DUE: THURSDAY, OCTOBER 8, 2009

1. Liboff, problem 11.27 on page 498.

2. Liboff, problem 11.39 on page 512.

3. Prove the following identities involving the Pauli spin matrices.

   (a) $\sigma_i \sigma_j = \delta_{ij} I + i \sum_{k=1}^{3} \epsilon_{ijk} \sigma_k$,

   (b) $(\vec{\boldsymbol{\sigma}} \cdot \vec{\boldsymbol{a}})(\vec{\boldsymbol{\sigma}} \cdot \vec{\boldsymbol{b}}) = (\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}}) I + i \vec{\boldsymbol{\sigma}} \cdot (\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}})$,

   (c) $\exp\left(-\tfrac{i}{2} \theta \hat{\boldsymbol{w}} \cdot \vec{\boldsymbol{\sigma}}\right) = I \cos(\theta/2) - i \hat{\boldsymbol{w}} \cdot \vec{\boldsymbol{\sigma}} \sin(\theta/2)$,

where I is the 2×2 identity matrix, $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$ are ordinary vectors, and $\hat{\boldsymbol{w}}$ is a unit vector. In part (a), the indices i and j can take on the values 1,2 and 3, which refer respectively to the x, y and z components of the "vector" $\vec{\boldsymbol{\sigma}} = (\sigma_x\,,\ \sigma_y\,,\ \sigma_z)$. The Levi-Civita tensor ϵ_{ijk} is defined such that:

$$\epsilon_{ijk} = \begin{cases} +1\,, & \text{if } i,j,k \text{ is an even permutation of 1,2,3}\,, \\ -1\,, & \text{if } i,j,k \text{ is an odd permutation of 1,2,3}\,, \\ 0\,, & \text{if } i,j,k \text{ are not distinct integers}\,. \end{cases}$$

HINTS: Part (b) follows easily from part (a) if you express the components of the cross product using the ϵ_{ijk} symbol. To solve part (c), expand the exponential in a Taylor series and use part (b). Then show that the resulting expression can be written as a linear combination of I and $\hat{\boldsymbol{w}} \cdot \boldsymbol{\sigma}$ and identify the corresponding coefficients as series in θ that can be resummed.

4. Consider the spinor $\alpha \equiv |1/2\,,\,1/2\rangle_{\hat{\boldsymbol{z}}}$ [cf. eq.(11.72) of Liboff]. I explicitly exhibit the subscript $\hat{\boldsymbol{z}}$ to emphasize that $|1/2\,,\,1/2\rangle_{\hat{\boldsymbol{z}}}$ is an eigenstate of S_z with eigenvalue $\frac{1}{2}\hbar$.

(a) Show (either geometrically or algebraically) that if the z-axis is rotated by an angle θ about a fixed axis $\hat{\boldsymbol{w}} = (-\sin\phi\,,\,\cos\phi\,,\,0)$ [where the right-hand rule defines the direction of the unit vector $\hat{\boldsymbol{w}}$ perpendicular to the rotation plane], then the z-axis will end up pointing along the direction given by

$$\hat{\boldsymbol{n}} = (\sin\theta\cos\phi\,,\,\sin\theta\sin\phi\,,\,\cos\theta)\,.$$

(b) Define $|1/2\,,\,1/2\rangle_{\hat{\boldsymbol{n}}}$ to be an eigenstate of $\vec{\boldsymbol{S}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}$ with eigenvalue $\frac{1}{2}\hbar$. Express $|1/2\,,\,1/2\rangle_{\hat{\boldsymbol{n}}}$ with respect to the basis spanned by $\alpha = |1/2\,,\,1/2\rangle_{\hat{\boldsymbol{z}}} = \binom{1}{0}$ and $\beta = |1/2\,,\,-1/2\rangle_{\hat{\boldsymbol{z}}} = \binom{0}{1}$.

(c) Define $R_{\hat{\boldsymbol{w}}}(\theta)$ to be the rotation operator that corresponds to a rotation of state vectors by an angle θ about the $\hat{\boldsymbol{w}}$ axis [defined in part (a)]. For a spin-1/2 particle, I claim that

$$R_{\hat{\boldsymbol{w}}}(\theta) = \exp\left(-\frac{i}{2}\theta\hat{\boldsymbol{w}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\right)\,.$$

Let us put this operator to the test. Prove that

$$R_{\hat{\boldsymbol{w}}}(\theta)\,|1/2\,,\,1/2\rangle_{\hat{\boldsymbol{z}}} = |1/2\,,\,1/2\rangle_{\hat{\boldsymbol{n}}}\,.$$

HINT: Evaluate $R_{\hat{\boldsymbol{w}}}(\theta)$ using the result of problem 3(c). Then, verify the above result with respect to the basis $\{\alpha\,,\,\beta\}$.

5. Consider a spin-1/2 particle of magnetic moment $\vec{\boldsymbol{\mu}} = \gamma\vec{\boldsymbol{S}}$, where $\gamma \equiv e/mc$. At time $t = 0$, the state of the system is given by $\alpha \equiv |1/2\,,\,1/2\rangle_{\hat{\boldsymbol{z}}}$ (i.e., spin-up).

(a) If the observable S_x is measured at time $t = 0$, what results can be found and with what probabilities?

(b) Instead of performing the measurement specified in part (a), we let the system evolve under the influence of a uniform magnetic field B parallel to the y-axis (i.e., $\vec{\boldsymbol{B}} = B\hat{\boldsymbol{y}}$). Calculate the state of the system at time t with respect to the $\{\alpha\,,\,\beta\}$ basis.

HINT: The time evolution is governed by the time evolution operator $\exp(-iHt/\hbar)$ as noted in problem 1 of this homework set. The Hamiltonian H is given below eq. (11.96) of Liboff. Using problem 3(c), one can explicitly evaluate the time evolution operator for this problem.

(c) At a fixed time $t = T$, we measure one of the observables S_x, S_y and S_z. For each case, what values can be found and with what probabilities? Is there any value of B (which may depend on T) such that one of the above measurements yields a unique result?