then one separately sets $r = r_1$ and $r = r_2$ to obtain $a_n(r_1)$ and $a_n(r_2)$, respectively. Note that $a_0(r_1)$ and $a_0(r_2)$ are arbitrary nonzero constants since the most general solution to eq. (1) is an arbitrary linear combination of $y_1(x)$ and $y_2(x)$.

Case 2: $r_1 = r_2$. In this case, there is only one independent Frobenius series,

$$y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n(r_1) x^n \,, \qquad \text{where } a_0(r_1) \neq 0 \,. \tag{12}$$

Moreover, since the indicial equation has a double root, it follows that $F(r) = (r - r_1)^2$, in which case

$$F(r_1) = \left(\frac{\partial F}{\partial r}\right)_{r=r_1} = 0 \,. \tag{13}$$

To find the second solution, we first set $r_1 = r_2$ in eq. (9) to obtain:

$$\frac{\partial^2 y}{\partial x^2} + \frac{p(x)}{x}\frac{\partial y}{\partial x} + \frac{q(x)}{x^2} y(x, r) = a_0 F(r) x^{r-2} = a_0 (r - r_1)^2 x^{r-2} \,, \tag{14}$$

[2] Taking a_0 to be independent of r is a matter of convenience. With this choice, the $a_n(r)$ that are determined from eq. (7) are well-defined functions of r for $n = 1, 2, 3, \ldots$.

where a_0 is an arbitrary nonzero constant that is *independent* of r. We now differentiate this equation with respect to r,[3]

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial y(x,r)}{\partial r}\right) + \frac{p(x)}{x}\frac{\partial}{\partial x}\left(\frac{\partial y(x,r)}{\partial r}\right) + \frac{q(x)}{x^2}\left(\frac{\partial y(x,r)}{\partial r}\right) = a_0 x^{r-2}\left[\frac{\partial F(r)}{\partial r} + F(r)\ln x\right].$$

Evaluating this equation at $r = r_1$ with the help of eq. (13) yields:

$$\frac{d^2}{dx^2}\left(\frac{\partial y(x,r)}{\partial r}\right)_{r=r_1} + \frac{p(x)}{x}\frac{d}{dx}\left(\frac{\partial y(x,r)}{\partial r}\right)_{r=r_1} + \frac{q(x)}{x^2}\left(\frac{\partial y(x,r)}{\partial r}\right)_{r=r_1} = 0\,.$$

This means that:

$$\left(\frac{\partial y(x,r)}{\partial r}\right)_{r=r_1}$$

is a solution to eq. (1) and can serve as the second linearly independent solution in the case under consideration. Using eq. (8),

$$\frac{\partial y(x,r)}{\partial r} = x^r \ln x \sum_{n=0}^{\infty} a_n(r)x^n + x^r \sum_{n=0}^{\infty} \frac{\partial a_n}{\partial r} x^n\,,$$

for $x > 0$. It follows that

$$y_2(x) = y_1(x)\ln x + x^{r_1}\sum_{n=1}^{\infty} b_n x^n\,, \quad \text{where } b_n \equiv \left(\frac{\partial a_n}{\partial r}\right)_{r=r_1} \quad \text{for } n = 1, 2, 3, \ldots. \tag{15}$$

Note that the sum starts at $n = 1$, since $a_0(r) \equiv a_0 \neq 0$ so that its partial derivative with respect to r vanishes. An arbitrary linear combination of eqs. (12) and (15) yields the most general solution of eq. (1).

Case 3: $r_1 \neq r_2$ and $r_1 - r_2$ is an integer. This case is subtle and requires a careful treatment. In some cases, the analysis of Case 1 can be implemented with no difficulty, and the two linearly independent solutions are given by eqs. (10) and (11). In other cases, only one Frobenius series can be derived, and the second linearly independent solutions resembles eq. (15) of Case 2. If you wish to explore the details of the Case 3 analysis, continue reading below. Otherwise, check out the examples at the end of these notes.

Without loss of generality, we assume that $r_1 > r_2$ so that $r_1 - r_2 \equiv m$ is a positive integer.[4] In this case, we again find that

$$y_1(x) = x^{r_1}\sum_{n=0}^{\infty} a_n(r_1)x^n\,, \qquad \text{where } a_0(r_1) \neq 0\,, \tag{16}$$

is a solution to eq. (1), where $a_n(r_1)$ is determined from eqs. (5) and (7). However, in contrast to Case 1 a second linearly independent Frobenius series solution to eq. (1) may not

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[3] Recall that $x^{r-2} = e^{(r-2)\ln x}$. Thus, taking a partial derivative with respect to r brings down a factor of $\ln x$.

[4] This case can only arise if r_1 and r_2 are real. Since the indicial equation is a real quadratic equation, if r_1 and r_2 are not real then they must be complex conjugates of each other, which corresponds to Case 1.

exist when $r_1 - r_2 = m$ is an integer. Indeed, there is a potential roadblock to determining $a_m(r_2)$ from eq. (7) in Case 3 due to the relation $F(r_2+m) = F(r_1) = 0$. A consistent solution for the coefficients $a_n(r)$ requires that the right-hand side of eq. (7) vanish for $n = m$. However there is no guarantee that this requirement can be satisfied, considering that we have assumed that $a_0(r_2) \neq 0$ in eq. (11). The only consistent alternative is to take $a_0(r_2) = 0$. In this latter case, eq. (7) immediately implies that:

$$a_0(r_2) = a_1(r_2) = \cdots = a_{m-1}(r_2) = 0\,. \tag{17}$$

Note that $a_m(r_2)$ can be nonzero since $F(r_2+m) = F(r_1) = 0$. In particular, eq. (7) yields

$$\begin{aligned} F(r_2+m+n)a_{n+m}(r_2) &= -\sum_{k=m}^{m+n-1} \left[(k+r_2)p_{n+m-k} + q_{n+m-k}\right] a_k(r_2) \\ &= -\sum_{k=0}^{n-1} \left[(k+m+r_2)p_{n-k} + q_{n-k}\right] a_{k+m}(r_2)\,. \end{aligned}$$

Using $r_1 = r_2 + m$, we end up with

$$F(r_1+n)a_{n+m}(r_2) = -\sum_{k=0}^{n-1} \left[(k+r_1)p_{n-k} + q_{n-k}\right] a_{k+m}(r_2)\,, \qquad \text{for } n = 1, 2, 3, \ldots$$

Comparing with eq. (7), we conclude that

$$a_{n+m}(r_2) = a_n(r_1)\,, \qquad \text{for } n = 1, 2, 3, \ldots, \tag{18}$$

in a normalization convention where $a_m(r_2) = a_0(r_1)$. Inserting eqs. (17) and (18) into eq. (11) yields

$$y_2(x) = x^{r_2}\sum_{n=m}^{\infty} a_n(r_2)x^n = x^{r_2}\sum_{n=0}^{\infty} a_{n+m}(r_2)x^{n+m} = x^{r_1}\sum_{n=0}^{\infty} a_n(r_1)x^n = y_1(x)\,. \tag{19}$$

That is, in this latter case we must seek a second linearly independent solution of eq. (1).

One can find the second linearly independent solution by the following trick. First, multiply eq. (9) by $r - r_2$ and define:

$$z(x,r) \equiv (r-r_2)y(x,r)\,, \qquad G(r) \equiv (r-r_2)F(r) = (r-r_1)(r-r_2)^2\,. \tag{20}$$

Then, eq. (9) becomes:

$$\frac{\partial^2 z}{\partial x^2} + \frac{p(x)}{x}\frac{\partial z}{\partial x} + \frac{q(x)}{x^2}z(x,r) = a_0 G(r)x^{r-2} = a_0(r-r_1)(r-r_2)^2 x^{r-2}\,. \tag{21}$$

Inspired by the analysis of Case 2, one differentiates this equation with respect to r,

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial z(x,r)}{\partial r}\right) + \frac{p(x)}{x}\frac{\partial}{\partial x}\left(\frac{\partial z(x,r)}{\partial r}\right) + \frac{q(x)}{x^2}\left(\frac{\partial z(x,r)}{\partial r}\right) = a_0 x^{r-2}\left[\frac{\partial G(r)}{\partial r} + G(r)\ln x\right]\,.$$

Evaluating this equation at $r = r_2$ with the help of

$$G(r_2) = \left(\frac{\partial G}{\partial r}\right)_{r=r_2} = 0\,,$$

it follows that

$$\frac{d^2}{dx^2}\left(\frac{\partial z(x,r)}{\partial r}\right)_{r=r_2} + \frac{p(x)}{x}\frac{d}{dx}\left(\frac{\partial z(x,r)}{\partial r}\right)_{r=r_2} + \frac{q(x)}{x^2}\left(\frac{\partial z(x,r)}{\partial r}\right)_{r=r_2} = 0\,.$$

This means that:

$$\left(\frac{\partial z(x,r)}{\partial r}\right)_{r=r_2}$$

is a solution to eq. (1) and can serve as the second linearly independent solution in the case under consideration. It is convenient to define

$$b_n(r) \equiv (r - r_2)a_n(r)\,. \tag{22}$$

Then, eqs. (8) and (20) imply that

$$z(x,r) = x^r \sum_{n=0}^{\infty} b_n(r)x^n\,, \qquad \text{where } b_0(r) \equiv (r - r_2)a_0\,, \tag{23}$$

Following the steps of Case 2,

$$\frac{\partial z(x,r)}{\partial r} = x^r \ln x \sum_{n=0}^{\infty} b_n(r)x^n + x^r\left[a_0 + \sum_{n=1}^{\infty}\frac{\partial b_n}{\partial r}x^n\right]\,.$$

In obtaining this result, we noted that $\partial b_0(r)/\partial r = a_0$. It follows that

$$\left(\frac{\partial z(x,r)}{\partial r}\right)_{r=r_2} = x^{r_2}\left\{\ln x \sum_{n=0}^{\infty} b_n x^n + a_0 + \sum_{n=1}^{\infty} c_n x^n\right\}\,, \tag{24}$$

where

$$b_n \equiv \lim_{r\to r_2} b_n(r)\,, \qquad c_n \equiv \left(\frac{\partial b_n}{\partial r}\right)_{r=r_2} \qquad \text{for } n = 0, 1, 2, 3, \ldots. \tag{25}$$

We have been careful to define b_n by a limiting process. In particular, eqs. (7) and (22) yield

$$b_n(r) = \frac{-1}{F(r+n)}\sum_{k=0}^{n-1}\left[(k+r)p_{n-k} + q_{n-k}\right]b_k(r)\,, \qquad \text{for } n = 1, 2, 3, \ldots, \tag{26}$$

where $b_0(r) = (r - r_2)a_0$. Taking the limit of $r \to r_2$, it follows that $b_0(r_2) = 0$. Hence,

$$b_n(r_2) = 0\,, \qquad \text{for } n = 0, 1, 2, 3, \ldots, m-1\,. \tag{27}$$

But when $n = m$ we have $F(r_2 + m) = F(r_1) = 0$, and we can no longer conclude that $b_m(r_2) = 0$. In fact, one must use L'Hospital's rule to evaluate $b_m(r_2)$. Let us denote the result by:

$$b_m(r_2) = \lim_{r \to r_2} b_m(r) \equiv b a_0 \,, \tag{28}$$

which defines the constant b. One can now use eq. (26) to compute $b_{m+n}(r_2)$ in terms of b for $n = 1, 2, 3, \ldots$,

$$b_{m+n}(r_2) = \frac{-1}{F(r_2 + m + n)} \sum_{\ell=m}^{m+n-1} \left[(\ell + r_2)p_{m+n-\ell} + q_{m+n-\ell}\right] b_\ell(r_2) \,.$$

If we relabel the summation index by $\ell = k + m$, then the sum runs from $k = 0$ to $k = n-1$. Using $r_1 = r_2 + m$, it follows that

$$F(r_1 + n) b_{m+n}(r_2) = -\sum_{k=0}^{n-1} \left[(k + r_1)p_{n-k} + q_{n-k}\right] b_{m+k}(r_2) \,. \tag{29}$$

Comparing this equation with eq. (7), it follows that

$$b_{m+n}(r_2) = b a_n(r_1) \,, \qquad \text{for } n = 0, 1, 2, 3, \ldots \,, \tag{30}$$

where b is the constant defined in eq. (28). Indeed for $n = 0$ we get $b_m(r_2) = b a_0$. Substituting eq. (30) into eq. (29), b cancels out and one reproduces eq. (7) as claimed.

Using eqs. (27) and (30) and $r_1 = r_2 + m$,

$$\begin{aligned} x^{r_2} \sum_{n=0}^{\infty} b_n(r_2) x^n &= x^{r_2} \sum_{n=m}^{\infty} b_n(r_2) x^n = x^{r_2} \sum_{n=0}^{\infty} b_{m+n}(r_2) x^{m+n} \\ &= b\, x^{r_2+m} \sum_{n=0}^{\infty} a_n(r_1) x^n = b\, x^{r_1} \sum_{n=0}^{\infty} a_n(r_1) x^n \\ &= b y_1(x) \,, \end{aligned} \tag{31}$$

after identifying the first solution given by eq. (16). Hence, eqs. (24) and (25) yields the second linearly independent solution of eq. (1),

$$y_2(x) = b\, y_1(x) \ln x + x^{r_2} \left[a_0 + \sum_{n=1}^{\infty} c_n x^n \right] , \quad \text{where } c_n \equiv \left(\frac{\partial b_n}{\partial r} \right)_{r=r_2} \text{ for } n = 0, 1, 2, 3, \ldots \tag{32}$$

For further details, the following references are useful:

1. Ravi P. Agarwal and Donal O'Regan, *Ordinary and Partial Differential Equations* (Springer Science, New York, 2009).

2. Earl D. Rainville, Phillip E. Bedient and Richard E. Bedien, *Elementary Differential Equations*, 8th edition (Pearson Education, Upper Saddle River, NJ, 1997).

Appendix A: Examples of logarithmic series solutions

Example 1. Obtain the two linearly independent solutions valid for $x > 0$ of

$$x^2 y'' - x(1+x)y + y = 0\,. \tag{33}$$

Comparing with eq. (1), we identify $p(x) = -1 - x$ and $q(x) = 1$. Thus, $p_0 \equiv p(0) = -1$ and $q_0 \equiv q(0) = 1$. The indicial equation given in eq. (5) is

$$F(r) = r(r-1) - r + 1 = (r-1)^2 = 0\,.$$

That is, there is a double root, with $r = 1$. Since $p_1 = -1$ and $p_{n+1} = q_n = 0$ for $n = 1, 2, 3, \ldots$, the recursion relation given by eq. (7) simplifies to

$$a_n(r) = \frac{a_{n-1}(r)}{n-1+r}\,, \qquad \text{for } n = 1, 2, 3, \ldots. \tag{34}$$

Inserting $r = 1$ in eq. (34), one easily deduces that

$$a_n = \frac{a_0}{n!}\,, \qquad \text{where } a_n \equiv a_n(1)\,.$$

Thus, using eq. (12), we obtain one of the solutions to eq. (33),

$$y_1(x) = a_0 x \sum_{n=0}^{\infty} \frac{x^n}{n!} = a_0 x e^x\,.$$

To obtain the second linearly independent solution, we employ eq. (15). Using eq. (34), it is a simple matter to obtain[5]

$$a_n'(r) = \frac{a_0'}{r(r+1)\cdots(r+n-1)}\,,$$

where $a_0' \equiv a_0'(r)$ is independent of r by convention [cf. eq. (8)]. Hence,

$$b_n' = \left(\frac{\partial a_n'}{\partial r}\right)_{r=1} = -\frac{a_0'}{n!}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)\,.$$

Thus eq. (15) yields the second linearly independent solution of eq. (33),

$$y_2(x) = a_0' x \left\{ e^x \ln x - \sum_{n=1}^{\infty} \frac{H_n x^n}{n!} \right\}\,,$$

where the *harmonic numbers*

$$H_n \equiv 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\,, \tag{35}$$

are defined to be the sum of the first n terms of the harmonic series.[6]

[5]In obtaining the second solution, we write a_n' in place of a_n to avoid confusion.

[6]It is convenient to define $H_0 \equiv 0$, in which case it follows that the harmonic numbers satisfy the recursion relation,

$$H_n = H_{n-1} + \frac{1}{n}\,, \tag{36}$$

for all positive integers n.

Example 2. Obtain the two linearly independent solutions valid for $x > 0$ of

$$xy'' + y = 0\,. \tag{37}$$

Comparing with eq. (1), we identify $p(x) = 0$ and $q(x) = x$. Thus, $p_0 \equiv p(0) = 0$ and $q_0 \equiv q(0) = 0$. The indicial equation given in eq. (5) is

$$F(r) = r(r-1) = 0\,,$$

and the corresponding roots are $r_1 = 1$ and $r_2 = 0$. Hence, $m \equiv r_1 - r_2 = 1$. Since $q_1 = 1$ and $p_n = q_{n+1} = 0$ for $n = 1, 2, 3, \ldots$, the recursion relation given by eq. (7) yields

$$(r+n)(r+n-1)a_n(r) = -a_{n-1}(r)\,, \qquad \text{for } n = 1, 2, 3, \ldots. \tag{38}$$

Inserting $r = 1$ in eq. (38), one easily deduces that

$$a_n = \frac{(-1)^n a_0}{n!(n+1)!}\,, \qquad \text{where } a_n \equiv a_n(1)\,.$$

Thus, using eq. (16), we obtain one of the solutions to eq. (37),

$$y_1(x) = a_0 x \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!(n+1)!}\,.$$

For the case of $r = 0$, eq. (38) is incompatible with $a_0(0) \neq 0$. Hence, we do not obtain a second linearly independent Forbenius series solution. Instead, following eq. (22) we define $b_n(r) = r a_n'(r)$ [cf. footnote 5]. Hence, eq. (38) yields

$$b_n(r) = \frac{-r a_{n-1}'(r)}{(r+n)(r+n-1)}\,, \qquad \text{for } n = 1, 2, 3, \ldots. \tag{39}$$

In particular, $b_0(0) = 0$ and $b_1(0) = -a_0'$, which implies that $b \equiv -1$ [cf. eq. (28)]. Hence,

$$b_n(r) = \frac{(-1)^n a_0'}{(r+1)(r+2)\cdots(r+n)(r+1)(r+2)\cdots(r+n-1)}\,.$$

Therefore, eq. (25) yields an expression involving the harmonic numbers H_n [cf. eqs. (35) and (36)],

$$c_n = \left(\frac{\partial b_n}{\partial r}\right)_{r=0} = -\,\frac{(-1)^n (H_n + H_{n-1}) a_0'}{n!(n-1)!}\,, \qquad \text{for } n = 1, 2, 3, \ldots.$$

Plugging into eq. (32), we obtain the second linearly independent solution of eq. (37),

$$y_2(x) = a_0' \left\{ -x \ln x \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!(n+1)!} + 1 - \sum_{n=1}^{\infty} \frac{(-1)^n (H_n + H_{n-1}) x^n}{n!(n-1)!} \right\}\,.$$

or equivalently,

$$y_2(x) = a_0' \left\{ 1 - \sum_{n=1}^{\infty} \frac{(-1)^n (H_n + H_{n-1} - \ln x) x^n}{n!(n-1)!} \right\}\,.$$

Appendix B: Applications to Bessel functions

Bessel's differential equation is given by:

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{\nu^2}{x^2}\right) y(x) = 0\,.$$

In the notation of eq. (1), $p(x) = 1$ and $q(x) = x^2 - \nu^2$. From eq. (3), we get $p_0 = 1$, $q_0 = -\nu^2$ and $q_2 = 1$ (and all other p_n and q_n vanish). The indicial equation is

$$F(r) = r^2 - \nu^2\,, \tag{40}$$

and the recurrence relations [cf. eq. (7)] are given by

$$F(1+r)a_1 = 0\,, \tag{41}$$

$$F(n+r)a_n = -a_{n-2}\,, \qquad \text{for } n = 2, 3, 4, \ldots. \tag{42}$$

If $F(1+r) \neq 0$, the the solution to the recurrence relations are:

$$a_{2n}(r) = \frac{(-1)^n a_0}{F(2n+r)F(2n-2+r)\cdots F(2+r)}\,, \qquad a_1 = a_{2n+1} = 0\,, \tag{43}$$

for $n = 1, 2, 3, \ldots$ and $a_0 \neq 0$. If $F(1+r) = 0$, the $a_1 \neq 0$ and we must replace $a_1 = a_{2n+1} = 0$ with

$$a_{2n+1}(r) = \frac{(-1)^n a_1}{F(2n+1+r)F(2n-1+r)\cdots F(3+r)}\,,$$

for $n = 1, 2, 3, \ldots$ and $a_1 \neq 0$.

Example 1: 2ν is not an integer. In this case, the results of Case 1 of these notes apply. We set $r = \pm\nu$ and note that $F(1 \pm \nu) \neq 0$. Hence,

$$a_{2n}(\nu) = \frac{(-1)^n a_0}{2^{2n} n!(\nu+n)(\nu+n-1)\cdots(\nu+1)}\,, \qquad a_1 = a_{2n+1} = 0\,,$$

for $n = 1, 2, 3, \ldots$ and $a_0 \neq 0$. It is conventional to define

$$a_0 \equiv \frac{1}{2^\nu \Gamma(\nu+1)}\,, \tag{44}$$

in which case the series solution to Bessel's equation is given by

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(\nu+n+1)} \frac{1}{n!} \left(\frac{x}{2}\right)^{2n+\nu}\,. \tag{45}$$

The analysis for $r = -\nu$ is almost identical and results in a second linearly independent solution,

$$J_{-\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(-\nu+n+1)} \frac{1}{n!} \left(\frac{x}{2}\right)^{2n-\nu}\,.$$

Example 2: $\boldsymbol{\nu = 0}$. In this case, the results of Case 2 of these notes apply. One of the solutions follows immediately by setting $\nu = 0$ in eq. (45),

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{[n!]^2} \left(\frac{x}{2}\right)^{2n}, \tag{46}$$

after using $\Gamma(n+1) = n!$. To obtain the second linearly independent solution, we employ eq. (15). Thus, we must compute the quantity $(\partial a_{2n}/\partial r)_{r=0}$ with $\nu = 0$. Eqs. (40) and (43) yield

$$a_{2n}(r) = \frac{(-1)^n a_0}{(2n+r)^2(2n+r-2)^2 \cdots (2+r)^2}, \qquad \text{for } n = 1, 2, 3, \ldots. \tag{47}$$

Then

$$\frac{\partial a_{2n}}{\partial r} = -2a_{2n}(0)\left[\frac{1}{2n+r} + \frac{1}{2n+r-2} + \cdots + \frac{1}{2+r}\right].$$

Using eqs. (44) and (47), it follows that

$$\left(\frac{\partial a_{2n}}{\partial r}\right)_{r=0} = -H_n a_{2n}(0)\,, \tag{48}$$

where the H_n are the harmonic numbers defined in eq. (35) and

$$a_{2n}(0) = \frac{(-1)^n}{2^{2n}[n!]^2}\,.$$

Note that eq. (48) also holds for $n = 0$ since $H_0 = 0$ (cf. footnote 6), which is to be expected as a_0 is independent of r. Hence, eq. (15) yields:[7]

$$y_2(x) = J_0(x)\ln x - \sum_{n=0}^{\infty} \frac{(-1)^n H_n}{[n!]^2}\left(\frac{x}{2}\right)^{2n}. \tag{49}$$

Indeed, this is a second linearly independent solution to Bessel's equation for $\nu = 0$. However, it is more traditional to employ a particular linear combination of $J_0(x)$ and $y_2(x)$ for the second solution to Besse's equation, which we denoted in class by $N_0(x)$. The series expansion of $N_0(x)$ involves the logarithmic derivative of the Gamma function evaluated at the positive integer $n+1$, which is given by:[8]

$$\psi(n+1) \equiv \frac{\Gamma'(n+1)}{\Gamma(n+1)} = -\gamma + H_n\,, \tag{50}$$

where γ is Euler's constant. Using the series expansions given in eqs. (46) and (49), the corresponding series expansion for $N_0(x)$ is given by:

$$N_0(x) = \frac{2}{\pi}\Big\{[y_2(x) + (\gamma - \ln 2)J_0(x)]\Big\} = \frac{2}{\pi}\left\{J_0(x)\ln\frac{x}{2} - \sum_{n=0}^{\infty}\frac{(-1)^n\psi(n+1)}{[n!]^2}\left(\frac{x}{2}\right)^{2n}\right\}.$$

[7]One can extend the summation index to $n = 0$ in eq. (49) since the summand vanishes when $n = 0$.
[8]Further details on the logarithmic derivative of the Gamma function can be found in Appendix C.

One can check that the series expansion for $N_0(x)$ given above matches the one obtained from employing L'Hospital's rule on the definition of $N_0(x)$,

$$N_0(x) = \lim_{\nu\to 0} \frac{J_\nu(x)\cos\pi\nu - J_{-\nu}(x)}{\sin\pi\nu} = \lim_{\nu\to 0} \frac{2}{\pi}\frac{\partial J_\nu(x)}{\partial\nu}\,,$$

and evaluating the last quantity using the series for $J_\nu(x)$ given in eq. (45).

Example 3a: $\boldsymbol{\nu = \pm\frac{1}{2}}$. In this case, the results of Case 3 of these notes apply, since $r_1 - r_2 = 1$ which is an integer. However, in this example no problems arise in computing the coefficients of the expansion from the recurrence relation, in which case we recover the results of Case 1. In more detail, for $r = \frac{1}{2}$, eqs. (40) and (43) yield

$$a_{2n}(\tfrac{1}{2}) = \frac{(-1)^n a_0}{(2n+1)!}\,, \qquad a_{2n+1}(\tfrac{1}{2}) = 0\,, \qquad \text{for } n = 0,1,2,3,\ldots.$$

For $r = -\frac{1}{2}$, we have $F(1+r) = F(\frac{1}{2}) = 0$ so that $a_1 \neq 0$. Thus, in this case, eqs. (40) and (43) yield

$$a_{2n}(-\tfrac{1}{2}) = \frac{(-1)^n a_0}{(2n)!}\,, \qquad a_{2n+1}(-\tfrac{1}{2}) = \frac{(-1)^n a_1}{(2n+1)!}\,, \qquad \text{for } n = 0,1,2,3,\ldots.$$

It therefore follows that,

$$y_1(x) = a_0 x^{1/2} \sum_{n=0}^\infty \frac{(-1)^n x^{2n}}{(2n+1)!} = a_0 x^{-1/2}\sin x\,,$$

$$y_2(x) = x^{-1/2}\left\{a_0 \sum_{n=0}^\infty \frac{(-1)^n x^{2n}}{(2n)!} + a_1 \sum_{n=0}^\infty \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right\} = x^{-1/2}\left[a_0\cos x + a_1 \sin x\right]\,.$$

Clearly, $y_1(x)$ and $y_2(x)$ are linearly independent, although it is more traditional to define a different linearly independent $y_2(x)$ which only has the cosine term. If we choose $a_0 = (2/\pi)^{1/2}$ according to eq. (44), then

$$y_1(x) = J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\,\sin x\,,$$

and the corresponding spherical Bessel function,

$$j_0(x) \equiv \sqrt{\frac{\pi}{2x}}\,J_{1/2}(x) = \frac{\sin x}{x}\,,$$

whereas $(\pi/2x)^{1/2} y_2(x)$ is a linear combination of $j_0(x)$ and $n_0(x) \equiv -\cos x/x$.

Example 3b: $\boldsymbol{\nu = \pm 1}$. In this case, the results of Case 3 of these notes apply, since $r_1 - r_2 = 2$ which is an integer. However, only one Frobenius series can be obtained,

$$J_1(x) = \sum_{n=0}^\infty \frac{(-1)^n}{n!(n+1)!}\left(\frac{x}{2}\right)^{2n+1}\,. \tag{51}$$

When $r=-1$, we have $F(n-1)=0$ for $n=2$. In this case, eq. (42) implies that $a_0=0$ and a_2 is arbitrary. Moreover, starting from a_2, one can use eq. (42) to derive a_{2n} for $n=2,3,4,\ldots$. It is easy to check that the resulting Frobenius series is proportional to $J_1(x)$, as expected from eqs. (18) and (19). (You are strongly encouraged to verify this statement, since the math in this example is simpler than in the general case!)

To obtain the second linearly independent solution, we employ eq. (32). First, we construct $b_n(r)=(r+1)a_n(r)$ following eq. (22). Then using eq. (42),

$$\begin{aligned} b_{2n}(r) &= \frac{(-1)^n(r+1)a_0}{[(2n+r)^2-1][(2n+r-2)^2-1]\cdots[(2+r)^2-1]} \\ &= \frac{(-1)^n a_0}{(r+3)[(2n+r)^2-1][(2n+r-2)^2-1]\cdots[(4+r)^2-1]}\,, \qquad \text{for } n=1,2,3,\ldots, \end{aligned}$$

after writing $(2+r)^2-1=r^2+4r+3=(r+1)(r+3)$ and canceling out the common factor $r+1$ from numerator and denominator. It is critical to take this step before computing $(\partial b_{2n}/\partial r)_{r=-1}$; otherwise you will be stymied by factors of zero appearing in the denominator. One can further simplify $b_{2n}(r)$ by factoring

$$(2k+r)^2-1=(2k+r+1)(2k+r-1)\,, \qquad \text{for } k=2,3,4,\ldots,n\,.$$

Hence,

$$b_{2n}(r)=\frac{(-1)^n a_0}{(2n+r+1)(2n+r-1)^2(2n+r-3)^2(2n+r-5)^2\cdots(5+r)^2(3+r)^2}\,, \tag{52}$$

where all denominator factors are squared except for $(2n+r+1)$. Taking the derivative with respect to r yields

$$\frac{\partial b_{2n}}{\partial r}=-b_{2n}(r)\left[\frac{1}{2n+r+1}+\frac{2}{2n+r-1}+\frac{2}{2n+r-3}+\frac{2}{2n+r-5}+\cdots+\frac{2}{5+r}+\frac{2}{3+r}\right].$$

Setting $r=-1$, we make use of eqs. (44) and (52) to obtain

$$b_{2n}(-1)=\frac{(-1)^n a_0}{2^{2n-1}n!(n-1)!}\,, \qquad a_0=\tfrac{1}{2}\,.$$

Hence, we end up with:

$$\begin{aligned} \left(\frac{\partial b_{2n}}{\partial r}\right)_{r=-1} &= -b_{2n}(-1)\left[1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-2}+\frac{1}{n-1}+\frac{1}{2n}\right], \\ &= \frac{(-1)^{n+1}}{2^{2n}n!(n-1)!}\left[H_{n-1}+\frac{1}{2n}\right], \\ &= \frac{(-1)^{n+1}(H_n+H_{n-1})}{2^{2n+1}n!(n-1)!}\,. \end{aligned} \tag{53}$$

In obtaining eq. (53), we have used eq. (36) to eliminate the factor of $1/(2n)$ above. Note that we can now use eq. (28) to determine the constant b. In this example $m=r_1-r_2=2$, in which case

$$b=b_2(-1)/a_0=-\tfrac{1}{2}\,.$$

Therefore, eq. (32) yields the following second linearly independent solution to Bessel's equation for $|\nu| = 1$,

$$y_2(x) = -\tfrac{1}{2}J_1(x)\ln x + \frac{1}{x}\left\{\frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(H_n + H_{n-1})}{2^{2n+1}n!(n-1)!}x^{2n}\right\}.$$

If we relabel the last sum by taking $n \to n+1$, we end up with:

$$y_2(x) = -\tfrac{1}{2}J_1(x)\ln x + \frac{1}{2x}\left\{1 + \sum_{n=0}^{\infty}\frac{(-1)^n(H_{n+1} + H_n)}{(n+1)!n!}\left(\frac{x}{2}\right)^{2n+2}\right\}.$$

Indeed, this is a second linearly independent solution to Bessel's equation for $|\nu| = 1$. However, it is more traditional to employ a particular linear combination of $J_1(x)$ and $y_2(x)$, which in class we denoted by $N_1(x)$,

$$\begin{aligned} N_1(x) &= -\frac{4}{\pi}\left[y_2(x) + \tfrac{1}{2}(\ln 2 - \gamma)J_1(x)\right] \\ &= \frac{2}{\pi}\left\{J_1(x)\ln\frac{x}{2} - \frac{1}{x} - \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)!n!}\left[\psi(n+2) + \psi(n+1)\right]\left(\frac{x}{2}\right)^{2n+1}\right\}, \end{aligned}$$

where we have used eqs. (50) and (51). One can check that this series expansion matches the one obtained from employing L'Hospital's rule on the definition of $N_1(x)$,

$$N_1(x) = \lim_{\nu\to 1}\frac{J_\nu(x)\cos\pi\nu - J_{-\nu}(x)}{\sin\pi\nu} = \frac{1}{\pi}\left[\lim_{\nu\to 1}\frac{\partial J_\nu(x)}{\partial\nu} - \lim_{\nu\to -1}\frac{\partial J_\nu(x)}{\partial\nu}\right],$$

and evaluating the last two quantities using the series for $J_\nu(x)$ given in eq. (45).

The general case of $\nu = \pm k$ for $k = 2, 3, 4, \ldots$ can be similarly analyzed. However, for the second linearly independent solution to Bessel's equation, it is simpler to employ

$$N_k(x) = \lim_{\nu\to k}\frac{J_\nu(x)\cos\pi\nu - J_{-\nu}(x)}{\sin\pi\nu} = \frac{1}{\pi}\left[\lim_{\nu\to k}\frac{\partial J_\nu(x)}{\partial\nu} + (-1)^k\lim_{\nu\to -k}\frac{\partial J_\nu(x)}{\partial\nu}\right],$$

and evaluate the last two quantities using the series for $J_\nu(x)$ given in eq. (45). For the record, the resulting series expansion is quoted below:

$$\begin{aligned} N_k(x) = \frac{1}{\pi}\Bigg\{&2J_k(x)\ln\frac{x}{2} - \sum_{n=0}^{k-1}\frac{(k-n-1)!}{n!}\left(\frac{x}{2}\right)^{2n-k} \\ &- \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+k)!n!}\left[\psi(n+k+1) + \psi(n+1)\right]\left(\frac{x}{2}\right)^{2n+k}\Bigg\}. \end{aligned} \quad (54)$$

This formula holds for all non-negative integers k (but if $k = 0$, the first sum above is absent).

Appendix C: The logarithmic derivative of the Gamma function

In this Appendix, I will sketch some of the main properties of the logarithmic derivative[9] of the Gamma function. The formal definition is given by:

$$\psi(x) \equiv \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)} \,,$$

where $\Gamma'(x)$ is the ordinary derivative of $\Gamma(x)$ with respect to x. We also define:

$$\lim_{x\to 1} \Gamma'(x) = -\gamma \,,$$

where $\gamma \simeq 0.5772156649\cdots$ is known as Euler's constant. It is not known whether γ is a rational or irrational number (although there is strong suspicion for the latter). By definition of $\psi(x)$, we see [since $\Gamma(1) = 1$] that:

$$\Psi(1) = -\gamma \,.$$

Using $\Gamma(x+1) = x\Gamma(x)$, we can differentiate this equation to derive a fundamental property of $\psi(x)$.

$$\Gamma'(x+1) = \Gamma(x) + x\Gamma'(x) \,,$$

$$\frac{\Gamma'(x+1)}{\Gamma(x)} = 1 + x\frac{\Gamma'(x)}{\Gamma(x)} \,.$$

Finally, writing $\Gamma(x) = \Gamma(x+1)/x$ on the left hand side above, and then dividing through by x, we find:

$$\psi(x+1) = \frac{1}{x} + \psi(x) \,. \tag{55}$$

Consider the case of $x = n = 0, 1, 2, \ldots$. Then, using eq. (55)

$$\psi(n+1) = \frac{1}{n} + \psi(n) = \frac{1}{n} + \frac{1}{n-1} + \psi(n-1) = \cdots \,,$$

until we reach $\psi(1) = -\gamma$. The end result is:

$$\psi(n+1) = -\gamma + \sum_{k=1}^{n} \frac{1}{k} \,.$$

It is convenient to rewrite this result in terms of the harmonic numbers defined in eq. (35),

$$\psi(n+1) = -\gamma + H_n \,, \qquad n = 0, 1, 2, 3, \ldots \,. \tag{56}$$

By employing the convention that $H_0 = 0$ [cf. footnote 6], we see that eq. (56) holds for $n = 0$. Thus, we have confirmed eq. (50).

[9]The *logarithmic derivative* of a function is defined as the derivative of the logarithm of the function.

We next examine the asymptotic behavior of $\psi(x)$ as $x \to \infty$. This is easily accomplished by making use of Stirling's formula:

$$\ln \Gamma(x+1) = (x+\tfrac{1}{2})\ln x - x + \tfrac{1}{2}\ln 2\pi + \mathcal{O}(x^{-1})\,, \qquad \text{as } x \to \infty\,.$$

Differentiating this formula yields the large x asymptotic behavior of $\psi(x+1)$:

$$\psi(x+1) = \ln x + \frac{1}{2x} + \mathcal{O}(x^{-2})\,, \qquad \text{as } x \to \infty\,. \tag{57}$$

In particular, it follows that for an integer n,

$$\lim_{n\to\infty} \psi(n+1) - \ln n = 0\,.$$

If we use result for $\psi(n+1)$ given in eq. (56), we conclude that:

$$\gamma = \lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{1}{k} - \ln n\right]\,.$$

This is a remarkable formula, which often serves as the definition of Euler's constant. (Textbooks that adopt this definition must spend some time proving that this limit exists and is finite.) The above results also imply that:

$$\sum_{k=1}^{n}\frac{1}{k} = \ln n + \gamma + \frac{1}{2n} + \mathcal{O}\left(\frac{1}{n^2}\right)\,.$$

This result provides the start of an asymptotic expansion for the finite harmonic sum as $n \to \infty$. Moreover, it tells us that the infinite harmonic sum diverges logarithmically (which explains the slow growth of the corresponding finite sums).

So far, we only have an explicit formula for $\psi(x)$ when x is a positive integer [eq. (56)]. We can derive a more general result as follows. In the same way that we derived eq. (56), we may use eq. (55) to obtain:

$$\begin{aligned}\psi(x+n) &= \frac{1}{x+n-1} + \frac{1}{x+n-2} + \cdots + \frac{1}{x} + \psi(x) \\ &= \sum_{k=0}^{n-1}\frac{1}{x+k} + \psi(x)\,,\end{aligned} \tag{58}$$

where n is a positive integer and x is arbitrary. Subtracting eq. (56) from eq. (58) yields:

$$\psi(x+n) - \psi(n+1) = \sum_{k=0}^{n-1}\left(\frac{1}{x+k} - \frac{1}{k+1}\right) + \gamma + \psi(x)\,. \tag{59}$$

Consider the $n \to \infty$ limit of eq. (59). Using eq. (57), it follows that

$$\lim_{n\to\infty}\psi(x+n) - \psi(n+1) = \mathcal{O}(n^{-1}) \to 0\,.$$

Hence, we conclude that:

$$\psi(x) = -\gamma - \sum_{k=0}^{\infty}\left(\frac{1}{x+k} - \frac{1}{k+1}\right)\,. \tag{60}$$

You should check that if x is a positive integer, then eq. (60) reduces to eq. (56).