1. Liboff, problem 11.27 on page 498.


3. Prove the following identities involving the Pauli spin matrices.

(a) \( \sigma_i \sigma_j = \delta_{ij} I + i \sum_{k=1}^{3} \epsilon_{ijk} \sigma_k \),

(b) \( (\vec{\sigma} \cdot \vec{a})(\vec{\sigma} \cdot \vec{b}) = (\vec{a} \cdot \vec{b}) I + i \vec{\sigma} \cdot (\vec{a} \times \vec{b}) \),

(c) \( \exp\left( -\frac{i}{2} \theta \hat{w} \cdot \vec{\sigma} \right) = I \cos(\theta/2) - i \hat{w} \cdot \vec{\sigma} \sin(\theta/2) \),

where \( I \) is the 2 \times 2 identity matrix, \( \vec{a} \) and \( \vec{b} \) are ordinary vectors, and \( \hat{w} \) is a unit vector. In part (a), the indices \( i \) and \( j \) can take on the values 1,2 and 3, which refer respectively to the \( x, y \) and \( z \) components of the “vector” \( \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \). The Levi-Civita tensor \( \epsilon_{ijk} \) is defined such that:

\[
\epsilon_{ijk} = \begin{cases} 
+1, & \text{if } i, j, k \text{ is an even permutation of } 1,2,3, \\
-1, & \text{if } i, j, k \text{ is an odd permutation of } 1,2,3, \\
0, & \text{if } i, j, k \text{ are not distinct integers}.
\end{cases}
\]

**HINTS:** Part (b) follows easily from part (a) if you express the components of the cross product using the \( \epsilon_{ijk} \) symbol. To solve part (c), expand the exponential in a Taylor series and use part (b). Then show that the resulting expression can be written as a linear combination of \( I \) and \( \hat{w} \cdot \vec{\sigma} \) and identify the corresponding coefficients as series in \( \theta \) that can be resummed.
4. Consider the spinor $\alpha \equiv |1/2, 1/2\rangle_\hat{z}$ [cf. eq.(11.72) of Liboff]. I explicitly exhibit the subscript $\hat{z}$ to emphasize that $|1/2, 1/2\rangle_\hat{z}$ is an eigenstate of $S_z$ with eigenvalue $\frac{1}{2}\hbar$.

(a) Show (either geometrically or algebraically) that if the $z$-axis is rotated by an angle $\theta$ about a fixed axis $\hat{w} = (-\sin \phi, \cos \phi, 0)$ [where the right-hand rule defines the direction of the unit vector $\hat{w}$ perpendicular to the rotation plane], then the $z$-axis will end up pointing along the direction given by

$$\hat{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta).$$

(b) Define $|1/2, 1/2\rangle_\hat{n}$ to be an eigenstate of $\vec{S} \cdot \hat{n}$ with eigenvalue $\frac{1}{2}\hbar$. Express $|1/2, 1/2\rangle_\hat{n}$ with respect to the basis spanned by $\alpha = |1/2, 1/2\rangle_\hat{z} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\beta = |1/2, -1/2\rangle_\hat{z} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(c) Define $R_{\hat{w}}(\theta)$ to be the rotation operator that corresponds to a rotation of state vectors by an angle $\theta$ about the $\hat{w}$ axis [defined in part (a)]. For a spin-1/2 particle, I claim that

$$R_{\hat{w}}(\theta) = \exp \left( -\frac{i}{2} \theta \hat{w} \cdot \vec{\sigma} \right).$$

Let us put this operator to the test. Prove that

$$R_{\hat{w}}(\theta) |1/2, 1/2\rangle_\hat{z} = |1/2, 1/2\rangle_\hat{n}.$$

**HINT:** Evaluate $R_{\hat{w}}(\theta)$ using the result of problem 3(c). Then, verify the above result with respect to the basis $\{\alpha, \beta\}$.

5. Consider a spin-1/2 particle of magnetic moment $\vec{\mu} = \gamma \vec{S}$, where $\gamma \equiv e/mc$. At time $t = 0$, the state of the system is given by $\alpha \equiv |1/2, 1/2\rangle_\hat{z}$ (i.e., spin-up).

(a) If the observable $S_x$ is measured at time $t = 0$, what results can be found and with what probabilities?

(b) Instead of performing the measurement specified in part (a), we let the system evolve under the influence of a uniform magnetic field $B$ parallel to the $y$-axis (i.e., $\vec{B} = B\hat{y}$). Calculate the state of the system at time $t$ with respect to the $\{\alpha, \beta\}$ basis.

**HINT:** The time evolution is governed by the time evolution operator $\exp(-iHt/\hbar)$ as noted in problem 1 of this homework set. The Hamiltonian $H$ is given below eq. (11.96) of Liboff. Using problem 3(c), one can explicitly evaluate the time evolution operator for this problem.

(c) At a fixed time $t = T$, we measure one of the observables $S_x$, $S_y$ and $S_z$. For each case, what values can be found and with what probabilities? Is there any value of $B$ (which may depend on $T$) such that one of the above measurements yields a unique result?