1. Liboff, problem 11.27 on page 498.

(a) Let A be an hermitian operator. We first demonstrate that

$$(e^{iA})^\dagger = e^{-iA}\,.$$

To prove this, we use the series expansion that defines the exponential,

$$e^{iA} = \sum_{n=0}^{\infty} \frac{(iA)^n}{n!}\,.$$

The sum converges for any operator A. Then,

$$e^{-iA} = \sum_{n=0}^{\infty} \frac{(-iA)^n}{n!} = \sum_{n=0}^{\infty} \frac{[(iA)^\dagger]^n}{n!} = \left[\sum_{n=0}^{\infty} \frac{(iA)^n}{n!}\right]^\dagger = (e^{iA})^\dagger\,,$$

where we have used the fact that $(iA)^\dagger = -iA^\dagger = -iA$, since A is hermitian. In the above derivation, we have also used the fact that $(A+B)^\dagger = A^\dagger + B^\dagger$, which also holds for the sum of an infinite number of terms, assuming that the sum converges.

A unitary operator $\widehat{U}$ is defined as an operator that satisfies $\widehat{U}\widehat{U}^\dagger = \widehat{U}^\dagger\widehat{U} = I$, where I is the identity operator. Defining $\widehat{U} \equiv e^{iA}$, we see that:

$$\widehat{U}\widehat{U}^\dagger = e^{iA}(e^{iA})^\dagger = e^{iA}e^{-iA} = e^{i(A-A)} = e^0 = I\,. \tag{1}$$

In this derivation, I used the fact that for any two operators A and B,[1]

$$e^A e^B = e^B e^A = e^{A+B}\,, \quad \text{if and only if} \quad AB = BA\,. \tag{2}$$

This result applies in eq. (1) since iA and $-iA$ clearly commute. Finally, we use the fact that $e^0 = I$, where 0 is the zero operator, which is again a consequence of the definition of the exponential of an operator via its series expansion.

Setting $A = -\widehat{H}t/\hbar$, we note that if $\widehat{H}$ is hermitian, then so is A. It then follows that $\widehat{U} = \exp(-i\widehat{H}t/\hbar)$ is unitary.

(b) Given $|\psi(t)\rangle = \widehat{U}\,|\psi(0)\rangle$, it follows that $\langle\psi(t)| = \langle\psi(0)|\,\widehat{U}^\dagger$. Hence, using the results of part (a),

$$\langle\psi(t)\,|\,\psi(t)\rangle = \langle\psi(0)|\,U^\dagger U\,|\psi(0)\rangle = \langle\psi(0)|\,I\,|\psi 0\rangle = \langle\psi(0)\,|\,\psi(0)\rangle\,.$$

That is, the normalization of ψ is independent of the time t.

[1] For a proof of eq. (2), see, e.g., Jacob T. Schwartz, *Introduction to Matrices and Vectors* (Dover Publications, Inc., Mineola, NY, 2001) pp. 157–159, which can be viewed at `http://books.google.com`.

2. Liboff, problem 11.39 on page 512.

(a) We are given a wave function of a rigid rotator,

$$\psi(t) = \frac{1}{\sqrt{14}}\begin{pmatrix}1\\2\\3\end{pmatrix} e^{-iEt/\hbar}\,, \qquad\qquad E \equiv \hbar^2/I\,.$$

Using the results of Liboff, problem 11.38, we can write at time $t=0$:

$$\frac{1}{\sqrt{14}}\begin{pmatrix}1\\2\\3\end{pmatrix} = -\frac{1}{\sqrt{7}}\,\xi_x^{(0)} + \frac{1}{\sqrt{7}}\,(\sqrt{2}-1)\,\xi_x^{(-1)} + \frac{1}{\sqrt{7}}\,(\sqrt{2}+1)\,\xi_x^{(1)}\,,$$

where,

$$\xi_x^{(0)} \equiv \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}, \qquad \xi_x^{(-1)} \equiv \frac{1}{2}\begin{pmatrix}1\\-\sqrt{2}\\1\end{pmatrix}, \qquad \xi_x^{(1)} \equiv \frac{1}{2}\begin{pmatrix}1\\\sqrt{2}\\1\end{pmatrix},$$

are eigenstates of $\widehat{L}_x$, with eigenvalues, 0, $-\hbar$ and $+\hbar$, respectively. Note that the $\xi_x^{(m)}$ are all eigenstates of $\vec{\boldsymbol{L}}^2$ with eigenvalue $2\hbar^2$. Thus,

$$\psi(t) = \left[-\frac{1}{\sqrt{7}}\,\xi_x^{(0)} + \frac{1}{\sqrt{7}}\,(\sqrt{2}-1)\,\xi_x^{(-1)} + \frac{1}{\sqrt{7}}\,(\sqrt{2}+1)\,\xi_x^{(1)}\right] e^{-iEt/\hbar}$$

since the energy E is independent of the value of m. The probability that a measurement L_x finds a value $-\hbar$ is simply the absolute square of the coefficient of $\xi_x^{(-1)}$, namely

$$\frac{1}{7}(\sqrt{2}-1)^2 = \frac{3-2\sqrt{2}}{7}\,.$$

(b) After the measurement finds the value of $L_x = -\hbar$, the state of the system is given simply by $\xi_x^{(-1)}$. Including the time-dependent factor, $e^{-iEt/\hbar}$, the column vector representation of the state is:

$$\frac{1}{2}\begin{pmatrix}1\\-\sqrt{2}\\1\end{pmatrix} e^{-iEt/\hbar}\,,$$

up to an overall unobservable phase factor.

3. Prove the following identities involving the Pauli spin matrices.

(a) $\sigma_i\sigma_j = \delta_{ij}I + i\sum_{k=1}^{3}\epsilon_{ijk}\sigma_k$,

This identity is verified by explicitly multiplying the 2×2 matrices, $\sigma_i\sigma_j$, in the nine possible cases for $i,j=1,2,3$. Using

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

it is easy to obtain:

$$\begin{aligned} &\sigma_1\sigma_2 = -\sigma_2\sigma_1 = i\sigma_3\,, \qquad \sigma_2\sigma_3 = -\sigma_3\sigma_2 = i\sigma_1\,, \qquad \sigma_3\sigma_1 = -\sigma_1\sigma_3 = i\sigma_2\,, \\ &(\sigma_1)^2 = (\sigma_2)^2 = (\sigma_3)^2 = I\,, \end{aligned}$$

where I is the 2×2 identity matrix. These nine equations are identical to identity (a) above.

(b) $(\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\vec{\boldsymbol{a}})(\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\vec{\boldsymbol{b}}) = (\vec{\boldsymbol{a}}\boldsymbol{\cdot}\vec{\boldsymbol{b}})I + i\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{b}})$,

Multiply identity (a) above by a_ib_j and sum over i and j, where each index can take on three values, 1, 2, and 3. Note the following results:

$$\sum_{i=1}^{3}\sigma_i a_i = \vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\vec{\boldsymbol{a}}\,, \qquad \sum_{j=1}^{3}\sigma_j b_j = \vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\vec{\boldsymbol{b}}\,, \qquad \sum_{i=1}^{3}\sum_{j=1}^{3}\delta_{ij}a_ib_j = \vec{\boldsymbol{a}}\boldsymbol{\cdot}\vec{\boldsymbol{b}}\,.$$

Moreover, using

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\epsilon_{ijk}a_ib_j = (\vec{\boldsymbol{a}}\times\vec{\boldsymbol{b}})_k\,,$$

it follows that:

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\epsilon_{ijk}a_ib_j\sigma_k = \vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{b}})\,,$$

and we immediately obtain identity (b). That is, identity (b) is really identity (a) in disguise!

(c) $\exp\left(-\tfrac{i}{2}\theta\hat{\boldsymbol{w}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\right) = I\cos(\theta/2) - i\hat{\boldsymbol{w}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\sin(\theta/2)$,

where I is the 2×2 identity matrix, $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$ are ordinary vectors, and $\hat{\boldsymbol{w}}$ is a unit vector.

Using the series expansion for the matrix exponential,

$$\exp\left(-\frac{i}{2}\theta\hat{\boldsymbol{w}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\right) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-i\theta}{2}\right)^n(\hat{\boldsymbol{w}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}})^n\,. \tag{3}$$

Using identity (b) above, it follows that

$$(\hat{\boldsymbol{w}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}})^2 = (\hat{\boldsymbol{w}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}})(\hat{\boldsymbol{w}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}) = I\,,$$

where we have used the fact that $\hat{\boldsymbol{w}}$ is a unit vector to obtain $\hat{\boldsymbol{w}} \cdot \hat{\boldsymbol{w}} = 1$ (and of course, $\hat{\boldsymbol{w}} \times \hat{\boldsymbol{w}} = 0$). Hence, for any integer n,

$$(\hat{\boldsymbol{w}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}})^{2n} = I\,, \qquad\qquad (\hat{\boldsymbol{w}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}})^{2n+1} = \hat{\boldsymbol{w}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\,.$$

Inserting these results back into the series [eq. (3) we obtain:

$$\begin{aligned}
\exp\left(-\frac{i}{2}\theta\hat{\boldsymbol{w}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\right) &= I\sum_{n=0}^{\infty}\frac{1}{(2n)!}\left(\frac{-i\theta}{2}\right)^{2n} + \hat{\boldsymbol{w}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left(\frac{-i\theta}{2}\right)^{2n+1} \\
&= I\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{\theta}{2}\right)^{2n} - i\hat{\boldsymbol{w}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{\theta}{2}\right)^{2n+1} \\
&= I\cos(\theta/2) - i\hat{\boldsymbol{w}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\sin(\theta/2)\,,
\end{aligned}$$

after summing up the well known series for the sine and cosine, and noting that $(-i)^{2n} = (-1)^n$.

4. Consider the spinor $\alpha \equiv \left|\frac{1}{2}\,,\,\frac{1}{2}\right\rangle_{\hat{\boldsymbol{z}}}$ [cf. eq.(11.72) of Liboff]. I explicitly exhibit the subscript $\hat{\boldsymbol{z}}$ to emphasize that $\left|\frac{1}{2}\,,\,\frac{1}{2}\right\rangle_{\hat{\boldsymbol{z}}}$ is an eigenstate of S_z with eigenvalue $\frac{1}{2}\hbar$.

(a) Show (either geometrically or algebraically) that if the z-axis is rotated by an angle θ about a fixed axis $\hat{\boldsymbol{w}} = (-\sin\phi\,,\,\cos\phi\,,\,0)$ [where the right-hand rule defines the direction of the unit vector $\hat{\boldsymbol{w}}$ perpendicular to the rotation plane], then the z-axis will end up pointing along the direction given by

$$\hat{\boldsymbol{n}} = (\sin\theta\cos\phi\,,\,\sin\theta\sin\phi\,,\,\cos\theta)\,.$$

Consider the active rotation of a unit vector, $\hat{\boldsymbol{z}}$, that ends up pointing along the $\hat{\boldsymbol{n}}$ direction, where $\hat{\boldsymbol{n}}$ is a vector that points in a direction with polar angle θ (with respect to the z-axis) and azimuthal angle ϕ (defined in the usual way by projecting $\hat{\boldsymbol{n}}$ in the x–y plane and measuring the angle with respect to the x-axis). Note that in performing this rotation, we have rotated $\hat{\boldsymbol{z}}$ by an angle θ about a fixed axis, $\hat{\boldsymbol{w}}$, which lies in the x–y plane perpendicular to the plane containing $\hat{\boldsymbol{z}}$ and $\hat{\boldsymbol{n}}$. The direction of $\hat{\boldsymbol{w}}$ is determined by the right hand rule. I claim that:

$$\hat{\boldsymbol{w}} = \frac{\hat{\boldsymbol{z}} \times \hat{\boldsymbol{n}}}{|\hat{\boldsymbol{z}} \times \hat{\boldsymbol{n}}|}\,,$$

where $|\hat{\boldsymbol{z}} \times \hat{\boldsymbol{n}}|$ is the length of the vector $\hat{\boldsymbol{z}} \times \hat{\boldsymbol{n}}$. This vector clearly satisfies the property that $\hat{\boldsymbol{z}} \times \hat{\boldsymbol{n}}$ is perpendicular to the plane containing $\hat{\boldsymbol{z}}$ and $\hat{\boldsymbol{n}}$ as required. Moreover, $\hat{\boldsymbol{z}} \times \hat{\boldsymbol{n}}$ points in a direction that is determined by the right-hand rule.

To complete the proof, simply compute $\hat{\boldsymbol{z}}\times\hat{\boldsymbol{n}}$.

$$\hat{\boldsymbol{z}}\times\hat{\boldsymbol{n}} = \det\begin{pmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ 0 & 0 & 1 \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{pmatrix}$$

$$= -\hat{\boldsymbol{x}}\sin\theta\sin\phi + \hat{\boldsymbol{y}}\sin\theta\cos\phi\,.$$

The length of the vector $\hat{\boldsymbol{z}}\times\hat{\boldsymbol{n}}$ is

$$|\hat{\boldsymbol{z}}\times\hat{\boldsymbol{n}}| = \left[\sin^2\theta\sin^2\phi + \sin^2\theta\cos^2\phi\right]^{1/2} = \sin\theta\,,$$

Hence, we conclude that

$$\hat{\boldsymbol{w}} = \frac{\hat{\boldsymbol{z}}\times\hat{\boldsymbol{n}}}{|\hat{\boldsymbol{z}}\times\hat{\boldsymbol{n}}|} = -\hat{\boldsymbol{x}}\sin\phi + \hat{\boldsymbol{y}}\cos\phi = (-\sin\phi\,,\,\cos\phi\,,\,0)\,.$$

(b) Define $\left|\tfrac{1}{2}\,,\,\tfrac{1}{2}\right\rangle_{\hat{\boldsymbol{n}}}$ to be an eigenstate of $\vec{\boldsymbol{S}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}$ with eigenvalue $\tfrac{1}{2}\hbar$. Express $\left|\tfrac{1}{2}\,,\,\tfrac{1}{2}\right\rangle_{\hat{\boldsymbol{n}}}$ with respect to the basis $\alpha = \left|\tfrac{1}{2}\,,\,\tfrac{1}{2}\right\rangle_{\hat{\boldsymbol{z}}} = \binom{1}{0}$ and $\beta = \left|\tfrac{1}{2}\,,\,-\tfrac{1}{2}\right\rangle_{\hat{\boldsymbol{z}}} = \binom{0}{1}$.

Using $\vec{\boldsymbol{S}} = \tfrac{1}{2}\hbar\vec{\boldsymbol{\sigma}}$, we can work out the dot product, $\vec{\boldsymbol{S}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}$:

$$\vec{\boldsymbol{S}}\boldsymbol{\cdot}\hat{\boldsymbol{n}} = \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}\,.$$

In class, we worked out the eigenvectors of $\vec{\boldsymbol{S}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}$. The eigenvector corresponding to eigenvalue $\tfrac{1}{2}\hbar$ is given by

$$\left|\tfrac{1}{2}\,,\,\tfrac{1}{2}\right\rangle_{\hat{\boldsymbol{n}}} = e^{i\delta}\begin{pmatrix} \cos(\theta/2)e^{-i\phi/2} \\ \sin(\theta/2)e^{i\phi/2} \end{pmatrix}\,, \tag{4}$$

where $e^{i\delta}$ is an arbitrary phase factor. This means that we can write:

$$\left|\tfrac{1}{2}\,,\,\tfrac{1}{2}\right\rangle_{\hat{\boldsymbol{n}}} = e^{i\delta}\left[\alpha\,\cos(\theta/2)e^{-i\phi/2} + \beta\,\sin(\theta/2)e^{i\phi/2}\right]\,.$$

(c) Define $R_{\hat{\boldsymbol{w}}}(\theta)$ to be the rotation operator that corresponds to a rotation of state vectors by an angle θ about the $\hat{\boldsymbol{w}}$ axis [defined in part (a)]. For a spin-$\tfrac{1}{2}$ particle, I claim that

$$R_{\hat{\boldsymbol{w}}}(\theta) = \exp\left(-\frac{i}{2}\theta\hat{\boldsymbol{w}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\right)\,.$$

Let us put this operator to the test. Prove that

$$R_{\hat{\boldsymbol{w}}}(\theta)\left|\tfrac{1}{2}\,,\,\tfrac{1}{2}\right\rangle_{\hat{\boldsymbol{z}}} = \left|\tfrac{1}{2}\,,\,\tfrac{1}{2}\right\rangle_{\hat{\boldsymbol{n}}}\,.$$

First, note that since $\hat{\boldsymbol{w}} = (-\sin\phi\,,\,\cos\phi\,,\,0)$, it follows that:

$$\hat{\boldsymbol{w}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}} = \begin{pmatrix} 0 & -\sin\phi - i\cos\phi \\ -sin\phi + i\cos\phi & 0 \end{pmatrix} = -i\begin{pmatrix} 0 & e^{-i\phi} \\ -e^{i\phi} & 0 \end{pmatrix}\,.$$

Using the result of problem 3(c),

$$R_{\hat{\boldsymbol{w}}}(\theta) = \exp\left(-\frac{i}{2}\theta\hat{\boldsymbol{w}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\right) = I\cos(\theta/2) - i\hat{\boldsymbol{w}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\sin(\theta/2)$$
$$= \begin{pmatrix} \cos(\theta/2) & -e^{-i\phi}\sin(\theta/2) \\ e^{i\phi}\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} .$$

Using the fact that the state $\left|\frac{1}{2}\,,\,\frac{1}{2}\right\rangle_{\hat{\boldsymbol{z}}}$ is represented by $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$, it follows that:

$$R_{\hat{\boldsymbol{w}}}(\theta)\left|\tfrac{1}{2}\,,\,\tfrac{1}{2}\right\rangle_{\hat{\boldsymbol{z}}} = \begin{pmatrix} \cos(\theta/2) & -e^{-i\phi}\sin(\theta/2) \\ e^{i\phi}\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$
$$= \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix} = e^{i\phi/2}\begin{pmatrix} \cos(\theta/2)e^{-i\phi/2} \\ \sin(\theta/2)e^{i\phi/2} \end{pmatrix} .$$

Indeed, $R_{\hat{\boldsymbol{w}}}(\theta)\left|\frac{1}{2}\,,\,\frac{1}{2}\right\rangle_{\hat{\boldsymbol{z}}}$ is an eigenstate of $\vec{\boldsymbol{S}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}$ with eigenvalue $\frac{1}{2}\hbar$. Thus, choosing $\delta = \phi/2$ in eq. (4), we conclude that:

$$R_{\hat{\boldsymbol{w}}}(\theta)\left|\tfrac{1}{2}\,,\,\tfrac{1}{2}\right\rangle_{\hat{\boldsymbol{z}}} = \left|\tfrac{1}{2}\,,\,\tfrac{1}{2}\right\rangle_{\hat{\boldsymbol{n}}} .$$

5. Consider a spin-$\frac{1}{2}$ particle of magnetic moment $\vec{\boldsymbol{\mu}} = \gamma\vec{\boldsymbol{S}}$, where $\gamma \equiv e/mc$. At time $t = 0$, the state of the system is given by $\alpha \equiv \left|\frac{1}{2}\,,\,\frac{1}{2}\right\rangle_{\hat{\boldsymbol{z}}}$ (i.e., spin-up).

(a) If the observable S_x is measured at time $t = 0$, what results can be found and with what probabilities?

To see what the possible results of a measurement of S_x, consider the initial state, $\left|\frac{1}{2}\,,\,\frac{1}{2}\right\rangle_{\hat{\boldsymbol{z}}}$. Expand this state in terms of the two possible states, $\left|\frac{1}{2}\,,\,\frac{1}{2}\right\rangle_{\hat{\boldsymbol{x}}}$ and $\left|\frac{1}{2}\,,\,-\frac{1}{2}\right\rangle_{\hat{\boldsymbol{x}}}$ that can result from a measurement of S_x. These two states are the normalized eigenstates of $S_x = \frac{1}{2}\hbar\sigma_x$, namely,

$$\left|\tfrac{1}{2}\,,\,\tfrac{1}{2}\right\rangle_{\hat{\boldsymbol{x}}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} , \qquad \left|\tfrac{1}{2}\,,\,-\tfrac{1}{2}\right\rangle_{\hat{\boldsymbol{x}}} = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix} . \tag{5}$$

Using the above results, it is easy to determine that:

$$\left|\tfrac{1}{2}\,,\,\tfrac{1}{2}\right\rangle_{\hat{\boldsymbol{z}}} = \begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\left|\tfrac{1}{2}\,,\,\tfrac{1}{2}\right\rangle_{\hat{\boldsymbol{x}}} - \frac{1}{\sqrt{2}}\left|\tfrac{1}{2}\,,\,-\tfrac{1}{2}\right\rangle_{\hat{\boldsymbol{x}}} .$$

The possible measurements of S_x yield either $\frac{1}{2}\hbar$ or $-\frac{1}{2}\hbar$, with equal probabilities of $\frac{1}{2}$ in the two cases (corresponding to the absolute square of the coefficients of $\left|\frac{1}{2}\,,\,\pm\frac{1}{2}\right\rangle_{\hat{\boldsymbol{x}}}$ above).

(b) Instead of performing the measurement specified in part (a), we let the system evolve under the influence of a uniform magnetic field B parallel to the

y-axis (i.e., $\vec{\boldsymbol{B}} = B\hat{\boldsymbol{y}}$ with $B > 0$). Calculate the state of the system at time t with respect to the $\{\alpha\,,\,\beta\}$ basis.

The Hamiltonian of the system is

$$H = -\gamma\vec{\boldsymbol{S}}\boldsymbol{\cdot}\vec{\boldsymbol{B}} = -\gamma B S_y = -\tfrac{1}{2}\hbar\gamma B\sigma_y\,.$$

Thus, $|\psi(t)\rangle = U(t)\,|\psi(0)\rangle$, where

$$U(t) = e^{-iHt/\hbar} = \exp\left(\tfrac{1}{2}i\gamma B\sigma_y t\right)\,.$$

Let us define $\omega_0 \equiv -\gamma B$. Then,

$$\begin{aligned} U(t) &= \exp\left(-\tfrac{1}{2}i\omega_0 t\sigma_y\right) = I\cos(\omega_0 t/2) - i\sigma_y\sin(\omega_0 t/2) \\ &= \begin{pmatrix} \cos(\omega_0 t/2) & -\sin(\omega_0 t/2) \\ \sin(\omega_0 t/2) & \cos(\omega_0 t/2) \end{pmatrix}\,. \end{aligned}$$

Here we have used the results of problem 3(c), and have identified $\hat{\boldsymbol{w}} = (0\,,\,1\,,\,0)$. Note that it is critical that $\hat{\boldsymbol{w}}$ is a *unit* vector. This requirement allows us to unambiguously identify $\theta \equiv \omega_0 t$.

Hence, the state of the system at time t is:

$$|\psi(t)\rangle = U(t)\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\cos(\omega_0 t/2)\\ \sin(\omega_0 t/2)\end{pmatrix} = \alpha\,\cos(\omega_0 t/2) + \beta\,\sin(\omega_0 t/2)\,. \qquad (6)$$

(c) At a fixed time $t = T > 0$, we measure one of the observables S_x, S_y and S_z. For each case, what values can be found and with what probabilities? Is there any value of B (which may depend on T) such that one of the above measurements yields a unique result?

We now measure either S_x, S_y or S_z. To address each case separately, we should expand the spinor obtained in part (b) in terms of a new basis consisting of the eigenstates of S_x, S_y or S_z, respectively.

First, we expand in terms of eigenstates of S_z. But, this has already been done in part (b):

$$|\psi(t)\rangle = \cos(\omega_0\theta/2)\left|\tfrac{1}{2}\,,\,\tfrac{1}{2}\right\rangle_{\hat{\boldsymbol{z}}} + \sin(\omega_0\theta/2)\left|\tfrac{1}{2}\,,\,-\tfrac{1}{2}\right\rangle_{\hat{\boldsymbol{z}}}\,.$$

Thus the possible results obtained by measuring S_z are:

$$\begin{aligned} &+\tfrac{1}{2}\hbar\,, \quad \text{with probability} \quad \cos^2(\omega_0 t/2) = \tfrac{1}{2}(1+\cos\omega_0 t)\,, \\ &-\tfrac{1}{2}\hbar\,, \quad \text{with probability} \quad \sin^2(\omega_0 t/2) = \tfrac{1}{2}(1-\cos\omega_0 t)\,, \end{aligned}$$

where the probabilities are identified as the absolute squares of the coefficients of $\left|\tfrac{1}{2}\,,\,\pm\tfrac{1}{2}\right\rangle_{\hat{\boldsymbol{z}}}$, respectively.

Second, we expand $|\psi(t)\rangle$ in terms of eigenstates of S_x:

$$|\psi(t)\rangle = c_+ \left|\tfrac{1}{2}\,,\,\tfrac{1}{2}\right\rangle_{\hat{\boldsymbol{x}}} + c_- \left|\tfrac{1}{2}\,,\,-\tfrac{1}{2}\right\rangle_{\hat{\boldsymbol{x}}}\,,$$

where[2]

$$c_\pm = {}_{\hat{\boldsymbol{x}}}\left\langle \tfrac{1}{2}, \pm\tfrac{1}{2}\right| \psi(t)\rangle\,.$$

Using eqs. (5) and (6), one can easily evaluate $c_\pm$,

$$c_+ = \begin{pmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \cos(\omega_0 t/2) \\ \sin(\omega_0 t/2) \end{pmatrix} = \frac{1}{\sqrt{2}} \left[\cos(\omega_0 t/2) + \sin(\omega_0 t/2)\right]\,,$$

$$c_- = \begin{pmatrix} -\dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \cos(\omega_0 t/2) \\ \sin(\omega_0 t/2) \end{pmatrix} = \frac{1}{\sqrt{2}} \left[-\cos(\omega_0 t/2) + \sin(\omega_0 t/2)\right]\,.$$

To get the relevant probabilities, we need to square these results:

$$|c_+|^2 = \tfrac{1}{2}\left[\sin^2(\omega_0 t/2) + \cos^2(\omega_0 t/2) + 2\sin(\omega_0 t/2)\cos(\omega_0 t/2)\right] = \tfrac{1}{2}\left[1 + \sin(\omega_0 t)\right]\,,$$
$$|c_-|^2 = \tfrac{1}{2}\left[\sin^2(\omega_0 t/2) + \cos^2(\omega_0 t/2) - 2\sin(\omega_0 t/2)\cos(\omega_0 t/2)\right] = \tfrac{1}{2}\left[1 - \sin(\omega_0 t)\right]\,.$$

Hence, the possible results obtained by measuring S_x are:

$$+\tfrac{1}{2}\hbar\,,\quad \text{with probability}\quad \tfrac{1}{2}(1 + \sin\omega_0 t)\,,$$
$$-\tfrac{1}{2}\hbar\,,\quad \text{with probability}\quad \tfrac{1}{2}(1 - \sin\omega_0 t)\,.$$

Finally, we expand $|\psi(t)\rangle$ in terms of eigenstates of S_y. Here, we note that the normalized eigenstates of $S_y = \frac{1}{2}\hbar\sigma_y$ are:

$$\left|\tfrac{1}{2}\,,\,\tfrac{1}{2}\right\rangle_{\hat{\boldsymbol{y}}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}\,, \qquad \left|\tfrac{1}{2}\,,\,-\tfrac{1}{2}\right\rangle_{\hat{\boldsymbol{y}}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}\,. \tag{7}$$

Following the previous computation, we write:

$$|\psi(t)\rangle = c_+ \left|\tfrac{1}{2}\,,\,\tfrac{1}{2}\right\rangle_{\hat{\boldsymbol{y}}} + c_- \left|\tfrac{1}{2}\,,\,-\tfrac{1}{2}\right\rangle_{\hat{\boldsymbol{y}}}\,,$$

where

$$c_\pm = {}_{\hat{\boldsymbol{y}}}\left\langle \tfrac{1}{2}, \pm\tfrac{1}{2}\right| \psi(t)\rangle\,.$$

Using eqs. (6) and (7), one can easily evaluate $c_\pm$. Keep in mind that in this computation, the adjoint spinor ${}_{\hat{\boldsymbol{y}}}\left\langle \tfrac{1}{2}, \pm\tfrac{1}{2}\right|$ appears, so we have to complex conjugate the results of eq. (7), which changes the sign of i. Hence,

$$c_+ = \begin{pmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{-i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \cos(\omega_0 t/2) \\ \sin(\omega_0 t/2) \end{pmatrix} = \frac{1}{\sqrt{2}} e^{-i\omega_0 t/2}\,,$$

$$c_- = \begin{pmatrix} -\dfrac{1}{\sqrt{2}} & \dfrac{i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \cos(\omega_0 t/2) \\ \sin(\omega_0 t/2) \end{pmatrix} = \frac{1}{\sqrt{2}} e^{i\omega_0 t/2}\,.$$

[2] Here, we are using the fact that we can expand any state $|\psi\rangle$ in terms of a complete basis $\{e_1\,,\,e_2\}$ as follows: $|\psi\rangle = \sum_{i=1}^{2} |e_i\rangle \langle e_i|\,\psi\rangle$. In this case, the complete basis is made up of $\left|\tfrac{1}{2}\,,\,\pm\tfrac{1}{2}\right\rangle_{\hat{\boldsymbol{x}}}$.

To get the relevant probabilities, we need to square these results, which yields $|c_\pm|^2 = \frac{1}{2}$. Hence, the possible results obtained by measuring S_y are:

$$\begin{array}{ll} +\tfrac{1}{2}\hbar\,, & \text{with probability} \quad \tfrac{1}{2}\,, \\ -\tfrac{1}{2}\hbar\,, & \text{with probability} \quad \tfrac{1}{2}\,. \end{array}$$

Thus, the measurement of S_y *never* gives a unique result., This could have been anticipated, since at $t = 0$, the initial state $\left|\frac{1}{2}\,,\,\frac{1}{2}\right\rangle_{\hat{\boldsymbol{z}}}$ is equally likely to be either spin-up or spin-down with respect to the y-axis. But, a B field applied in the $\hat{\boldsymbol{y}}$ direction does not modify the y-component of the spin.

However, the measurement of S_x and S_z can give unique results at some fixed time $t = T \geq 0$, if B (or equivalently, $\omega_0 \equiv -\gamma B$) takes on particular values. Noting that for any integer n,

$$\cos(n\pi) = (-1)^n\,, \qquad\qquad \sin[(n + \tfrac{1}{2})\pi] = (-1)^n\,,$$

we see that:

- If $\omega_0 T = n\pi$, then the measurement of S_z yields:
  1. pure spin up, if n is even,
  2. pure spin down, if n is odd.
- If $\omega_0 T = (n + \frac{1}{2})\pi$, then the measurement of S_x yields:
  1. pure spin up, if n is even,
  2. pure spin down, if n is odd.

Using $\omega_0 \equiv -\gamma B$, it follows that the measurement of S_x is unique if

$$B = \frac{n + \frac{1}{2}}{|\gamma| T}\,, \qquad \text{for} \quad n = 0, 1, 2, \ldots\,,$$

and the measurement of S_z is unique if

$$B = \frac{n\pi}{|\gamma| T}\,, \qquad \text{for} \quad n = 1, 2, 3, \ldots\,.$$

Note that the integer n is restricted to non-negative values in the first case and positive values in the second case, since by assumption $T \geq 0$ (and $B > 0$). Finally, as previously noted, the measurement of S_y is never unique.