1. Liboff, problem 11.27 on page 498.

(a) Let $A$ be an hermitian operator. We first demonstrate that

\[(e^{iA})^\dagger = e^{-iA}.\]

To prove this, we use the series expansion that defines the exponential,

\[e^{iA} = \sum_{n=0}^{\infty} \frac{(iA)^n}{n!}.\]

The sum converges for any operator $A$. Then,

\[e^{-iA} = \sum_{n=0}^{\infty} \frac{(-iA)^n}{n!} = \sum_{n=0}^{\infty} \frac{[(iA)\dagger]^n}{n!} = \left[\sum_{n=0}^{\infty} \frac{(iA)^n}{n!}\right]^\dagger = (e^{iA})^\dagger,
\]

where we have used the fact that $(iA)^\dagger = -iA^\dagger = -iA$, since $A$ is hermitian. In the above derivation, we have also used the fact that $(A + B)^\dagger = A^\dagger + B^\dagger$, which also holds for the sum of an infinite number of terms, assuming that the sum converges.

A unitary operator $\hat{U}$ is defined as an operator that satisfies $\hat{U}\hat{U}^\dagger = \hat{U}^\dagger\hat{U} = I$, where $I$ is the identity operator. Defining $\hat{U} \equiv e^{iA}$, we see that:

\[\hat{U}\hat{U}^\dagger = e^{iA}(e^{iA})^\dagger = e^{iA}e^{-iA} = e^{i(A-A)} = e^0 = I.\]  \hspace{1cm} (1)

In this derivation, I used the fact that for any two operators $A$ and $B$,

\[e^Ae^B = e^Be^A = e^{A+B}, \quad \text{if and only if} \quad AB = BA.\]  \hspace{1cm} (2)

This result applies in eq. (1) since $iA$ and $-iA$ clearly commute. Finally, we use the fact that $e^0 = I$, where 0 is the zero operator, which is again a consequence of the definition of the exponential of an operator via its series expansion.

Setting $A = -\hat{H}t/\hbar$, we note that if $\hat{H}$ is hermitian, then so is $A$. It then follows that $\hat{U} = \exp(-i\hat{H}t/\hbar)$ is unitary.

(b) Given $|\psi(t)\rangle = \hat{U}|\psi(0)\rangle$, it follows that $\langle \psi(t)| = (\psi(0)| \hat{U}^\dagger$. Hence, using the results of part (a),

\[\langle \psi(t)| \psi(t)\rangle = \langle \psi(0)| \hat{U}^\dagger \hat{U} |\psi(0)\rangle = \langle \psi(0)| I |\psi(0)\rangle = \langle \psi(0)| \psi(0)\rangle.\]

That is, the normalization of $\psi$ is independent of the time $t$.

\[\text{---1\footnote{For a proof of eq. (2), see, e.g., Jacob T. Schwartz, Introduction to Matrices and Vectors (Dover Publications, Inc., Mineola, NY, 2001) pp. 157–159, which can be viewed at http://books.google.com.}}\]

(a) We are given a wave function of a rigid rotator,

$$\psi(t) = \frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} e^{-iEt/\hbar}, \quad E \equiv \hbar^2/I.$$

Using the results of Liboff, problem 11.38, we can write at time $t = 0$:

$$\frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = -\frac{1}{\sqrt{7}} \xi_x^{(0)} + \frac{1}{\sqrt{7}} (\sqrt{2} - 1) \xi_x^{(-1)} + \frac{1}{\sqrt{7}} (\sqrt{2} + 1) \xi_x^{(1)},$$

where,

$$\xi_x^{(0)} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \xi_x^{(-1)} \equiv \frac{1}{2} \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}, \quad \xi_x^{(1)} \equiv \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix},$$

are eigenstates of $\hat{L}_x$, with eigenvalues, 0, $-\hbar$ and $+\hbar$, respectively. Note that the $\xi_x^{(m)}$ are all eigenstates of $\hat{L}_x^2$ with eigenvalue $2\hbar^2$. Thus,

$$\psi(t) = \left[ -\frac{1}{\sqrt{7}} \xi_x^{(0)} + \frac{1}{\sqrt{7}} (\sqrt{2} - 1) \xi_x^{(-1)} + \frac{1}{\sqrt{7}} (\sqrt{2} + 1) \xi_x^{(1)} \right] e^{-iEt/\hbar}$$

since the energy $E$ is independent of the value of $m$. The probability that a measurement $L_x$ finds a value $-\hbar$ is simply the absolute square of the coefficient of $\xi_x^{(-1)}$, namely

$$\frac{1}{7} (\sqrt{2} - 1)^2 = \frac{3 - 2\sqrt{2}}{7}.$$

(b) After the measurement finds the value of $L_x = -\hbar$, the state of the system is given simply by $\xi_x^{(-1)}$. Including the time-dependent factor, $e^{-iEt/\hbar}$, the column vector representation of the state is:

$$\frac{1}{2} \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} e^{-iEt/\hbar},$$

up to an overall unobservable phase factor.
3. Prove the following identities involving the Pauli spin matrices.

(a) \( \sigma_i \sigma_j = \delta_{ij} I + i \sum_{k=1}^{3} \epsilon_{ijk} \sigma_k \),

This identity is verified by explicitly multiplying the 2 \( \times \) 2 matrices, \( \sigma_i \sigma_j \), in the nine possible cases for \( i, j = 1, 2, 3 \). Using

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

it is easy to obtain:

\[
\sigma_1 \sigma_2 = -\sigma_2 \sigma_1 = i \sigma_3, \quad \sigma_2 \sigma_3 = -\sigma_3 \sigma_2 = i \sigma_1, \quad \sigma_3 \sigma_1 = -\sigma_1 \sigma_3 = i \sigma_2,
\]

\( (\sigma_1)^2 = (\sigma_2)^2 = (\sigma_3)^2 = I \),

where \( I \) is the 2 \( \times \) 2 identity matrix. These nine equations are identical to identity (a) above.

(b) \( (\vec{\sigma} \cdot \vec{a})(\vec{\sigma} \cdot \vec{b}) = (\vec{a} \cdot \vec{b})I + i\vec{\sigma} \cdot (\vec{a} \times \vec{b}) \),

Multiply identity (a) above by \( a_i b_j \) and sum over \( i \) and \( j \), where each index can take on three values, 1, 2, and 3. Note the following results:

\[
\sum_{i=1}^{3} \sigma_i a_i = \vec{\sigma} \cdot \vec{a}, \quad \sum_{j=1}^{3} \sigma_j b_j = \vec{\sigma} \cdot \vec{b}, \quad \sum_{i=1}^{3} \sum_{j=1}^{3} \delta_{ij} a_i b_j = \vec{a} \cdot \vec{b}.
\]

Moreover, using

\[
\sum_{i=1}^{3} \sum_{j=1}^{3} \epsilon_{ijk} a_i b_j = (\vec{a} \times \vec{b})_k,
\]

it follows that:

\[
\sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \epsilon_{ijk} a_i b_j \sigma_k = \vec{\sigma} \cdot (\vec{a} \times \vec{b}),
\]

and we immediately obtain identity (b). That is, identity (b) is really identity (a) in disguise!

(c) \( \exp \left( -\frac{i}{2} \theta \hat{w} \cdot \vec{\sigma} \right) = I \cos(\theta/2) - i\hat{w} \cdot \vec{\sigma} \sin(\theta/2) \),

where \( I \) is the 2 \( \times \) 2 identity matrix, \( \vec{a} \) and \( \vec{b} \) are ordinary vectors, and \( \hat{w} \) is a unit vector.

Using the series expansion for the matrix exponential,

\[
\exp \left( -\frac{i}{2} \theta \hat{w} \cdot \vec{\sigma} \right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{-i \theta}{2} \right)^n (\hat{w} \cdot \vec{\sigma})^n.
\]
Using identity (b) above, it follows that

\[(\hat{w} \cdot \vec{\sigma})^2 = (\hat{w} \cdot \vec{\sigma})(\hat{w} \cdot \vec{\sigma}) = I,\]

where we have used the fact that \(\hat{w}\) is a unit vector to obtain \(\hat{w} \cdot \hat{w} = 1\) (and of course, \(\hat{w} \times \hat{w} = 0\)). Hence, for any integer \(n\),

\[(\hat{w} \cdot \vec{\sigma})^2 = I, \quad (\hat{w} \cdot \vec{\sigma})^{2n+1} = \hat{w} \cdot \vec{\sigma}.\]

Inserting these results back into the series [eq. (3) we obtain:

\[
\exp\left(-\frac{i}{2} \hat{w} \cdot \vec{\sigma}\right) = I \sum_{n=0}^{\infty} \frac{1}{(2n)!} \left(-\frac{i\theta}{2}\right)^{2n} + \hat{w} \cdot \vec{\sigma} \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \left(-\frac{i\theta}{2}\right)^{2n+1}
\]

\[
= I \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{\theta}{2}\right)^{2n} - i\hat{w} \cdot \vec{\sigma} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{\theta}{2}\right)^{2n+1}
\]

\[
= I \cos(\theta/2) - i\hat{w} \cdot \vec{\sigma} \sin(\theta/2),
\]

after summing up the well known series for the sine and cosine, and noting that \((-i)^{2n} = (-1)^n\).

4. Consider the spinor \(\alpha \equiv |\frac{1}{2}, \frac{1}{2}\rangle\) [cf. eq.(11.72) of Liboff]. I explicitly exhibit the subscript \(\hat{z}\) to emphasize that \(|\frac{1}{2}, \frac{1}{2}\rangle\) is an eigenstate of \(S_z\) with eigenvalue \(\frac{1}{2}\hbar\).

(a) Show (either geometrically or algebraically) that if the \(z\)-axis is rotated by an angle \(\theta\) about a fixed axis \(\hat{w} = (-\sin \phi, \cos \phi, 0)\) [where the right-hand rule defines the direction of the unit vector \(\hat{w}\) perpendicular to the rotation plane], then the \(z\)-axis will end up pointing along the direction given by

\[
\hat{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta).
\]

Consider the active rotation of a unit vector, \(\hat{z}\), that ends up pointing along the \(\hat{n}\) direction, where \(\hat{n}\) is a vector that points in a direction with polar angle \(\theta\) (with respect to the \(z\)-axis) and azimuthal angle \(\phi\) (defined in the usual way by projecting \(\hat{n}\) in the \(x-y\) plane and measuring the angle with respect to the \(x\)-axis). Note that in performing this rotation, we have rotated \(\hat{z}\) by an angle \(\theta\) about a fixed axis, \(\hat{w}\), which lies in the \(x-y\) plane perpendicular to the plane containing \(\hat{z}\) and \(\hat{n}\). The direction of \(\hat{w}\) is determined by the right hand rule. I claim that:

\[
\hat{w} = \frac{\hat{z} \times \hat{n}}{|\hat{z} \times \hat{n}|},
\]

where \(|\hat{z} \times \hat{n}|\) is the length of the vector \(\hat{z} \times \hat{n}\). This vector clearly satisfies the property that \(\hat{z} \times \hat{n}\) is perpendicular to the plane containing \(\hat{z}\) and \(\hat{n}\) as required. Moreover, \(\hat{z} \times \hat{n}\) points in a direction that is determined by the right-hand rule.
To complete the proof, simply compute \( \hat{z} \times \hat{n} \).

\[
\hat{z} \times \hat{n} = \det \begin{pmatrix}
\hat{x} & \hat{y} & \hat{z} \\
0 & 0 & 1 \\
\sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta 
\end{pmatrix}
\]

\[= -\hat{x} \sin \theta \sin \phi + \hat{y} \sin \theta \cos \phi.\]

The length of the vector \( \hat{z} \times \hat{n} \) is

\[|\hat{z} \times \hat{n}| = \left[ \sin^2 \theta \sin^2 \phi + \sin^2 \theta \cos^2 \phi \right]^{1/2} = \sin \theta,
\]

Hence, we conclude that

\[
\hat{w} = \frac{\hat{z} \times \hat{n}}{|\hat{z} \times \hat{n}|} = -\hat{x} \sin \phi + \hat{y} \cos \phi = (-\sin \phi, \cos \phi, 0).
\]

(b) Define \(|\frac{1}{2}, \frac{1}{2}\rangle_{\hat{n}}\) to be an eigenstate of \(\vec{S} \cdot \hat{n}\) with eigenvalue \(\frac{1}{2} \hbar\). Express \(|\frac{1}{2}, \frac{1}{2}\rangle_{\hat{n}}\) with respect to the basis \(\alpha = |\frac{1}{2}, \frac{1}{2}\rangle_{\hat{z}} = |1\rangle\) and \(\beta = |\frac{1}{2}, -\frac{1}{2}\rangle_{\hat{z}} = |0\rangle\).

Using \(\vec{S} = \frac{\hbar}{2} \vec{\sigma}\), we can work out the dot product, \(\vec{S} \cdot \hat{n}\):

\[
\vec{S} \cdot \hat{n} = \frac{\hbar}{2} \begin{pmatrix}
\cos \theta & \sin \theta e^{-i\phi} \\
\sin \theta e^{i\phi} & -\cos \theta
\end{pmatrix}.
\]

In class, we worked out the eigenvectors of \(\vec{S} \cdot \hat{n}\). The eigenvector corresponding to eigenvalue \(\frac{1}{2} \hbar\) is given by

\[
|\frac{1}{2}, \frac{1}{2}\rangle_{\hat{n}} = e^{i\delta} \begin{pmatrix}
\cos(\theta/2)e^{-i\phi/2} \\
\sin(\theta/2)e^{i\phi/2}
\end{pmatrix},
\]

where \(e^{i\delta}\) is an arbitrary phase factor. This means that we can write:

\[
|\frac{1}{2}, \frac{1}{2}\rangle_{\hat{n}} = e^{i\delta} \left[ \alpha \cos(\theta/2)e^{-i\phi/2} + \beta \sin(\theta/2)e^{i\phi/2} \right].
\]

(c) Define \(R_{\hat{w}}(\theta)\) to be the rotation operator that corresponds to a rotation of state vectors by an angle \(\theta\) about the \(\hat{w}\) axis [defined in part (a)]. For a spin-\(\frac{1}{2}\) particle, I claim that

\[
R_{\hat{w}}(\theta) = \exp \left( -\frac{i}{2} \theta \hat{w} \cdot \vec{\sigma} \right).
\]

Let us put this operator to the test. Prove that

\[
R_{\hat{w}}(\theta) |\frac{1}{2}, \frac{1}{2}\rangle_{\hat{z}} = |\frac{1}{2}, \frac{1}{2}\rangle_{\hat{n}}.
\]

First, note that since \(\hat{w} = (-\sin \phi, \cos \phi, 0)\), it follows that:

\[
\hat{w} \cdot \vec{\sigma} = \begin{pmatrix}
0 & -\sin \phi - i \cos \phi \\
-\sin \phi + i \cos \phi & 0
\end{pmatrix} = -i \begin{pmatrix}
0 & e^{-i\phi} \\
e^{i\phi} & 0
\end{pmatrix}.
\]
Using the result of problem 3(c),

\[ R(\theta) = \exp \left( -\frac{i}{2} \theta \hat{\mathbf{w}} \cdot \mathbf{\sigma} \right) = I \cos(\theta/2) - i\hat{\mathbf{w}} \cdot \mathbf{\sigma} \sin(\theta/2) \]

\[ = \begin{pmatrix} \cos(\theta/2) & -e^{-i\phi} \sin(\theta/2) \\ e^{i\phi} \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}. \]

Using the fact that the state \( |\frac{1}{2}, \frac{1}{2}\rangle_\xi \) is represented by \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \), it follows that:

\[ R(\theta) |\frac{1}{2}, \frac{1}{2}\rangle_\xi = \begin{pmatrix} \cos(\theta/2) & e^{-i\phi} \sin(\theta/2/2) \\ e^{i\phi} \sin(\theta/2/2) & \cos(\theta/2) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = e^{i\phi/2} |\frac{1}{2}, \frac{1}{2}\rangle_\xi. \]

Indeed, \( R(\theta) |\frac{1}{2}, \frac{1}{2}\rangle_\xi \) is an eigenstate of \( \mathbf{\hat{S}} \cdot \mathbf{\hat{n}} \) with eigenvalue \( \frac{1}{2} \hbar \). Thus, choosing \( \delta = \phi/2 \) in eq. (4), we conclude that:

\[ R(\theta) |\frac{1}{2}, \frac{1}{2}\rangle_\xi = |\frac{1}{2}, \frac{1}{2}\rangle_\xi. \]

5. Consider a spin-\( \frac{1}{2} \) particle of magnetic moment \( \mathbf{\mu} = \gamma \mathbf{\hat{S}} \), where \( \gamma \equiv e/mc \). At time \( t = 0 \), the state of the system is given by \( \alpha \equiv |\frac{1}{2}, \frac{1}{2}\rangle_\xi \) (i.e., spin-up).

(a) If the observable \( S_x \) is measured at time \( t = 0 \), what results can be found and with what probabilities?

To see what the possible results of a measurement of \( S_x \), consider the initial state, \( |\frac{1}{2}, \frac{1}{2}\rangle_\xi \). Expand this state in terms of the two possible states, \( |\frac{1}{2}, \frac{1}{2}\rangle_\xi \) and \( |\frac{1}{2}, -\frac{1}{2}\rangle_\xi \) that can result from a measurement of \( S_x \). These two states are the normalized eigenstates of \( S_x = \frac{1}{2} \hbar \sigma_x \), namely,

\[ |\frac{1}{2}, \frac{1}{2}\rangle_\xi = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |\frac{1}{2}, -\frac{1}{2}\rangle_\xi = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \]

Using the above results, it is easy to determine that:

\[ |\frac{1}{2}, \frac{1}{2}\rangle_\xi = \frac{1}{\sqrt{2}} |\frac{1}{2}, \frac{1}{2}\rangle_\xi - \frac{1}{\sqrt{2}} |\frac{1}{2}, -\frac{1}{2}\rangle_\xi. \]

The possible measurements of \( S_x \) yield either \( \frac{1}{2} \hbar \) or \( -\frac{1}{2} \hbar \), with equal probabilities of \( \frac{1}{2} \) in the two cases (corresponding to the absolute square of the coefficients of \( |\frac{1}{2}, \pm\frac{1}{2}\rangle_\xi \) above).

(b) Instead of performing the measurement specified in part (a), we let the system evolve under the influence of a uniform magnetic field \( B \) parallel to the
y-axis (i.e., $\vec{B} = B\hat{y}$ with $B > 0$). Calculate the state of the system at time $t$ with respect to the \{\alpha, \beta\} basis.

The Hamiltonian of the system is

$$H = -\gamma \vec{S} \cdot \vec{B} = -\gamma BS_y = -\frac{1}{2}\hbar\gamma B\sigma_y.$$  

Thus, $|\psi(t)\rangle = U(t) |\psi(0)\rangle$, where

$$U(t) = e^{-iHt/\hbar} = \exp\left(\frac{1}{2}i\gamma B\sigma_y t\right).$$

Let us define $\omega_0 \equiv -\gamma B$. Then,

$$U(t) = \exp\left(-\frac{1}{2}i\omega_0 t\sigma_y\right) = I \cos(\omega_0 t/2) - i\sigma_y \sin(\omega_0 t/2) = \begin{pmatrix} \cos(\omega_0 t/2) & -\sin(\omega_0 t/2) \\ \sin(\omega_0 t/2) & \cos(\omega_0 t/2) \end{pmatrix}.$$ 

Here we have used the results of problem 3(c), and have identified $\hat{\omega} = (0, 1, 0)$. Note that it is critical that $\hat{\omega}$ is a unit vector. This requirement allows us to unambiguously identify $\theta \equiv \omega_0 t$.

Hence, the state of the system at time $t$ is:

$$|\psi(t)\rangle = U(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(\omega_0 t/2) \\ \sin(\omega_0 t/2) \end{pmatrix} = \alpha \cos(\omega_0 t/2) + \beta \sin(\omega_0 t/2). \quad (6)$$

(c) At a fixed time $t = T > 0$, we measure one of the observables $S_x, S_y$ and $S_z$. For each case, what values can be found and with what probabilities? Is there any value of $B$ (which may depend on $T$) such that one of the above measurements yields a unique result?

We now measure either $S_x, S_y$ or $S_z$. To address each case separately, we should expand the spinor obtained in part (b) in terms of a new basis consisting of the eigenstates of $S_x, S_y$ or $S_z$, respectively.

First, we expand in terms of eigenstates of $S_z$. But, this has already been done in part (b):

$$|\psi(t)\rangle = \cos(\omega_0 \theta/2) |\frac{1}{2}, \frac{1}{2}\rangle_\hat{z} + \sin(\omega_0 \theta/2) |\frac{1}{2}, -\frac{1}{2}\rangle_\hat{z}.$$ 

Thus the possible results obtained by measuring $S_z$ are:

$$+\frac{1}{2}\hbar, \quad \text{with probability} \quad \cos^2(\omega_0 t/2) = \frac{1}{2}(1 + \cos\omega_0 t),$$

$$-\frac{1}{2}\hbar, \quad \text{with probability} \quad \sin^2(\omega_0 t/2) = \frac{1}{2}(1 - \cos\omega_0 t),$$

where the probabilities are identified as the absolute squares of the coefficients of $|\frac{1}{2}, \pm\frac{1}{2}\rangle_\hat{z}$, respectively.
Second, we expand $|\psi(t)\rangle$ in terms of eigenstates of $S_x$:

$$|\psi(t)\rangle = c_+ |\frac{1}{2}, \frac{1}{2}\rangle + c_- |\frac{1}{2}, -\frac{1}{2}\rangle,$$

where

$$c_\pm = \hat{a} \langle \frac{1}{2}, \pm \frac{1}{2} | \psi(t) \rangle.$$

Using eqs. (5) and (6), one can easily evaluate $c_\pm$,

$$c_+ = \left( \frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}} \right) \begin{pmatrix} \cos(\omega_0 t/2) \\ \sin(\omega_0 t/2) \end{pmatrix} = \frac{1}{\sqrt{2}} \left[ \cos(\omega_0 t/2) + \sin(\omega_0 t/2) \right],$$

$$c_- = \left( -\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}} \right) \begin{pmatrix} \cos(\omega_0 t/2) \\ \sin(\omega_0 t/2) \end{pmatrix} = \frac{1}{\sqrt{2}} \left[ -\cos(\omega_0 t/2) + \sin(\omega_0 t/2) \right].$$

To get the relevant probabilities, we need to square these results:

$$|c_+|^2 = \frac{1}{2} \left[ \sin^2(\omega_0 t/2) + \cos^2(\omega_0 t/2) + 2 \sin(\omega_0 t/2) \cos(\omega_0 t/2) \right] = \frac{1}{2} \left[ 1 + \sin(\omega_0 t) \right],$$

$$|c_-|^2 = \frac{1}{2} \left[ \sin^2(\omega_0 t/2) + \cos^2(\omega_0 t/2) - 2 \sin(\omega_0 t/2) \cos(\omega_0 t/2) \right] = \frac{1}{2} \left[ 1 - \sin(\omega_0 t) \right].$$

Hence, the possible results obtained by measuring $S_x$ are:

$$+\frac{1}{2} \hbar, \quad \text{with probability} \quad \frac{1}{2} \left[ 1 + \sin(\omega_0 t) \right],$$

$$-\frac{1}{2} \hbar, \quad \text{with probability} \quad \frac{1}{2} \left[ 1 - \sin(\omega_0 t) \right].$$

Finally, we expand $|\psi(t)\rangle$ in terms of eigenstates of $S_y$. Here, we note that the normalized eigenstates of $S_y = \frac{1}{2} \hbar \sigma_y$ are:

$$|\frac{1}{2}, \frac{1}{2}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad |\frac{1}{2}, -\frac{1}{2}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}. \quad (7)$$

Following the previous computation, we write:

$$|\psi(t)\rangle = c_+ |\frac{1}{2}, \frac{1}{2}\rangle + c_- |\frac{1}{2}, -\frac{1}{2}\rangle,$$

where

$$c_\pm = \hat{y} \langle \frac{1}{2}, \pm \frac{1}{2} | \psi(t) \rangle.$$

Using eqs. (6) and (7), one can easily evaluate $c_\pm$. Keep in mind that in this computation, the adjoint spinor $\hat{y} \langle \frac{1}{2}, \frac{1}{2} \rangle$ appears, so we have to complex conjugate the results of eq. (7), which changes the sign of $i$. Hence,

$$c_+ = \left( \frac{1}{\sqrt{2}} \quad -\frac{i}{\sqrt{2}} \right) \begin{pmatrix} \cos(\omega_0 t/2) \\ \sin(\omega_0 t/2) \end{pmatrix} = \frac{1}{\sqrt{2}} e^{-i\omega_0 t/2},$$

$$c_- = \left( -\frac{1}{\sqrt{2}} \quad \frac{i}{\sqrt{2}} \right) \begin{pmatrix} \cos(\omega_0 t/2) \\ \sin(\omega_0 t/2) \end{pmatrix} = \frac{1}{\sqrt{2}} e^{i\omega_0 t/2}.$$

\(^2\)Here, we are using the fact that we can expand any state $|\psi\rangle$ in terms of a complete basis $\{e_1, e_2\}$ as follows: $|\psi\rangle = \sum_{i=1}^{2} |e_i\rangle \langle e_i | \psi\rangle$. In this case, the complete basis is made up of $|\frac{1}{2}, \frac{1}{2}\rangle$.\)
To get the relevant probabilities, we need to square these results, which yields $|c_\pm|^2 = \frac{1}{2}$. Hence, the possible results obtained by measuring $S_y$ are:

\[ +\frac{\hbar}{2}, \quad \text{with probability } \frac{1}{2}; \]
\[ -\frac{\hbar}{2}, \quad \text{with probability } \frac{1}{2}. \]

Thus, the measurement of $S_y$ never gives a unique result. This could have been anticipated, since at $t = 0$, the initial state $\frac{1}{2}, \frac{1}{2} \rangle \hat{z}$ is equally likely to be either spin-up or spin-down with respect to the $y$-axis. But, a $B$ field applied in the $\hat{y}$ direction does not modify the $y$-component of the spin.

However, the measurement of $S_x$ and $S_z$ can give unique results at some fixed time $t = T \geq 0$, if $B$ (or equivalently, $\omega_0 \equiv -\gamma B$) takes on particular values. Noting that for any integer $n$,

\[
\cos(n\pi) = (-1)^n, \quad \sin[(n + \frac{1}{2})\pi] = (-1)^n,
\]

we see that:

- If $\omega_0 T = n\pi$, then the measurement of $S_z$ yields:
  1. pure spin up, if $n$ is even,
  2. pure spin down, if $n$ is odd.

- If $\omega_0 T = (n + \frac{1}{2})\pi$, then the measurement of $S_x$ yields:
  1. pure spin up, if $n$ is even,
  2. pure spin down, if $n$ is odd.

Using $\omega_0 \equiv -\gamma B$, it follows that the measurement of $S_x$ is unique if

\[ B = \frac{n + \frac{1}{2}}{|\gamma|T}, \quad \text{for } n = 0, 1, 2, \ldots, \]

and the measurement of $S_z$ is unique if

\[ B = \frac{n\pi}{|\gamma|T}, \quad \text{for } n = 1, 2, 3, \ldots. \]

Note that the integer $n$ is restricted to non-negative values in the first case and positive values in the second case, since by assumption $T \geq 0$ (and $B > 0$). Finally, as previously noted, the measurement of $S_y$ is never unique.