Relationship of the commutation rules to classical-like representations of quantum angular momenta addition

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We perform an analysis of a graphical representation for the addition of two angular momenta, focusing our attention on the angle δ between the xy components of two angular momenta. Then we propose a new complete set of commuting observables, which differ from the usual sets that are connected by the Clebsch–Gordan coefficients. This set shows that the angle δ can be a well-defined variable in quantum mechanics. An empirical analysis of the graphical representations of the angular momenta relations, which may include the angle δ, followed by quantum mechanical considerations, leads to the vanishing of certain quantum angular momentum commutators for specific states. Therefore, although the commutators are not null in general, the quantum addition of angular momenta may be represented using classical-like diagrams.

I. INTRODUCTION

Nowadays it is a well-established practice in modern physics courses to introduce the quantum theory of angular momentum as early as possible instead of the vector model of the atom (VM). This is possible to some extent because now students become acquainted with the mathematical tools used in this theory almost from the beginning of their studies. But to a large extent it is the proven capability of the quantum theory of angular momentum to describe matter which has caused the VM to decline from the panorama of modern physics.

Nevertheless, quantum mechanics books (see, for instance, Cohen-Tannoudji *et al.*,[1] Schiff,[2] and Zare[3]) usually include a brief mention of the vector model of the atom when discussing angular momentum. Cohen-Tannoudji *et al.* (pp. 668–670) discuss briefly why that model has some realistic aspects and also describe its limitations. Zare (pp. 12–17) discusses the applicability of the VM to systems with large angular momentum. Biedenharn and Louck[4,5] also use the VM to discuss the physical interpretation of Wigner and Racah coefficients.

Although in some specialized treatises (Brink and Satchler,[6] Varshalovich *et al.*[7]) the VM is scarcely or simply not mentioned, some recently written texts include the VM as in Harter's book.[8] This author uses the VM to illustrate modern quantitative applications of angular momentum in the study of molecular systems with high angular momenta.

As can be seen, the VM is still presented in many introductory and advanced courses. We can understand this if we recall that in the evolution of the angular momentum theory, as in many other theories in physics, empirical knowledge came first and later on a general theory organized and explained the multitude of empirical facts. We think that in the frame of this transformation the VM has changed from an empirical model to a graphical representation of the geometrical interpretation of well-founded facts in quantum mechanics. In this sense we understand Wigner's assertion: ''This 'vector addition model' is of very general validity.''[9] In this article we will demonstrate that the validity of some graphical representations of the VM can be derived from the values of the commutators between angular momentum operators.

We would like to address the following question: How do commutation relations in quantum mechanics allow some graphical representations but exclude many others? To answer this question we will begin the analysis of a statement about the VM made in Gasiorowicz's book.[10] This discussion will lead to an extension of the VM into a graphical representation that we name ''angular coupled addition'' (ACA). We will start playing empirically with the ACA and then we will analyze it from the point of view of quantum physics. ACA is not truly important in itself, though it gives us the opportunity to apply quantum mechanics to simple examples, which show pedagogically the importance of the commutation relations in this theory.

To analyze the ACA we first reproduce the basic aspects of quantum mechanics that are useful to explain the VM. Then we propose a complete set of commuting observables (CSCO) well adapted to describe the ACA. This set is not discussed in textbooks. It differs from the ordinary set associated with the Clebsch–Gordan coefficients in that the square of the total angular momentum of two particles is replaced by the scalar product defined by Eq. (9). The subsequent analysis of the transformation matrix between the basis functions of this set and the basis functions of the ordinary sets, explains why and when the ACA will give a correct graphical representation of the addition of two angular momenta. Furthermore we show that some of the correct ACA geometrical relations in the example discussed in this article ($j_1=1$, $j_2=1$), are also well-founded in quantum mechanics for any couple j_1, j_2 of quantum numbers. Most important, the discussion demonstrates that the particular commutativity between operators is at the heart of the validity of the graphical representation.

II. VECTOR MODEL

In one of his books, Gasiorowicz (see Ref. 10, p. 293), suggests the impossibility of getting a correct answer when the VM is used to add quantum angular momentum. He uses

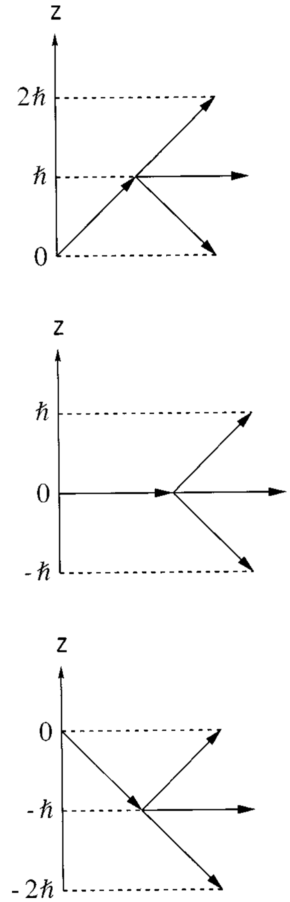

Fig. 1. Nine possible positions for the addition of the angular momentum for two particles with quantum numbers $j_1=1$ and $j_2=1$. The magnitude of each angular momentum is $\sqrt{2}\hbar$ and the value of the z projections are: $\hbar$, 0, and $-\hbar$. According to the VM the value of the angular momentum addition corresponding to a z component equal to $2\hbar$ is $2\sqrt{2}\hbar$; quantum theory predicts a value of $\sqrt{6}\hbar$.

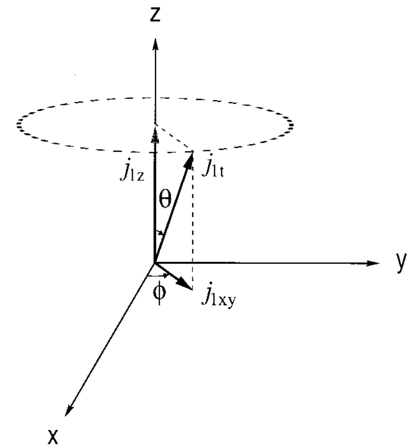

Fig. 2. Vector model for a particle with total angular momentum j_{1t}, z component equal to j_{1z}, and component on the xy plane equal to j_{1xy}.

as an example the addition of two angular momenta with quantum numbers $j_1=1$ and $j_2=1$. According to quantum mechanics each momentum has a value $\hbar\sqrt{j_i(j_i+1)}=\sqrt{2}\hbar$, $i=1,2$, and must have a z component equal to one of the following values: $\hbar$, 0, $-\hbar$; where $\hbar$ is Planck's constant divided by 2π. In Fig. 1 the argument from Gasiorowicz is illustrated: From the nine possible addition arrangements of the two momenta, there is no way to obtain a vector sum with a z component equal to $2\hbar$ and with a length equal to $\hbar\sqrt{2(2+1)}=\sqrt{6}\hbar$, which corresponds to the total angular momentum with a quantum number $j=2$ and z projection equal to $2\hbar$.

Now, we will show that a small change in the VM makes the above statement, in the case discussed by Gasiorowicz, not quite accurate.

Figure 2 shows the so-called vector model for the angular momentum of a particle, which we will call particle 1, with quantum numbers $j_1=1$ and $m_1=1$. The magnitude of the angular momentum $j_{1t}=\hbar\sqrt{j_1(j_1+1)}=\sqrt{2}\hbar$ and its z projection $j_{1z}=\hbar$ are perfectly determined and hence the angle θ. However, the angle ϕ is completely undetermined and sometimes is considered either as a random variable or changing continuously as if the angular momentum were precessing around the z axis. From the same figure it is easy to see that the size of the projection of the angular momentum into the xy plane $j_{1xy}=\sqrt{j_{1t}^2-j_{1z}^2}$ is also a well-defined magnitude, in this case equal to $\hbar$.

In order to add two angular momenta using the vector model, there is no difficulty in calculating the z component because that component is the sum of the two z projections: $j_z=j_{1z}+j_{2z}$; but the situation is less clear for the two components j_{1xy} and j_{2xy}. If their angular positions are not known, how can they be added?

Let us try to answer this question with the following proposition: Add these two components maintaining the angle δ between the j_{1xy} and j_{2xy} projections constant, but still allow them to change their angular position randomly. In this way, the resultant and the two projections have a fixed angular position among them, but note that this proposition does not fix the angle ϕ either for the individual projections or for the resultant one.

Later on we will show that this angle δ is well defined in quantum mechanics. But before doing this, let us figure out what angle this is.

Quantum mechanics predicts a resultant angular momentum $j_t=\hbar\sqrt{j(j+1)}=\sqrt{6}\hbar$ when the resultant z component is $j_z=2\hbar$ and the individual angular momenta have quantum numbers $j_1=1$ and $j_2=1$. To fit the VM to this quantum prediction, angle δ must be 90°, as shown in Fig. 3. For δ=90° the resultant values for the angular momenta are: $j_{xy}=\sqrt{2}\hbar$, $j_z=2\hbar$, and $j_t=\hbar\sqrt{j(j+1)}=\sqrt{6}\hbar$, which correspond to the quantum numbers: $j=2$ and $m=2$.

Figure 4 shows another view of the vectors in Fig. 3. The vector for the angular momentum of the first particle (on the zy plane) is the same as in Fig. 1; but now the vector for the angular momentum of the second particle can move on the surface of the cone generated by its own movement, while it maintains its z component equal to $\hbar$ and the sum of the two z components equal to $2\hbar$. In Fig. 1 the same vector was restricted to lie on the zy plane.

As shown in Fig. 4, there is a position on the cone for the second vector where the addition of the two angular momenta produces a vector sum with a length equal to $\sqrt{6}\hbar$. This unique position on the cone (and its symmetrical realization on the negative side of the x axis) corresponds in Fig. 3 to the angle δ equal to 90°. To leave the angle ϕ of all

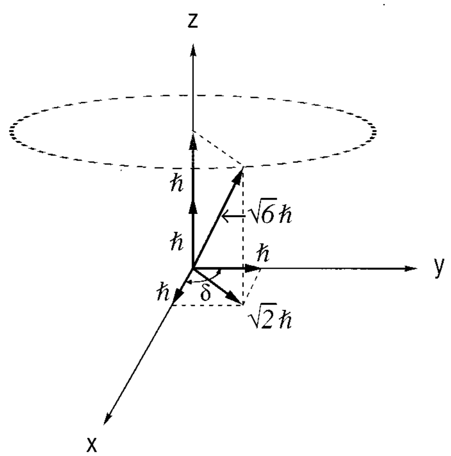

Fig. 3. ACA addition of the angular momentum for two particles with quantum numbers $j_1=1$, $j_2=1$, $m_1=1$, and $m_2=1$. The resultant angular momentum is $j_t=\sqrt{6}\hbar$ with z component $j_z=2\hbar$ and xy component $j_{xy}=\sqrt{2}\hbar$. The components $j_{xy}=\sqrt{2}\hbar$, $j_{1xy}=\hbar$, and $j_{2xy}=\hbar$ can have any arbitrary direction but the relative angular position between them must remain fixed. The angle between j_{1xy} and j_{2xy} is $\delta=90°$.

vectors undetermined, the whole figure can rotate randomly around the z axis. The triangle shown in Fig. 4 is the same as that obtained by addition with the usual vector model, but note that all z projections in the figure are fixed since it is not permitted that individual angular momentum vectors rotate around the resultant.

At this stage, the failure of the VM to pass Gasiorowicz's test can be understood: It is produced by the artificial constriction imposed on the two angular momentum vectors to lie on the same plane ($\delta=0$). We name our procedure to add angular momenta ''angular coupled addition'' because δ is a variable that links the angular momentum xy projections.

Now let us see if by using ACA we may draw all the vectorial diagrams that correspond to the case $j_1=1$ and $j_2=1$.

The addition of two angular momenta with quantum numbers $j_1=1$ and $j_2=1$ can give a total angular momenta equal to $\sqrt{6}\hbar$, $\sqrt{2}\hbar$, or 0; with z components for the first: $2\hbar$, $\hbar$, 0, $-\hbar$, and $-2\hbar$; for the second: $\hbar$, 0, $-\hbar$, and 0 for the last,

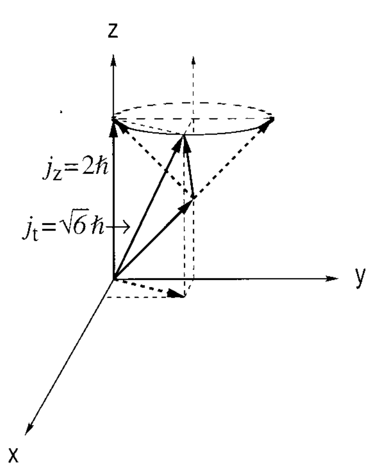

Fig. 4. ACA addition of two angular momenta with quantum numbers $j_1=1$ and $j_2=1$.

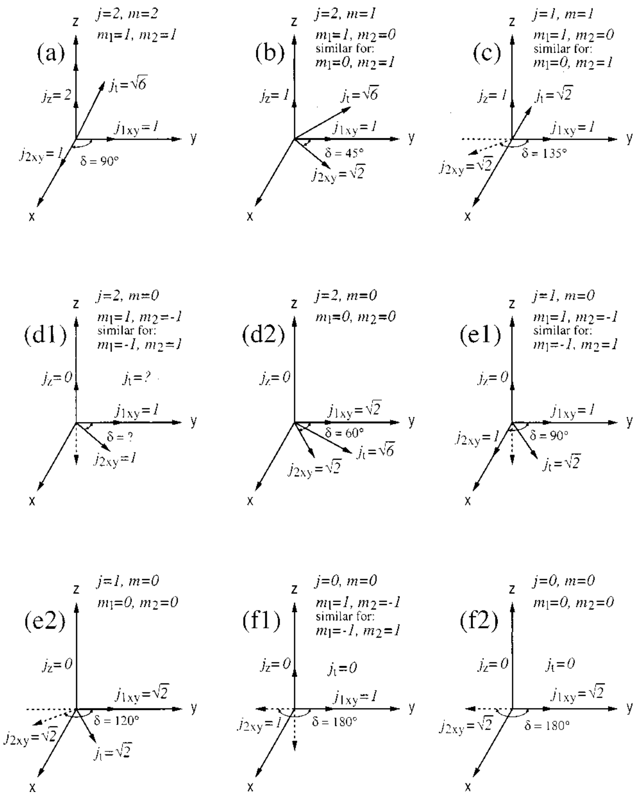

Fig. 5. ACA applied to six of the nine possible combinations for the addition of the angular momenta of two particles with quantum numbers $j_1=1$ and $j_2=1$. The remaining three would correspond to $j=2$, $m=-2$; $j=2$, $m=-1$ and $j=1$, $m=-1$. All lengths are in units of $\hbar$.

respectively. (See Gasiorowicz's book (Ref. 10), or any other reliable text on modern physics or quantum mechanics.)

In Fig. 5, six of the nine cases are shown indicating the quantum numbers m_1 and m_2 that correspond to the z components $m_1\hbar$ and $m_2\hbar$ of the angular momenta of each particle. The three remaining cases not shown (corresponding to $m=-2$ and $m=-1$), are the negative z component version of some of the above. The well-defined j_{xy} component for each particle (named j_{1xy} and j_{2xy}) is also shown. The quantum numbers j and m for the total angular momentum and the length $j_t=\hbar\sqrt{j(j+1)}=(j_z^2+j_{xy}^2)^{1/2}$ of the resultant momentum are shown for each case. The value of the total z component j_z in terms of the individual components is simply $j_{1z}+j_{2z}$. All lengths of the components are in units of $\hbar$.

In all figures the ACA is applied in order to find the angle δ that matches the results with the quantum theory. As indicated in the figures, in all cases it was possible to find this angle, except the case in Fig. 5(d1), which corresponds to $j=2$, $m=0$, $m_1=1$, and $m_2=-1$. Note that Fig. 5(a) corresponds to Gasiorowicz's case already discussed.

III. THE QUANTUM BASIS OF THE VECTOR MODEL

The physical variables like angular momentum are represented as operators in quantum mechanics and their properties are expressed in terms of relations among them. But what we wish to represent geometrically are relations among eigenvalues, not among operators. The eigenvalues are the quantities which may be measured from a physical system.

Let us begin our discussion by remembering that the noncommutativity of the Cartesian angular momentum components $\hat{\mathbf{J}}_x$, $\hat{\mathbf{J}}_y$, and $\hat{\mathbf{J}}_z$ (boldface with caret means operator), i.e.,

$$[\hat{\mathbf{J}}_x,\hat{\mathbf{J}}_y]=i\hbar\hat{\mathbf{J}}_z,$$
$$[\hat{\mathbf{J}}_y,\hat{\mathbf{J}}_z]=i\hbar\hat{\mathbf{J}}_x, \quad (1)$$
$$[\hat{\mathbf{J}}_z,\hat{\mathbf{J}}_x]=i\hbar\hat{\mathbf{J}}_y,$$

implies that no common eigenfunctions can be found for two of them. The immediate consequence of this fact is that it is not possible to obtain from the operator equation (the arrow means vector)

$$\hat{\vec{\mathbf{J}}}=\mathbf{i}\hat{\mathbf{J}}_x+\mathbf{j}\hat{\mathbf{J}}_y+\mathbf{k}\hat{\mathbf{J}}_z, \quad (2)$$

the equation relating eigenvalues (ERE): $\vec{J}=\mathbf{i}J_x+\mathbf{j}J_y+\mathbf{k}J_z$, where J_α with $\alpha=x,y,z$ would be the eigenvalues of $\hat{\mathbf{J}}_\alpha$ and $\vec{J}$ the eigenvalue of $\hat{\vec{\mathbf{J}}}$; $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the unit vectors. Since if it were possible to get a common eigenfunction $\Phi\neq 0$ for all $\hat{\mathbf{J}}_\alpha$, then Eq. (2) applied to this function would give the above ERE. Similarly, it is not possible to obtain from the operator equation

$$\hat{\vec{\mathbf{J}}}\cdot\hat{\vec{\mathbf{J}}}=\hat{\mathbf{J}}^2=\hat{\mathbf{J}}_x^2+\hat{\mathbf{J}}_y^2+\hat{\mathbf{J}}_z^2 \quad (3)$$

the corresponding ERE: $J^2=J_x^2+J_y^2+J_z^2$.

The above considerations are a way to understand why the eigenvalue of angular momentum cannot be represented as a vector in quantum mechanics.

In general one can say that it is possible to write the ERE $f(P_\beta)=0$, with $\beta=1,2,...$, and P_β the corresponding eigenvalue of $\hat{\mathbf{P}}_\beta$, such that it preserves the form of the operator equation: $f(\hat{\mathbf{P}}_\beta)=0$, if the operators $\hat{\mathbf{P}}_\beta$ have a common eigenfunction. Equations (2) and (3) are examples in which there are no corresponding ERE. On the contrary, when quantum operators commute, the ERE is not forbidden, since a common eigenfunction for them may exist.

For instance, let us analyze the following operator equations:

$$\hat{\vec{\mathbf{J}}}_{xy}=\mathbf{i}\hat{\mathbf{J}}_x+\mathbf{j}\hat{\mathbf{J}}_y, \quad (4)$$
$$\hat{\mathbf{J}}^2=\hat{\mathbf{J}}_{xy}^2+\hat{\mathbf{J}}_z^2, \quad (5)$$

where $\hat{\mathbf{J}}_{xy}^2=\hat{\mathbf{J}}_x^2+\hat{\mathbf{J}}_y^2$.

Since $\hat{\mathbf{J}}_x$ and $\hat{\mathbf{J}}_y$ do not commute, the first operator equation does not admit the ERE $\vec{J}_{xy}=\mathbf{i}J_x+\mathbf{j}J_y$ and the angle $\phi=\tan^{-1}(J_y/J_x)$ is undetermined.

To analyze Eq. (5) let us remember that the operators $\hat{\mathbf{J}}^2$ and $\hat{\mathbf{J}}_z$ commute, and have the eigenfunctions ψ_{jm} that satisfy the eigenvalue equations

$$\hat{\mathbf{J}}^2\psi_{jm}=j(j+1)\hbar^2\psi_{jm}, \quad \hat{\mathbf{J}}_z\psi_{jm}=m\hbar\psi_{jm},$$

with eigenvalues $j(j+1)\hbar^2$ and $m\hbar$. j and m are integers or half integers and $m=j,j-1,...,-j$.

Since $\hat{\mathbf{J}}^2$ and $\hat{\mathbf{J}}_z^2$ commute, any two operators in Eq. (5) also commute. It is easy to show that any common eigenfunction of those operators is also an eigenfunction of the operator $\hat{\mathbf{J}}_{xy}^2$. Indeed, denoting $\hat{J}_{xy}^2$ as the eigenvalue of $\hat{\mathbf{J}}_{xy}^2$ and applying the operator Eq. (5) to one of these eigenfunctions, ψ_{jm}, the following eigenvalue equation is obtained:

$$\hat{\mathbf{J}}_{xy}^2\psi_{jm}=(J^2-J_z^2)\psi_{jm},$$

and hence the ERE

$$J^2=J_{xy}^2+J_z^2. \quad (6)$$

Because it will be useful later, let us generalize the preceding result, recalling that any operator expressed as an addition of commuting operators also commute with them, and any common eigenfunction of them is also an eigenfunction of the former.

Interpreting Eq. (6) as the Pythagorean theorem, its graphical representation is a triangle. This triangle is that of Fig. 2, since J^2 corresponds to j_{1t}^2, J_{xy}^2 to j_{1xy}^2, J_z^2 to j_{1z}^2, and $\cos\theta=J_z/J$. Note that Eq. (6) is obtained from a quantum consideration; therefore it is a quantum equation that has the same form as the classical one.

To summarize the above discussion we can say that the possibility of drawing a classical-like geometrical diagram, as that of the Fig. 2, is derived from the commutativity of its associated quantum operators. In other words, the triangle with angle θ in Fig. 2 can be considered as a correct quantum diagram since it was obtained from a quantum equation. In Sec. IV, we will extend this consideration to the addition of angular momenta.

IV. ANGULAR MOMENTA ADDITION WITH THE ACA

For our study of the ACA we are interested in a system composed of two independent angular momenta, like the orbital angular momentum of particle 1 and that of particle 2, or like the angular momentum of a particle and its spin. For the purposes of this discussion, we use the two particle system, it being understood that the conclusions apply to both systems.

The operator associated with the total angular momentum is

$$\hat{\vec{\mathbf{J}}}=\hat{\vec{\mathbf{J}}}_1+\hat{\vec{\mathbf{J}}}_2,$$

where $\hat{\vec{\mathbf{J}}}_1$ and $\hat{\vec{\mathbf{J}}}_2$ are the angular momentum operators associated with particle 1 and particle 2, respectively. Expressing the operators of the previous equation in terms of their Cartesian components, we get for the z component $\hat{\mathbf{J}}_z=\hat{\mathbf{J}}_{1z}+\hat{\mathbf{J}}_{2z}$.

The four commuting operators $\hat{\mathbf{J}}_\gamma^2$ and $\hat{\mathbf{J}}_{\gamma z}$, where $\gamma=1,2$, satisfy the eigenvalue equations

$$\hat{\mathbf{J}}_1^2\psi_{j_1m_1}=j_1(j_1+1)\hbar^2\psi_{j_1m_1}, \quad \hat{\mathbf{J}}_{1z}\psi_{j_1m_1}=m_1\hbar\psi_{j_1m_1},$$
$$\hat{\mathbf{J}}_2^2\psi_{j_2m_2}=j_2(j_2+1)\hbar^2\psi_{j_2m_2}, \quad \hat{\mathbf{J}}_{2z}\psi_{j_2m_2}=m_2\hbar\psi_{j_2m_2};$$

where j_γ and m_γ are integers or half integers and $m_\gamma=j_\gamma,j_\gamma-1,...,-j_\gamma$. The system of two particles can be represented by a linear combination of products of the eigenfunctions, which are, $\psi_{j_1m_1}\psi_{j_2m_2}$. This set of functions forms a basis set which is called the uncoupled representation.

If, instead, one measures the square and the z component of the total angular momentum, and the squared magnitude of each individual angular momentum, one gets the eigenvalues of the four commuting operators $\hat{\mathbf{J}}^2=(\hat{\vec{\mathbf{J}}}_1+\hat{\vec{\mathbf{J}}}_2)^2$, $\hat{\mathbf{J}}_z$, $\hat{\mathbf{J}}_1^2$, and $\hat{\mathbf{J}}_2^2$. The eigenvalue equations for the first two operators are

$$\hat{\mathbf{J}}^2\psi_{jmj_1j_2}=j(j+1)\hbar^2\psi_{jmj_1j_2},$$

$$\hat{\mathbf{J}}_z\psi_{jmj_1j_2}=m\hbar\psi_{jmj_1j_2},$$

where $j=j_1+j_2,j_1+j_2-1,...,|j_1-j_2|$, and $m=-j,\dots,+j$. Their eigenfunctions form a basis set called the coupled representation. It is important to remember that $\hat{\mathbf{J}}_{1z}$ and $\hat{\mathbf{J}}_{2z}$ do not commute with $\hat{\mathbf{J}}^2$.

$\hat{\mathbf{J}}^2$ can be expressed as

$$\hat{\mathbf{J}}^2=\hat{\mathbf{J}}_1^2+\hat{\mathbf{J}}_2^2+2\hat{\vec{\mathbf{J}}}_1\cdot\hat{\vec{\mathbf{J}}}_2. \tag{7}$$

According to discussion in the former section, the operator $2\hat{\vec{\mathbf{J}}}_1\cdot\hat{\vec{\mathbf{J}}}_2$ has common eigenfunctions with the operators $\hat{\mathbf{J}}^2$, $\hat{\mathbf{J}}_1^2$, and $\hat{\mathbf{J}}_2^2$. Then, it is possible to write the following ERE:

$$J^2=J_1^2+J_2^2+2(\hat{\vec{\mathbf{J}}}_1\cdot\hat{\vec{\mathbf{J}}}_2)_e, \tag{8}$$

where $(\hat{\vec{\mathbf{J}}}_1\cdot\hat{\vec{\mathbf{J}}}_2)_e$ is the eigenvalue of the operator in parentheses. This equation admits a geometrical interpretation as the law of the cosines by noting that $(\hat{\vec{\mathbf{J}}}_1\cdot\hat{\vec{\mathbf{J}}}_2)_e$ can be written as $J_1J_2\cos\gamma$.

The graphical representation of Eq. (8) is the well-known triangle used to add angular momenta, i.e., the vector addition model. Like Eq. (6), Eq. (8) is also obtained from a quantum consideration, therefore the geometrical representation of the involved quantum variables is a correct one.

Now let us focus our attention on Fig. 3. The eigenvalues represented in this figure are J^2, J_{1xy}, J_{2xy}, J_{1z}, J_{2z}, and the angle δ. Using Eq. (4), and defining the operator $\hat{\mathbf{J}}_{\cdot xy}$ as

$$\hat{\mathbf{J}}_{\cdot xy}=\hat{\vec{\mathbf{J}}}_{1xy}\cdot\hat{\vec{\mathbf{J}}}_{2xy}=\hat{\mathbf{J}}_{1x}\hat{\mathbf{J}}_{2x}+\hat{\mathbf{J}}_{1y}\hat{\mathbf{J}}_{2y}, \tag{9}$$

which is related to the angle δ of ACA, $\hat{\mathbf{J}}^2$ can be written as

$$\begin{aligned}\hat{\mathbf{J}}^2&=(\hat{\vec{\mathbf{J}}}_{1xy}+\mathbf{k}\hat{\mathbf{J}}_{1z}+\hat{\vec{\mathbf{J}}}_{2xy}+\mathbf{k}\hat{\mathbf{J}}_{2z})^2\\&=\hat{\mathbf{J}}_{1xy}^2+\hat{\mathbf{J}}_{2xy}^2+2\hat{\mathbf{J}}_{\cdot xy}+\hat{\mathbf{J}}_{1z}^2+\hat{\mathbf{J}}_{2z}^2+2\hat{\mathbf{J}}_{1z}\hat{\mathbf{J}}_{2z}.\end{aligned} \tag{10}$$

We note that this operator equation would involve all the mentioned eigenvalues if a common eigenfunction were to exist for operators in Eq. (10). Since $\hat{\mathbf{J}}^2$ does not commute with all the operators to the right, it will not be possible in general to write the corresponding ERE,

$$J^2=J_{1xy}^2+J_{2xy}^2+2(\hat{\mathbf{J}}_{\cdot xy})_e+J_{1z}^2+J_{2z}^2+2J_{1z}J_{2z}, \tag{11}$$

of which the graphical representation is Fig. 3, where $(\hat{\mathbf{J}}_{\cdot xy})_e$ is the eigenvalue of the operator in parentheses. Then, apparently, the ACA as shown in Fig. 3 is not a correct graphical representation. To show that this is not the case, and to explain Figs. 3 and 5, we need first to describe the operator $\hat{\mathbf{J}}_{\cdot xy}$.

As shown in the Appendix, this operator and $\hat{\mathbf{J}}_z$, $\hat{\mathbf{J}}_1^2$, and $\hat{\mathbf{J}}_2^2$, form a set of four commuting operators. We write the eigenvalue equation of the new operator as

$$\hat{\mathbf{J}}_{\cdot xy}\psi_{J_{\cdot xy}mj_1j_2}=J_{\cdot xy}\hbar^2\psi_{J_{\cdot xy}mj_1j_2}, \tag{12}$$

with $(\hat{\mathbf{J}}_{\cdot xy})_e=J_{\cdot xy}\hbar^2$.

In units of $\hbar$ the eigenvalues for their nine common eigenfunctions, when $j_1=1$ and $j_2=1$, are $J_{\cdot xy}=-1,-1,0,0,0,1,1,-\sqrt{2},\sqrt{2}$. The eigenfunctions form a basis set named ''angular coupled representation.'' As we will see later, this basis set is well adapted to discuss the angle δ. To make the connection between quantum mechanics and the ACA simpler, in what follows we will restrict the discussion to the case when $j_1=1$ and $j_2=1$, as in the previous figures. Note that these two quantum numbers are common to all three representations.

As we have three basis sets of eigenfunctions, any function taken from one set can be expanded as a linear combination of the functions taken from another set. We show the matrices that connect the three possible pairs of basis sets in Tables I, II, and III (see the appendix). From Tables I to III we can expand any coupled state in terms of uncoupled and angular coupled states, and from Table IV we can calculate the result of applying the operator $\hat{\mathbf{J}}_{\cdot xy}$ to any uncoupled state.

With these tools we proceed to analyze Fig. 5. We adopt the notation $|n_1,n_2\rangle_s$ for the state vectors, where $s=u,c,a$; $n_1=m_1,j,J_{\cdot xy}$; $n_2=m_2,m,m$, for the uncoupled, coupled, and angular coupled representations, respectively. For the next discussion, eigenvalues will be in units of $\hbar$.

Let us consider in detail Fig. 3 [Figs. 4 and 5(a) show the same case], which represents the case $j=2$, $m=2$. From the tables we can see that the expansion of the coupled state $|2,2\rangle_c$ in any of the other two bases reduces to only one term,

$$|2,2\rangle_c=|1,1\rangle_u=|0,2\rangle_a.$$

This state is a common eigenfunction for the three representations and consequently all quantum numbers for the three representations are well defined, allowing us to draw j_{1z}, j_{1xy}, j_{2z}, j_{2xy}, j_z, j_{xy}, and j_t, as was done in Figs. 3, 4, and 5(a). Since there is a common eigenfunction for all the operators in Eq. (10), Eq. (11) is a correct relation. We can understand this by noting that although operators in Eq. (10) do not commute in general they commute for this particular state. Once again the ERE allows us to find the graphical representation, since Eq. (11) describes the relations between the sides of the triangles in Fig. 3.

When discussing the vector model, textbooks normally present a diagram where the vectors of the individual angular momenta are precessing around the resultant vector (see, for instance, Zare, Ref. 3, p. 52). This precession is introduced to have an indeterminacy of the individual z components. Note that for the particular case just discussed, the diagram in Zare's book would be incorrect because the z projections of the angular momentum vectors of each particle are well defined. If it is desired to show graphically the indeterminacy in the x and y components, it is the whole Fig. 3 (or Fig. 4) that may rotate around the z axis.

What remains is to calculate the angle between the xy plane components, which can be obtained using the value $J_{\cdot xy}=0$. Indeed, since

$$\hat{\mathbf{J}}_{\cdot xy}|2,2\rangle_c=\hat{\mathbf{J}}_{\cdot xy}|1,1\rangle_u=\hat{\mathbf{J}}_{\cdot xy}|0,2\rangle_a=0|0,2\rangle_a,$$

Table I. Transformation matrix between the uncoupled basis and the coupled basis.

		$m_1 m_2$								
j	m	1 1	1 0	0 1	1 −1	0 0	−1 1	−1 0	0 −1	−1 −1
2	2	1	0	0	0	0	0	0	0	0
2	1	0	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0	0	0	0	0	0
1	1	0	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0	0	0	0	0	0
2	0	0	0	0	$\frac{1}{\sqrt{6}}$	$\sqrt{\frac{2}{3}}$	$\frac{1}{\sqrt{6}}$	0	0	0
1	0	0	0	0	$\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$	0	0	0
0	0	0	0	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	0	0	0
1	−1	0	0	0	0	0	0	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0
2	−1	0	0	0	0	0	0	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0
2	−1	0	0	0	0	0	0	0	0	1

and noting that for the state $|1,1\rangle_u$, $J_{1z}=J_{2z}=1$, then using Eq. (6) with $J_1^2=J_2^2=2$ we obtain $J_{1xy}=J_{2xy}=1$. Writing the eigenvalue

$$J_{\cdot xy}=J_{1xy}J_{2xy}\cos\delta,$$

then $J_{1xy}J_{2xy}\cos\delta=0$ and we obtain that $\delta=\pi/2$, which is in agreement with Fig. 5(a).

Now let us analyze the case illustrated in Fig. 5(b), which corresponds to $j=2$, $m=1$. We have

$$|2,1\rangle_c=\frac{1}{\sqrt{2}}|1,0\rangle_u+\frac{1}{\sqrt{2}}|0,1\rangle_u=|1,1\rangle_a. \tag{13}$$

Neither the z projections nor the xy projections of the individual angular momenta are well defined in this case. On the contrary, the product $\hat{\mathbf{J}}_{\cdot xy}$ is well defined with eigenvalue equal to one, as can be seen in Table III.

Rewriting Eq. (10) as

Table II. Transformation matrix between the angular coupled basis and the uncoupled basis.

		$J_{\cdot xy}m$								
m_1	m_2	−1 −1	−1 1	0 −2	0 0	0 2	1 −1	1 1	$-\sqrt{2}$ 0	$\sqrt{2}$ 0
1	1	0	0	0	0	1	0	0	0	0
1	0	0	$-\frac{1}{\sqrt{2}}$	0	0	0	0	$\frac{1}{\sqrt{2}}$	0	0
0	1	0	$\frac{1}{\sqrt{2}}$	0	0	0	0	$\frac{1}{\sqrt{2}}$	0	0
1	−1	0	0	0	$-\frac{1}{\sqrt{2}}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$
0	0	0	0	0	0	0	0	0	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
−1	1	0	0	0	$\frac{1}{\sqrt{2}}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$
−1	0	$-\frac{1}{\sqrt{2}}$	0	0	0	0	$\frac{1}{\sqrt{2}}$	0	0	0
0	−1	$\frac{1}{\sqrt{2}}$	0	0	0	0	$\frac{1}{\sqrt{2}}$	0	0	0
−1	−1	0	0	1	0	0	0	0	0	0

Table III. Transformation matrix between the angular coupled basis and the coupled basis.

		$J_{\cdot xy} m$								
j	m	$-1\ -1$	$-1\ 1$	$0\ -2$	$0\ 0$	$0\ 2$	$1\ -1$	$1\ 1$	$-\sqrt{2}\ 0$	$\sqrt{2}\ 0$
2	2	0	0	0	0	1	0	0	0	0
2	1	0	0	0	0	0	0	1	0	0
1	1	0	1	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	$-\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{6}}$	$\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{6}}$
1	0	0	0	0	-1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	$\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{6}}$	$\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{6}}$
1	-1	1	0	0	0	0	0	0	0	0
2	-1	0	0	0	0	0	1	0	0	0
2	-2	0	0	1	0	0	0	0	0	0

$$\hat{\mathbf{J}}^2 = (\hat{\mathbf{J}}_{1xy}^2 + \hat{\mathbf{J}}_{2xy}^2) + 2\hat{\mathbf{J}}_{\cdot xy} + \hat{\mathbf{J}}_z^2, \tag{14}$$

and considering the operators grouped by parentheses as one operator, then we note that the state in Eq. (13) is a common eigenfunction of all the operators in the above equation, making it possible to write the ERE

$$J^2 = (\hat{\mathbf{J}}_{1xy}^2 + \hat{\mathbf{J}}_{2xy}^2)_e + 2J_{\cdot xy} + J_z^2, \tag{15}$$

with $(\hat{\mathbf{J}}_{1xy}^2 + \hat{\mathbf{J}}_{2xy}^2)_e$ the eigenvalue of the operator in parentheses.

Indeed, remembering that

$$\hat{\mathbf{J}}_{1xy}^2 = (\hat{\mathbf{J}}_1^2 - \hat{\mathbf{J}}_{1z}^2), \quad \hat{\mathbf{J}}_{2xy}^2 = (\hat{\mathbf{J}}_2^2 - \hat{\mathbf{J}}_{2z}^2), \tag{16}$$

then

$$(\hat{\mathbf{J}}_{1xy}^2 + \hat{\mathbf{J}}_{2xy}^2)\left(\frac{1}{\sqrt{2}}|1,0\rangle_u + \frac{1}{\sqrt{2}}|0,1\rangle_u\right) = 3\left(\frac{1}{\sqrt{2}}|1,0\rangle_u + \frac{1}{\sqrt{2}}|0,1\rangle_u\right).$$

Also note that (it will be useful later)

$$\hat{\mathbf{J}}_{1xy}^2 \hat{\mathbf{J}}_{2xy}^2\left(\frac{1}{\sqrt{2}}|1,0\rangle_u + \frac{1}{\sqrt{2}}|0,1\rangle_u\right) = 2\left(\frac{1}{\sqrt{2}}|1,0\rangle_u + \frac{1}{\sqrt{2}}|0,1\rangle_u\right).$$

Therefore the state also has a well-defined product $(\hat{\mathbf{J}}_{1xy}^2 \hat{\mathbf{J}}_{2xy}^2)_e = 2$ and sum $(\hat{\mathbf{J}}_{1xy}^2 + \hat{\mathbf{J}}_{2xy}^2)_e = 3$. Writing the eigenvalue

$$J_{\cdot xy} = \sqrt{(\hat{\mathbf{J}}_{1xy}^2 \hat{\mathbf{J}}_{2xy}^2)_e} \cos\delta = 1,$$

we can infer that $\delta = 45°$. This is the value found by ACA which is shown in Fig. 5(b).

Note that the eigenvalues J_{1xy} and J_{2xy} are not well defined. This is expected since J_{1z} and J_{2z} are not well defined, as can be seen from the state in Eq. (13). We may interpret this indefiniteness by writing the above eigenvalues in terms of J_{1xy} and J_{2xy} to get the simultaneous equations $J_{1xy}^2 + J_{2xy}^2 = 3$, $J_{1xy}^2 J_{2xy}^2 = 2$. What we found is that there is no unique solution, but two of them: for the value $J_{1xy} = 1$, corresponds a value of $J_{2xy} = \sqrt{2}$; the other solution corresponds to the inverted values. With these numbers it is not difficult to see that the ERE in Eq. (15) has its graphical representation in Fig. 5(b).

The analysis for Fig. 5(c) follows a similar procedure. Figure 5(e1) and (e2) corresponds to the case $j=1$, $m=0$. This more complicated case is simplified upon observing in Table I that the coefficient for the state $|0,0\rangle_u$ is zero. Then, discarding Fig. 5(e2), the analysis becomes similar to that of Fig. 5(b) and (c).

What about Fig. 5(d1) and (d2) for $j=2$, $m=0$? From Tables I and III, the expansion of the coupled state gives us

Table IV. Matrix elements of $\hat{\mathbf{J}}_{\cdot xy}$ between uncoupled states.

		$m_1 m_2$								
m_1	m_2	$1\ 1$	$1\ 0$	$0\ 1$	$1\ -1$	$0\ 0$	$-1\ 1$	$-1\ 0$	$0\ -1$	$-1\ -1$
1	1	0	0	0	0	0	0	0	0	0
1	0	0	0	1	0	0	0	0	0	0
0	1	0	1	0	0	0	0	0	0	0
1	-1	0	0	0	0	1	0	0	0	0
0	0	0	0	0	1	0	1	0	0	0
-1	1	0	0	0	0	1	0	0	0	0
-1	0	0	0	0	0	0	0	0	1	0
0	-1	0	0	0	0	0	0	1	0	0
-1	-1	0	0	0	0	0	0	0	0	0

$$|2,0\rangle_c=\frac{1}{\sqrt{6}}|1,-1\rangle_u+\sqrt{\frac{2}{3}}|0,0\rangle_u+\frac{1}{\sqrt{6}}|-1,1\rangle_u$$
$$=\left(\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{6}}\right)|\sqrt{2},0\rangle_a+\left(\frac{-1}{\sqrt{3}}+\frac{1}{\sqrt{6}}\right)|-\sqrt{2},0\rangle_a.$$

For this case neither the scalar product $\hat{\mathbf{J}}_{\cdot xy}$ nor the individual z components are well defined. The former magnitude can have the values $\pm\sqrt{2}$, and the latter 1, 0, and -1. This explains that we cannot find a picture for the (d1) case and shows that Fig. 5(d2) is incorrect. A similar analysis can be made for Fig. 5(f1) and (f2).

For these cases, the ACA loses its use and we cannot draw a picture. Even in the cases when a drawing can be made [Fig. 5(e2), (d2), (f1), and (f2)], it is meaningless.

V. DISCUSSION

In Sec. IV it was shown that the cases where the ACA gives a correct graphical representation of the addition of two angular momenta, correspond to the existence of a common eigenfunction of operators that do not commute in general. For instance, the state $|2,2\rangle_c=|1,1\rangle_u=|0,2\rangle_a$ is a common eigenfunction for the noncommutative operators $\hat{\mathbf{J}}^2$, $\hat{\mathbf{J}}_{1z}$, $\hat{\mathbf{J}}_{2z}$, and $\hat{\mathbf{J}}_{\cdot xy}$. Applying the commutators of Eqs. (A4) and (A5) of the Appendix to that state, we obtain for this particular case a null result for each one (note that $\hat{\mathbf{J}}_{1z}$ and $\hat{\mathbf{J}}_{2z}$ always commute). Therefore these operators can have that common eigenfunction, and also for this case the ACA works especially well since all the magnitudes involved are well defined [see Fig. 5(a)]. Gasiorowicz's example is one of these cases.

A generalization of the ACA to other angular momentum quantum numbers can be made by observing that the commutators were canceled because the state $|2,2\rangle_c=|1,1\rangle_u=|0,2\rangle_a$ has extremum values for both quantum numbers m_1 and m_2 ($m_1=m_2=1$ are extremum since $j_1=j_2=1$). Expression (A2) shows that for $m_\alpha=j_\alpha$ the raising operator gives a null value. Similarly for $m_\alpha=-j_\alpha$ the lowering operator also gives a null value. With this consideration, it is not difficult to verify that all those commutators [in Eqs. (A4)–(A7)], are canceled when numbers m_1 and m_2 are both maximum or both minimum. Then those commutators are canceled when they are applied to any state with those extremum values of m_1 and m_2. For those states all eigenvalues of the operator $\hat{\mathbf{J}}_{\cdot xy}$ are null, as can be seen from Eq. (A3). Then expressing these eigenvalues as $J_{1xy}J_{2xy}\cos\delta$ we obtain $\delta=90°$ for any state with extremum values m_1 and m_2. Hence the ACA drawing is correct, from the point of view of quantum mechanics, for any angular momentum with maximum projection.

As an example of the above assertion, the interested reader can easily verify its validity by adding the angular momenta for two spin 1/2 particles, to obtain the states $|+,+\rangle$ or $|-,-\rangle$ of the triplet. For these cases the ACA works well and predicts $\delta=90°$.

With respect to the states $|2,1\rangle_c$, $|1,1\rangle_c$, and $|1,0\rangle_c$ that also admitted a useful drawing, the commutators $[\hat{\mathbf{J}}^2,\hat{\mathbf{J}}_{\cdot xy}]$, $[\hat{\mathbf{J}}^2,\hat{\mathbf{J}}^2_{1xy}\hat{\mathbf{J}}^2_{2xy}]$, $[\hat{\mathbf{J}}_{\cdot xy},\hat{\mathbf{J}}^2_{1xy}\hat{\mathbf{J}}^2_{2xy}]$, $[\hat{\mathbf{J}}_{\cdot xy},\hat{\mathbf{J}}^2_{1xy}+\hat{\mathbf{J}}^2_{2xy}]$, and $[\hat{\mathbf{J}}^2,\hat{\mathbf{J}}^2_{1xy}+\hat{\mathbf{J}}^2_{2xy}]$ are null, but $[\hat{\mathbf{J}}^2,\hat{\mathbf{J}}_{1z}]$, $[\hat{\mathbf{J}}^2,\hat{\mathbf{J}}_{2z}]$ are not null. Whereas for $|2,0\rangle_c$ and $|0,0\rangle_c$ that did not admit a useful drawing, all are different from zero. All these facts clearly show that the possibility of making correct drawings with the ACA depends on the particular commutativity of the operators.

To review some of the results just discussed, let us note that one particular commutativity allows an operator equation to change in an ERE. For instance, when the commutators in Eq. (1) are applied to the eigenfunction with $j_1=j_2=0$, a null result is obtained. Then, the operators in the equation

$$\hat{\mathbf{J}}^2=\hat{\mathbf{J}}^2_{1x}+\hat{\mathbf{J}}^2_{1y}+\hat{\mathbf{J}}^2_{1z}+\hat{\mathbf{J}}^2_{2x}+\hat{\mathbf{J}}^2_{2y}+\hat{\mathbf{J}}^2_{2z}$$
$$+2\hat{\mathbf{J}}_{1x}\hat{\mathbf{J}}_{2x}+2\hat{\mathbf{J}}_{1y}\hat{\mathbf{J}}_{2y}+2\hat{\mathbf{J}}_{1z}\hat{\mathbf{J}}_{2z}, \tag{17}$$

obtained developing Eq. (7), commute between them when applied to that state with $j_1=0=j_2$. Applying both sides of Eq. (17) to this state give us the corresponding ERE,

$$J^2=J^2_{1x}+J^2_{1y}+J^2_{1z}+J^2_{2x}+J^2_{2y}+J^2_{2z}$$
$$+2J_{1x}J_{2x}+2J_{1y}J_{2y}+2J_{1z}J_{2z}. \tag{18}$$

Of course all the numerical values of the above eigenvalues are zero and the picture associated with the latter equation reduces to one point.

Following the same reasoning that we have just applied to the above trivial case, we also realize that the operators $\hat{\mathbf{J}}^2$, $\hat{\mathbf{J}}^2_{1xy}$, $\hat{\mathbf{J}}^2_{2xy}$, $\hat{\mathbf{J}}_{\cdot xy}$, $\hat{\mathbf{J}}^2_{1z}$, $\hat{\mathbf{J}}^2_{2z}$, $\hat{\mathbf{J}}_{1z}$ and $\hat{\mathbf{J}}_{2z}$ commute between them when applied to the states in which quantum numbers m_1 and m_2 are both extremum. If we associate the operators addition $\hat{\mathbf{J}}^2_{1x}+\hat{\mathbf{J}}^2_{1y}$ as $\hat{\mathbf{J}}^2_{1xy}$, $\hat{\mathbf{J}}^2_{2x}+\hat{\mathbf{J}}^2_{2y}$ as $\hat{\mathbf{J}}^2_{2xy}$ and $2\hat{\mathbf{J}}_{1x}\hat{\mathbf{J}}_{2x}+2\hat{\mathbf{J}}_{1y}\hat{\mathbf{J}}_{2y}$ as $2\hat{\mathbf{J}}_{\cdot xy}$ in Eq. (18), then all the operators in the rearranged equation [see Eq. (10)] commute amongst themselves when applied to these states. The picture associated with the corresponding ERE [see Eq. (11)] for the case $j_1=j_2=1$ is shown as Fig. 5(a).

Continuing with this line of reasoning we also demonstrate that the operators $\hat{\mathbf{J}}^2$, $\hat{\mathbf{J}}^2_{1xy}+\hat{\mathbf{J}}^2_{2xy}$, $\hat{\mathbf{J}}_{\cdot xy}$ and $\hat{\mathbf{J}}^2_z$ commute amongst themselves when applied to the state $|1,0\rangle_\mu+|0,1\rangle_\mu$ (of course they also commute when applied to the above-mentioned states with extremum quantum numbers). If we make a new rearrangement of Eq. (18) by associating operators $\hat{\mathbf{J}}^2_{1xy}+\hat{\mathbf{J}}^2_{2xy}$ and $\hat{\mathbf{J}}^2_{1z}+\hat{\mathbf{J}}^2_{2z}+2\hat{\mathbf{J}}_{1z}\hat{\mathbf{J}}_{2z}$ as $\hat{\mathbf{J}}^2_z$, we get the operator Eq. (14) and its corresponding ERE, Eq. (15). The geometrical interpretation of this equation is shown in Fig. 5(b). A similar analysis can be made for the case shown in Fig. 5(c).

Finally we recall that the operators $\hat{\mathbf{J}}^2$, $\hat{\mathbf{J}}^2_{xy}$, and $\hat{\mathbf{J}}^2_z$ commute for states with any quantum numbers. A new rearrangement of Eq. (18) by adding $\hat{\mathbf{J}}^2_{1xy}+\hat{\mathbf{J}}^2_{2xy}$ and $2\hat{\mathbf{J}}_{\cdot xy}$ as $\hat{\mathbf{J}}^2_{xy}$ leads us to the operator Eq. (5) and its corresponding ERE, Eq. (6), whose graphical representation is Fig. 2. Similarly, a rearrangement of Eq. (18) produces the operator Eq. (7) which has its corresponding ERE Eq. (8), whose graphical representation is just the vector addition model.

VI. SUMMARY AND CONCLUSIONS

In this article we first generate some empirical graphical representations of angular momentum relations. These representations lead us to define the new operator $\hat{\mathbf{J}}_{\cdot xy}$. Then we construct a CSCO with this and the operators $\hat{\mathbf{J}}_z$, $\hat{\mathbf{J}}^2_1$, $\hat{\mathbf{J}}^2_2$.

With this set we were able to show that the angle δ which emerged from the empirical analysis is a physical variable in quantum mechanics. Finally we show that particular commutativity of the operators allows classical-like graphical representations of the addition of quantum angular momenta.

In addition, it is shown that the classical-like graphical representations, or pictures as they are usually called, of the quantum angular momentum are diagrams with a limited validity. However, in this article we find that they give a correct representation for some specific cases, which are analyzed in this work.

The discussion allows an understanding of the origin of the vector model and, what is more important, leads to a full appreciation of the power of the theory of angular momentum. Besides, we think that with the elements provided through the development of this work, the ambiguous phrases used in textbooks referring to the inadequacy of the vector model could be substituted by a more precise description.

Furthermore, the analysis of the ACA made in this work shows vividly the fundamental role played by the commutation relations in quantum mechanics; it also shows that no vector model can represent all the relations among angular momentum variables as quantum theory does.

We believe that this analysis gives students some new additional elements to have a better appreciation of the important role played by the commutation relations in the quantum theory, and also to gain a better insight into the elementary angular momentum theory.

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APPENDIX

In this Appendix we calculate the commutation relationships among the set of operators that generate the angular coupled basis as well as the transformation matrices between different bases for $j_1=1$ and $j_2=1$. All results are expressed in units of $\hbar$.

As the operator $\hat{\mathbf{J}}_{\cdot xy}$ can be written equivalently as $\hat{\mathbf{J}}_{1x}\hat{\mathbf{J}}_{2x}+\hat{\mathbf{J}}_{1y}\hat{\mathbf{J}}_{2y}$ and the commutation relations between the Cartesian components of $\hat{\mathbf{J}}_\alpha$ and $\hat{\mathbf{J}}_\alpha^2$ (see Cohen-Tannoudji *et al.*, Ref. 1),

$$[\hat{\mathbf{J}}_\alpha^2,\hat{\mathbf{J}}_{\alpha\beta}]=0,$$

where $\alpha=1,2$ and $\beta=x,y,z$; then the commutators

$$\begin{aligned}[\hat{\mathbf{J}}_{\cdot xy},\hat{\mathbf{J}}_\alpha^2]&=[\hat{\mathbf{J}}_{1x}\hat{\mathbf{J}}_{2x}+\hat{\mathbf{J}}_{1y}\hat{\mathbf{J}}_{2y},\hat{\mathbf{J}}_\alpha^2]\\&=[[\hat{\mathbf{J}}_{1x},\hat{\mathbf{J}}_\alpha^2]\hat{\mathbf{J}}_{2x}+[\hat{\mathbf{J}}_{1y},\hat{\mathbf{J}}_\alpha^2]\hat{\mathbf{J}}_{2y}]=0;\end{aligned}$$

and

$$\begin{aligned}[\hat{\mathbf{J}}_{\cdot xy},\hat{\mathbf{J}}_z]&=[\hat{\mathbf{J}}_{1x}\hat{\mathbf{J}}_{2x}+\hat{\mathbf{J}}_{1y}\hat{\mathbf{J}}_{2y},\hat{\mathbf{J}}_{1z}+\hat{\mathbf{J}}_{2z}]\\&=\hat{\mathbf{J}}_{1x}[\hat{\mathbf{J}}_{2x},\hat{\mathbf{J}}_{2z}]+[\hat{\mathbf{J}}_{1x},\hat{\mathbf{J}}_{1z}]\hat{\mathbf{J}}_{2x}\\&\quad+\hat{\mathbf{J}}_{1y}[\hat{\mathbf{J}}_{2y},\hat{\mathbf{J}}_{2z}]+[\hat{\mathbf{J}}_{1y},\hat{\mathbf{J}}_{1z}]\hat{\mathbf{J}}_{2y}\\&=-\hat{\mathbf{J}}_{1x}\hat{\mathbf{J}}_{2y}-\hat{\mathbf{J}}_{1y}\hat{\mathbf{J}}_{2x}+\hat{\mathbf{J}}_{1y}\hat{\mathbf{J}}_{2x}+\hat{\mathbf{J}}_{1x}\hat{\mathbf{J}}_{2y}=0.\end{aligned}$$

Note that the commutativity of the operators corresponding to different particles was taken into account. Finally the commutator

$$[\hat{\mathbf{J}}_\alpha^2,\hat{\mathbf{J}}_z]=0.$$

We can conclude that the four operators $\hat{\mathbf{J}}_{\cdot xy}$, $\hat{\mathbf{J}}_z$, $\hat{\mathbf{J}}_1^2$, and $\hat{\mathbf{J}}_2^2$ commute by pairs and form a CSCO for our system of two particles. They generate the angular coupled basis of wave functions.

Now we proceed to look at the transformations between the basis sets used in the main text. The transformation matrix between the uncoupled and coupled representations (Clebsch–Gordan coefficients), is calculated in several places (see, for instance, Zare, Ref. 3, and Schulten and Gordon[11]) and is reproduced in Table I. This matrix, called **U**, satisfies the equation

$$\mathbf{M}_C=\mathbf{U}\mathbf{M}_U, \tag{A1}$$

where $\mathbf{M}_C$ and $\mathbf{M}_U$ are column vectors with elements $|j,m\rangle_c$ and $|m_1,m_2\rangle_u$, respectively.

Now we will develop some relations that are useful for the main text and for the calculation of the transformation matrices.

Defining the operators

$$\hat{\vec{\mathbf{J}}}_{1xy}=\mathbf{i}\hat{\mathbf{J}}_{1x}+\mathbf{j}\hat{\mathbf{J}}_{1y},\quad \hat{\vec{\mathbf{J}}}_{2xy}=\mathbf{i}\hat{\mathbf{J}}_{2x}+\mathbf{j}\hat{\mathbf{J}}_{2y},$$

and using the raising and lowering operators (also known as spherical components) of $\hat{\mathbf{J}}_\alpha$, which can be written as

$$\hat{\mathbf{J}}_{\alpha\pm}=\hat{\mathbf{J}}_{\alpha x}\pm i\hat{\mathbf{J}}_{\alpha y},$$

with

$$\hat{\mathbf{J}}_{\alpha\pm}|j_\alpha m_\alpha\rangle=\sqrt{j_\alpha(j_\alpha+1)-m_\alpha(m_\alpha\pm1)}|j_\alpha m_\alpha\pm1\rangle, \tag{A2}$$

and $|j_\alpha m_\alpha\rangle$ the individual angular momentum with the usual quantum numbers j_α and m_α; we can show that the scalar product

$$\begin{aligned}\hat{\vec{\mathbf{J}}}_{1xy}\cdot\hat{\vec{\mathbf{J}}}_{2xy}=\hat{\mathbf{J}}_{\cdot xy}&=\hat{\mathbf{J}}_{1x}\hat{\mathbf{J}}_{2x}+\hat{\mathbf{J}}_{1y}\hat{\mathbf{J}}_{2y}\\&=\tfrac{1}{4}[(\hat{\mathbf{J}}_{1+}+\hat{\mathbf{J}}_{1-})(\hat{\mathbf{J}}_{2+}+\hat{\mathbf{J}}_{2-})\\&\quad-(\hat{\mathbf{J}}_{1+}-\hat{\mathbf{J}}_{1-})(\hat{\mathbf{J}}_{2+}-\hat{\mathbf{J}}_{2-})]\\&=\tfrac{1}{2}(\hat{\mathbf{J}}_{1+}\hat{\mathbf{J}}_{2-}+\hat{\mathbf{J}}_{1-}\hat{\mathbf{J}}_{2+}).\end{aligned} \tag{A3}$$

With $\hat{\vec{\mathbf{J}}}=\hat{\vec{\mathbf{J}}}_1+\hat{\vec{\mathbf{J}}}_2$, $\hat{\mathbf{J}}_{1xy}^2=\hat{\mathbf{J}}_{1x}^2+\hat{\mathbf{J}}_{1y}^2$, $\hat{\mathbf{J}}_{2xy}^2=\hat{\mathbf{J}}_{2x}^2+\hat{\mathbf{J}}_{2y}^2$ and previous definitions, and after some algebra, we can obtain the following commutators:

$$\begin{aligned}[\hat{\mathbf{J}}^2,\hat{\mathbf{J}}_{1z}]=-[\hat{\mathbf{J}}^2,\hat{\mathbf{J}}_{2z}]&=2[\hat{\mathbf{J}}_{\cdot xy},\hat{\mathbf{J}}_{1z}]\\&=-2[\hat{\mathbf{J}}_{\cdot xy},\hat{\mathbf{J}}_{2z}]\\&=\hat{\mathbf{J}}_{1-}\hat{\mathbf{J}}_{2+}-\hat{\mathbf{J}}_{1+}\hat{\mathbf{J}}_{2-},\end{aligned} \tag{A4}$$

$$\begin{aligned}[\hat{\mathbf{J}}^2,\hat{\mathbf{J}}_{\cdot xy}]&=(\hat{\mathbf{J}}_{2z}-\hat{\mathbf{J}}_{1z})[\hat{\mathbf{J}}^2,\hat{\mathbf{J}}_{1z}]+2\hat{\mathbf{J}}_{\cdot xy}\\&=(\hat{\mathbf{J}}_{1z}-\hat{\mathbf{J}}_{2z})[\hat{\mathbf{J}}^2,\hat{\mathbf{J}}_{2z}]+2\hat{\mathbf{J}}_{\cdot xy},\end{aligned} \tag{A5}$$

$$[\hat{\mathbf{J}}^2,\hat{\mathbf{J}}_{1xy}^2\hat{\mathbf{J}}_{2xy}^2]=2[\hat{\mathbf{J}}_{\cdot xy},\hat{\mathbf{J}}_{1xy}^2\hat{\mathbf{J}}_{2xy}^2]$$
$$=\tfrac{1}{4}(\hat{\mathbf{J}}_{1+}^2\hat{\mathbf{J}}_{1-}\hat{\mathbf{J}}_{2-}^2\hat{\mathbf{J}}_{2+}+\hat{\mathbf{J}}_{1+}^2\hat{\mathbf{J}}_{1-}\hat{\mathbf{J}}_{2-}\hat{\mathbf{J}}_{2+}\hat{\mathbf{J}}_{2-}+\hat{\mathbf{J}}_{1+}\hat{\mathbf{J}}_{1-}\hat{\mathbf{J}}_{1+}\hat{\mathbf{J}}_{2-}^2\hat{\mathbf{J}}_{2+}+\hat{\mathbf{J}}_{1-}^2\hat{\mathbf{J}}_{1+}\hat{\mathbf{J}}_{2+}^2\hat{\mathbf{J}}_{2-}$$
$$-\hat{\mathbf{J}}_{1-}\hat{\mathbf{J}}_{1+}^2\hat{\mathbf{J}}_{2+}\hat{\mathbf{J}}_{2-}^2+\hat{\mathbf{J}}_{1-}\hat{\mathbf{J}}_{1+}\hat{\mathbf{J}}_{1-}\hat{\mathbf{J}}_{2+}^2\hat{\mathbf{J}}_{2-}+\hat{\mathbf{J}}_{1-}^2\hat{\mathbf{J}}_{1+}\hat{\mathbf{J}}_{2+}\hat{\mathbf{J}}_{2-}\hat{\mathbf{J}}_{2+}-\hat{\mathbf{J}}_{1+}\hat{\mathbf{J}}_{1-}\hat{\mathbf{J}}_{1+}\hat{\mathbf{J}}_{2+}\hat{\mathbf{J}}_{2-}^2$$
$$-\hat{\mathbf{J}}_{1-}\hat{\mathbf{J}}_{1+}^2\hat{\mathbf{J}}_{2-}\hat{\mathbf{J}}_{2+}\hat{\mathbf{J}}_{2-}-\hat{\mathbf{J}}_{1+}\hat{\mathbf{J}}_{1-}^2\hat{\mathbf{J}}_{2-}\hat{\mathbf{J}}_{2+}^2-\hat{\mathbf{J}}_{1+}\hat{\mathbf{J}}_{1-}^2\hat{\mathbf{J}}_{2+}\hat{\mathbf{J}}_{2-}\hat{\mathbf{J}}_{2+}-\hat{\mathbf{J}}_{1-}\hat{\mathbf{J}}_{1+}\hat{\mathbf{J}}_{1-}\hat{\mathbf{J}}_{2-}\hat{\mathbf{J}}_{2+}^2), \quad \text{(A6)}$$

$$[\hat{\mathbf{J}}^2,\hat{\mathbf{J}}_{1xy}^2+\hat{\mathbf{J}}_{2xy}^2]=2[\hat{\mathbf{J}}_{\cdot xy},\hat{\mathbf{J}}_{1xy}^2+\hat{\mathbf{J}}_{2xy}^2]=2\hat{\mathbf{J}}_{1+}\hat{\mathbf{J}}_{2z}\hat{\mathbf{J}}_{2-}-2\hat{\mathbf{J}}_{1+}\hat{\mathbf{J}}_{1z}\hat{\mathbf{J}}_{2-}+\hat{\mathbf{J}}_{1z}\hat{\mathbf{J}}_{1-}\hat{\mathbf{J}}_{2+}-\hat{\mathbf{J}}_{1-}\hat{\mathbf{J}}_{2+}\hat{\mathbf{J}}_{2z}. \quad \text{(A7)}$$

By using the matrix elements of the spherical components in the uncoupled basis which can be calculated from Eq. (A2), we can directly obtain the matrix elements $\langle m_2' m_1' | \hat{\mathbf{J}}_{\cdot xy} | m_1 m_2 \rangle$. The result for $j_1=1$ and $j_2=1$, which we call matrix **C**, is shown in Table IV. Also we define $M_U^\dagger$, as a row vector with elements $\langle m_2 m_1|$. We can diagonalize matrix **C** by multiplying the operator $\hat{\mathbf{J}}_{\cdot xy}$ both to the left and to the right by the unit matrix $\mathbf{BB}^{-1}$, so that

$$\mathbf{B}^{-1}\hat{\mathbf{J}}_{\cdot xy}\mathbf{B}=J_{\cdot xy}\mathbf{I},$$

where **I** is the unit matrix. The solution of this equation gives rise to the eigenvalues $J_{\cdot xy}$ [written below Eq. (12)] and the eigenvectors which we may call $\mathbf{M}_A$. The matrix **B**, shown in Table II, connects the uncoupled and the angular coupled bases:

$$\mathbf{M}_U=\mathbf{B}\mathbf{M}_A.$$

Introducing this expression into Eq. (A1), the transformation matrix **A** relating the coupled and the angular coupled bases, is obtained as

$$\mathbf{M}_C=\mathbf{U}(\mathbf{B}\mathbf{M}_A)\equiv\mathbf{A}\mathbf{M}_A.$$

$\mathbf{M}_C$ is a column matrix with elements $|j,m\rangle_c$. The explicit form of matrix **A** is written in Table III.

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REAL SCIENCE IS HARD

The work of real science is hard and often for long intervals frustrating. You have to be a bit compulsive to be a productive scientist. Keep in mind that new ideas are commonplace, and almost always wrong. Most flashes of insight lead nowhere; statistically, they have a half-life of hours or maybe days. Most experiments to follow up the surviving insights are tedious and consume large amounts of time, only to yield negative or (worse!) ambiguous results.

Edward O. Wilson, ''Scientists, Scholars, Knaves and Fools,'' Am. Scientist **86** (1), 6–7 (1998).