DUE: THURSDAY, NOVEMBER 5, 2009

1. Consider a particle of mass m attached to a rigid massless rod of fixed length R whose other end is fixed at the origin. The rod is free to rotate about the origin. Classical mechanics teaches us that the Hamiltonian of this system is given by

$$H = \frac{\vec{\boldsymbol{L}}^2}{2I},$$

where $I = mR^2$ is the moment of inertia, and $\vec{\boldsymbol{L}} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{p}}$ is the angular momentum. In quantum mechanics, $\vec{\boldsymbol{L}}$ is an operator, and the Schrodinger equation for the energy levels of the rigid rotator is given by $H\,|\psi\rangle = E\,|\psi\rangle$.

(a) What are the possible energy eigenvalues of the system?

(b) Suppose that the particle of mass m has no internal spin degree of freedom, but carries an electric charge $+e$. It is placed in a uniform magnetic field $\vec{\boldsymbol{B}}$. Using the principle of minimal substitution, write down the Schrodinger equation for the charged rigid rotator.

(c) Compute the energy levels of the system described in part (b), assuming that the magnetic field is weak (i.e., assume that the term in the Hamiltonian that is quadratic in $\vec{\boldsymbol{B}}$ can be neglected).

2. Positronium is a bound state of two spin 1/2 particles: an electron (e^-) and a positron (e^+). Consider the Hamiltonian for the system, where we focus only on the spin degrees of freedom. In the presence of a uniform external magnetic field, we may take:

$$H = A(I - \vec{\boldsymbol{\sigma}}_{\mathbf{1}} \cdot \vec{\boldsymbol{\sigma}}_{\mathbf{2}}) + \mu_B(\vec{\boldsymbol{\sigma}}_{\mathbf{1}} - \vec{\boldsymbol{\sigma}}_{\mathbf{2}}) \cdot \vec{\boldsymbol{B}},$$

where A is a constant, I is the identity matrix in the direct product space of the two spin-$\frac{1}{2}$ Hilbert spaces, $\mu_B \equiv e\hbar/(2mc)$, and m is the electron mass. The labels 1 and 2 refer to the e^- and the e^+ respectively.

(a) Consider separately the case of zero magnetic field and the case of a uniform magnetic field pointing in the z-direction. In each case, list the complete set of simultaneously commuting angular momentum operators that also commute with the Hamiltonian H given above.

HINT: The total spin operator is $\vec{\boldsymbol{S}} \equiv \vec{\boldsymbol{S}}_{\mathbf{1}} + \vec{\boldsymbol{S}}_{\mathbf{2}}$, where $\vec{\boldsymbol{S}}_{\boldsymbol{i}} \equiv \frac{1}{2}\hbar\,\vec{\boldsymbol{\sigma}}_{\boldsymbol{i}}$ $(i = 1, 2)$. Note that the index i refers to the identity of the particle (either the electron or positron).

(b) In zero magnetic field, a transition is observed to occur from the $S = 1$ state to the $S = 0$ state (which is the ground state). The emitted photon is observed to have a frequency of 2×10^5 MHz. Compute the energy levels of the system (for $B = 0$), and then evaluate the constant A that appears in the Hamiltonian.

3. Liboff, problem 12.5 on page 584.

4. Liboff, problem 13.4 on page 689.

5. Consider the Hamiltonian for positronium in the presence of a uniform external magnetic field, given in problem 2 above. Assume that the magnetic field points in the z-direction.

(a) Treating the magnetic field as a perturbation, compute the energy eigenvalues to second order in B and the energy eigenstates to first order in B. Sketch the energy levels as a function of B.

(b) Repeat the calculation of part (a), but now solve the problem exactly. Expand out your solutions in a power series in B, and verify that the results of part (a) are indeed correct.