How do the connection coefficients transform under a general coordinate transformation?

1. Definition of the connection coefficients and the covariant derivative

The existence of an abstract vector $\vec{\boldsymbol{V}}$ that lives in a curved spacetime does not depend on the coordinates chosen to describe the location of spacetime points. But, once a coordinate system is chosen, one can also establish a set of linearly independent basis vectors $\vec{\boldsymbol{e}}_\mu$, where $\mu = 0, 1, 2, 3$ labels the four basis vectors (the maximally allowed number in a four-dimensional spacetime). We can then expand any abstract vector $\vec{\boldsymbol{V}}$ in terms of the four basis vectors,

$$\vec{\boldsymbol{V}} = V^\mu \vec{\boldsymbol{e}}_\mu \,, \tag{1}$$

where there is an implicit sum over μ following the Einstein summation convention. The four numbers V^μ are the *contravariant components* of the abstract vector $\vec{\boldsymbol{V}}$ with respect to the chosen coordinate system.[1] Eq. (1) refers to a particular point in the spacetime. We can just as well examine vector defined at another point in the spacetime. We can again make use of eq. (1), keeping in mind that the orientation of the basis vectors can be different. That is, in general $\vec{\boldsymbol{e}}_\mu$ depends on the location in spacetime. The basis vectors are orthonormal with respect to the metric tensor, which means that

$$\vec{\boldsymbol{e}}_\mu \cdot \vec{\boldsymbol{e}}_\nu = g_{\mu\nu} \,. \tag{2}$$

It follows from eqs. (1) and (2) that the dot product of two vectors $\vec{\boldsymbol{V}}$ and $\vec{\boldsymbol{W}}$ is given by

$$\vec{\boldsymbol{V}} \cdot \vec{\boldsymbol{W}} = V^\mu W^\nu \, \vec{\boldsymbol{e}}_\mu \cdot \vec{\boldsymbol{e}}_\nu = g_{\mu\nu} V^\mu W^\nu \,,$$

as expected.

We can now define how vectors in the spacetime change as one moves from one spacetime point to another. A simple computation yields

$$\begin{aligned} d\vec{\boldsymbol{V}} &= d(V^\mu \vec{\boldsymbol{e}}_\mu) = dV^\mu \vec{\boldsymbol{e}}_\mu + V^\mu d\vec{\boldsymbol{e}}_\mu \\ &= \frac{\partial V^\mu}{\partial x^\alpha} dx^\alpha \vec{\boldsymbol{e}}_\mu + V^\mu d\vec{\boldsymbol{e}}_\mu \,, \end{aligned} \tag{3}$$

where the chain rule has been employed in the final step. To complete the analysis, we must determine how $\vec{\boldsymbol{e}}_\mu$ changes as one moves from one spacetime point to another. Since $\vec{\boldsymbol{e}}_\mu$

[1] In contrast, the subscript μ employed by $\vec{\boldsymbol{e}}_\mu$ refers to one of the four possible basis vectors. In this sense, each of the four $\vec{\boldsymbol{e}}_\mu$ are abstract vectors in the same way that $\vec{\boldsymbol{V}}$ is an abstract vector.

depends on the location in spacetime (which is specified by the spacetime coordinate x^α), the change in $\vec{\boldsymbol{e}}_\mu$ as one moves from one spacetime point to another is encoded in the partial derivatives, $\partial\vec{\boldsymbol{e}}_\mu/\partial x^\alpha$. This latter quantity is also an abstract vector, so we can express it as a linear combination of basis vectors, in analogy with eq. (1),

$$\frac{\partial\vec{\boldsymbol{e}}_\mu}{\partial x^\alpha} = \Gamma^\beta_{\alpha\mu}\vec{\boldsymbol{e}}_\beta\,, \tag{4}$$

where the coefficients $\Gamma^\beta_{\alpha\mu}$, called the *connection coefficients*, determine how the basis changes as one moves from one spacetime point to another. That is, eq. (4) serves as the *definition* of the connection coefficients.[2] Using the chain rule, it follows that

$$d\vec{\boldsymbol{e}}_\mu = \frac{\partial\vec{\boldsymbol{e}}_\mu}{\partial x^\alpha}dx^\alpha = \Gamma^\beta_{\alpha\mu}dx^\alpha\vec{\boldsymbol{e}}_\beta\,. \tag{5}$$

Inserting eq. (5) into eq. (3) yields

$$d\vec{\boldsymbol{V}} = \frac{\partial V^\mu}{\partial x^\alpha}dx^\alpha\vec{\boldsymbol{e}}_\mu + \Gamma^\beta_{\alpha\mu}V^\mu dx^\alpha\vec{\boldsymbol{e}}_\beta\,. \tag{6}$$

Note that $\Gamma^\beta_{\alpha\mu}V^\mu dx^\alpha\vec{\boldsymbol{e}}_\beta = \Gamma^\mu_{\alpha\beta}V^\beta dx^\alpha\vec{\boldsymbol{e}}_\mu$, after relabeling the two pairs of dummy indices. Inserting this result back into eq. (6) yields

$$d\vec{\boldsymbol{V}} = \left(\frac{\partial V^\mu}{\partial x^\alpha} + \Gamma^\mu_{\alpha\beta}V^\beta\right)dx^\alpha\,\vec{\boldsymbol{e}}_\mu\,. \tag{7}$$

This result motivates the definition of the *covariant derivative*, $D_\alpha V^\mu$, as follows:

$$D_\alpha V^\mu \equiv \frac{\partial V^\mu}{\partial x^\alpha} + \Gamma^\mu_{\alpha\beta}V^\beta\,. \tag{8}$$

Inserting eq. (8) back into eq. (7), it follows that

$$d\vec{\boldsymbol{V}} = (D_\alpha V^\mu)dx^\alpha\,\vec{\boldsymbol{e}}_\mu\,. \tag{9}$$

The importance of eq. (9) is as follows. Given the abstract vector given by eq. (1), we identify the contravariant components V^μ, which transforms under a general coordinate transformation, $x' = x'(x)$ as follows,

$$V'^\mu = \frac{\partial x'^\mu}{\partial x^\alpha}V^\alpha\,. \tag{10}$$

Likewise, $d\vec{\boldsymbol{V}}$ is an abstract vector given by eq. (9). Thus, we identify the contravariant components of this vector by $(D_\alpha V^\mu)dx^\alpha$. This means that $D_\alpha V^\mu$ transforms under a general coordinate transformation as a second rank mixed tensor.[3]

[2]The definition of the connection coefficients employed in eq. (4) follows the conventions established by our textbook. Other textbooks define $\partial\vec{\boldsymbol{e}}_\mu/\partial x^\alpha = \Gamma^\beta_{\mu\alpha}\vec{\boldsymbol{e}}_\beta$ which has the order of the two lower indices of the connection coefficients switched with respect to eq. (4). Ultimately it will not matter, since in general relativity, the connection coefficients are symmetric under the interchange of the two lower indices [cf. eq. (39)].

[3]This conclusion is implicitly obvious, since it is ensured by the tensor notation and the rules for manipulating indices. However, it is formally a result of a theorem of tensor algebra known as the quotient theorem. This theorem is alluded to in problem 12.6 on pp. 296–297 of our textbook. For a more thorough discussion, see, e.g. p. 532 of Boas.

2. Transformation properties of the connection coefficients

Under a general coordinate transformation, the contravariant components of $\vec{\boldsymbol{V}}$ and the basis vectors $\vec{\boldsymbol{e}}$ must transform in such a way that the abstract vector $\vec{\boldsymbol{V}}$ (which exists independently of the choice of coordinates) is unaffected. That is,

$$\vec{\boldsymbol{V}} = V'^{\mu}\vec{\boldsymbol{e}}'_{\mu} = V^{\mu}\vec{\boldsymbol{e}}_{\mu} \,.$$

In light of eq. (10), we must have

$$\vec{\boldsymbol{e}}'_{\mu} = \frac{\partial x^{\beta}}{\partial x'^{\mu}}\vec{\boldsymbol{e}}_{\beta} \,, \tag{11}$$

since in this case

$$\vec{\boldsymbol{V}} = V'^{\mu}\vec{\boldsymbol{e}}'_{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x^{\beta}}{\partial x'^{\mu}}V^{\alpha}\vec{\boldsymbol{e}}_{\beta} = \frac{\partial x^{\beta}}{\partial x^{\alpha}}V^{\alpha}\vec{\boldsymbol{e}}_{\beta} = \delta^{\beta}_{\alpha}V^{\alpha}\vec{\boldsymbol{e}}_{\beta} = V^{\alpha}\vec{\boldsymbol{e}}_{\alpha} \,,$$

after making use of the chain rule.

To derive the transformation law of the connection coefficients, we employ eq. (4) in the primed coordinate system,

$$\frac{\partial\vec{\boldsymbol{e}}'_{\mu}}{\partial x'^{\alpha}} = \Gamma'^{\rho}_{\alpha\mu}\vec{\boldsymbol{e}}'_{\rho} \,, \tag{12}$$

Using eq. (11) on both sides of eq. (12),

$$\frac{\partial}{\partial x'^{\alpha}}\left(\frac{\partial x^{\beta}}{\partial x'^{\mu}}\vec{\boldsymbol{e}}_{\beta}\right) = \Gamma'^{\rho}_{\alpha\mu}\frac{\partial x^{\beta}}{\partial x'^{\rho}}\vec{\boldsymbol{e}}_{\beta} \,. \tag{13}$$

Employing the product rule on the left hand side of eq. (13),

$$\frac{\partial^2 x^{\beta}}{\partial x'^{\alpha}\partial x'^{\mu}}\vec{\boldsymbol{e}}_{\beta} + \frac{\partial x^{\beta}}{\partial x'^{\mu}}\frac{\partial\vec{\boldsymbol{e}}_{\beta}}{\partial x'^{\alpha}} = \Gamma'^{\rho}_{\alpha\mu}\frac{\partial x^{\beta}}{\partial x'^{\rho}}\vec{\boldsymbol{e}}_{\beta} \,. \tag{14}$$

Applying the chain rule to the second term on the left hand side of eq. (14).

$$\frac{\partial^2 x^{\beta}}{\partial x'^{\alpha}\partial x'^{\mu}}\vec{\boldsymbol{e}}_{\beta} + \frac{\partial x^{\beta}}{\partial x'^{\mu}}\frac{\partial x^{\tau}}{\partial x'^{\alpha}}\frac{\partial\vec{\boldsymbol{e}}_{\beta}}{\partial x^{\tau}} = \Gamma'^{\rho}_{\alpha\mu}\frac{\partial x^{\beta}}{\partial x'^{\rho}}\vec{\boldsymbol{e}}_{\beta} \,.$$

Using the definition of the connection coefficients given in eq. (4),

$$\frac{\partial^2 x^{\beta}}{\partial x'^{\alpha}\partial x'^{\mu}}\vec{\boldsymbol{e}}_{\beta} + \frac{\partial x^{\beta}}{\partial x'^{\mu}}\frac{\partial x^{\tau}}{\partial x'^{\alpha}}\Gamma^{\rho}_{\tau\beta}\vec{\boldsymbol{e}}_{\rho} = \Gamma'^{\rho}_{\alpha\mu}\frac{\partial x^{\beta}}{\partial x'^{\rho}}\vec{\boldsymbol{e}}_{\beta} \,.$$

Relabeling the indices of the second term above, $\beta \to \rho$ and $\rho \to \beta$,

$$\frac{\partial^2 x^{\beta}}{\partial x'^{\alpha}\partial x'^{\mu}}\vec{\boldsymbol{e}}_{\beta} + \frac{\partial x^{\rho}}{\partial x'^{\mu}}\frac{\partial x^{\tau}}{\partial x'^{\alpha}}\Gamma^{\beta}_{\tau\rho}\vec{\boldsymbol{e}}_{\beta} = \Gamma'^{\rho}_{\alpha\mu}\frac{\partial x^{\beta}}{\partial x'^{\rho}}\vec{\boldsymbol{e}}_{\beta} \,.$$

We can now factor out a common factor, and we end up with

$$\left(\Gamma'^{\,\rho}_{\alpha\mu}\frac{\partial x^\beta}{\partial x'^{\,\rho}} - \frac{\partial x^\rho}{\partial x'^{\,\mu}}\frac{\partial x^\tau}{\partial x'^{\,\alpha}}\Gamma^\beta_{\tau\rho} - \frac{\partial^2 x^\beta}{\partial x'^{\,\alpha}\partial x'^{\,\mu}}\right)\vec{\boldsymbol{e}}_\beta = 0\,. \tag{15}$$

Since the $\vec{\boldsymbol{e}}_\beta$ are four linearly independent basis vectors, we conclude that the expression inside the parentheses in eq. (15) must vanish. That is,

$$\Gamma'^{\,\rho}_{\alpha\mu}\frac{\partial x^\beta}{\partial x'^{\,\rho}} = \frac{\partial x^\rho}{\partial x'^{\,\mu}}\frac{\partial x^\tau}{\partial x'^{\,\alpha}}\Gamma^\beta_{\tau\rho} + \frac{\partial^2 x^\beta}{\partial x'^{\,\alpha}\partial x'^{\,\mu}}\,. \tag{16}$$

Finally, we multiply both sides of eq. (16) by $\partial x'^{\,\sigma}/\partial x^\beta$, and make use of the identity

$$\frac{\partial x'^{\,\sigma}}{\partial x^\beta}\frac{\partial x^\beta}{\partial x'^{\,\rho}} = \frac{\partial x'^{\,\sigma}}{\partial x'^{\,\rho}} = \delta^\sigma_\rho\,,$$

which is a consequence of the chain rule. The end result after summing over the repeated index ρ,

$$\boxed{\Gamma'^{\,\sigma}_{\alpha\mu} = \frac{\partial x'^{\,\sigma}}{\partial x^\beta}\frac{\partial x^\rho}{\partial x'^{\,\mu}}\frac{\partial x^\tau}{\partial x'^{\,\alpha}}\Gamma^\beta_{\tau\rho} + \frac{\partial x'^{\,\sigma}}{\partial x^\beta}\frac{\partial^2 x^\beta}{\partial x'^{\,\alpha}\partial x'^{\,\mu}}\,.} \tag{17}$$

There is an alternate form of eq. (17) that is sometimes useful. To derive it, we begin with an identity that is a consequence of the chain rule,

$$\frac{\partial x'^{\,\beta}}{\partial x^\mu}\frac{\partial x^\mu}{\partial x'^{\,\alpha}} = \frac{\partial x'^{\,\beta}}{\partial x'^{\,\alpha}} = \delta^\beta_\alpha\,, \tag{18}$$

Take the partial derivative of this equation with respect to $x'^{\,\rho}$. Since δ^β_α is the Kronecker delta (in particular, it does not depend on the location in spacetime), it follows that $\partial\delta^\beta_\alpha/\partial x'^{\,\rho} = 0$. Hence,

$$\frac{\partial}{\partial x'^{\,\rho}}\left(\frac{\partial x'^{\,\beta}}{\partial x^\mu}\frac{\partial x^\mu}{\partial x'^{\,\alpha}}\right) = 0\,. \tag{19}$$

Using the product rule, eq. (19) yields

$$\left[\frac{\partial}{\partial x'^{\,\rho}}\left(\frac{\partial x'^{\,\beta}}{\partial x^\mu}\right)\right]\frac{\partial x^\mu}{\partial x'^{\,\alpha}} + \frac{\partial x'^{\,\beta}}{\partial x^\mu}\frac{\partial^2 x^\mu}{\partial x'^{\,\rho}\partial x'^{\,\alpha}} = 0\,. \tag{20}$$

We now evaluate the term in brackets above by invoking the chain rule,

$$\frac{\partial}{\partial x'^{\,\rho}}\left(\frac{\partial x'^{\,\beta}}{\partial x^\mu}\right) = \frac{\partial x^\tau}{\partial x'^{\,\rho}}\frac{\partial}{\partial x^\tau}\left(\frac{\partial x'^{\,\beta}}{\partial x^\mu}\right) = \frac{\partial x^\tau}{\partial x'^{\,\rho}}\frac{\partial^2 x'^{\,\beta}}{\partial x^\tau\partial x^\mu}\,.$$

Using this result in eq. (20) leads to the following identity,

$$\frac{\partial x'^{\,\beta}}{\partial x^\mu}\frac{\partial^2 x^\mu}{\partial x'^{\,\rho}\partial x'^{\,\alpha}} = -\frac{\partial x^\tau}{\partial x'^{\,\rho}}\frac{\partial^2 x'^{\,\beta}}{\partial x^\tau\partial x^\mu}\frac{\partial x^\mu}{\partial x'^{\,\alpha}}\,. \tag{21}$$

Let us relabel the indices in eq. (21) as follows: $\beta \to \sigma$, $\mu \to \beta$, $\rho \to \alpha$ and $\alpha \to \mu$. The result of this index gymnastics is

$$\frac{\partial x'^{\,\sigma}}{\partial x^\beta}\frac{\partial^2 x^\beta}{\partial x'^{\,\alpha}\partial x'^{\,\mu}} = -\frac{\partial x^\tau}{\partial x'^{\,\alpha}}\,\frac{\partial^2 x'^{\,\sigma}}{\partial x^\tau \partial x^\beta}\,\frac{\partial x^\beta}{\partial x'^{\,\mu}}\,. \tag{22}$$

The left hand side of eq. (22) is precisely the last term that appears in eq. (17). Thus, an alternate form of the transformation law for the connection coefficients under a general coordinate transformation is

$$\boxed{\Gamma'^{\,\sigma}_{\alpha\mu} = \frac{\partial x'^{\,\sigma}}{\partial x^\beta}\frac{\partial x^\rho}{\partial x'^{\,\mu}}\frac{\partial x^\tau}{\partial x'^{\,\alpha}}\Gamma^\beta_{\tau\rho} - \frac{\partial x^\tau}{\partial x'^{\,\alpha}}\,\frac{\partial^2 x'^{\,\sigma}}{\partial x^\tau \partial x^\beta}\,\frac{\partial x^\beta}{\partial x'^{\,\mu}}\,.} \tag{23}$$

Both forms of the transformation law given in eqs. (17) and (23) imply that $\Gamma^\sigma_{\alpha\mu}$ is not a tensor. In particular, under a general coordinate transformation, a third rank tensor $T^\sigma_{\alpha\mu}$ would transform as,

$$T'^{\,\sigma}_{\alpha\mu} = \frac{\partial x'^{\,\sigma}}{\partial x^\beta}\frac{\partial x^\rho}{\partial x'^{\,\mu}}\frac{\partial x^\tau}{\partial x'^{\,\alpha}}T^\beta_{\tau\rho}\,. \tag{24}$$

Comparing eq. (24) with the transformation law for the connection coefficients, we see that it is the presence of the inhomogeneous term[4] that is the origin of the non-tensorial property of $\Gamma^\sigma_{\alpha\mu}$.

3. Proof that the covariant derivative of a vector transforms like a tensor

An inhomogeneous term also appears in the transformation law of the ordinary partial derivative, $\partial V^\mu/\partial x^\alpha$, with respect to general coordinate transformations. This should already be obvious in light of eq. (3). But, let us prove this assertion directly by taking the derivative of eq. (10) with respect to $x'^{\,\alpha}$.

$$\frac{\partial V'^{\,\mu}}{\partial x'^{\,\alpha}} = \frac{\partial}{\partial x'^{\,\alpha}}\left(\frac{\partial x'^{\,\mu}}{\partial x^\rho}V^\rho\right) = \frac{\partial x^\beta}{\partial x'^{\,\alpha}}\,\frac{\partial}{\partial x^\beta}\left(\frac{\partial x'^{\,\mu}}{\partial x^\rho}V^\rho\right)\,, \tag{25}$$

where we have made use of the chain rule,

$$\frac{\partial}{\partial x'^{\,\alpha}} = \frac{\partial x^\beta}{\partial x'^{\,\alpha}}\,\frac{\partial}{\partial x^\beta}\,.$$

Evaluating the partial derivative with respect to x^β in eq. (25) using the product rule, the end result is

$$\frac{\partial V'^{\,\mu}}{\partial x'^{\,\alpha}} = \frac{\partial x^\beta}{\partial x'^{\,\alpha}}\,\frac{\partial x'^{\,\mu}}{\partial x^\rho}\,\frac{\partial V^\rho}{\partial x^\beta} + \frac{\partial x^\beta}{\partial x'^{\,\alpha}}\,\frac{\partial^2 x'^{\,\mu}}{\partial x^\beta \partial x^\rho}\,V^\rho\,. \tag{26}$$

[4]In this context, the inhomogeneous term refers to the last term in eq. (17) or eq. (23), which is independent of the connection coefficients. In contrast, the transformation law given in eq. (24) is homogeneous since the tensor T appears on both sides of the equation.

This should be compared with the transformation law for a mixed second rank tensor, $T_\alpha{}^\mu$, with respect to a general coordinate transformations,

$$T'^{\ \mu}_\alpha = \frac{\partial x^\beta}{\partial x'^{\,\alpha}} \frac{\partial x'^{\,\mu}}{\partial x^\rho} T_\beta{}^\rho \,. \tag{27}$$

In particular, $\partial V^\mu/\partial x^\mu$ fails to have the correct tensorial transformation law under a general coordinate transformation due to the presence of the inhomogeneous term (i.e., the last term) in eq. (26).

It is remarkable that the covariant derivative of a vector transforms like a tensor since

$$D_\alpha V^\mu \equiv \frac{\partial V^\mu}{\partial x^\alpha} + \Gamma^\mu_{\alpha\beta} V^\beta \,.$$

is the sum of two quantities whose separate transformation laws are not tensorial with respect to general coordinate transformations. Let us see how this is possible. We compute

$$\begin{aligned} \frac{\partial V'^{\,\mu}}{\partial x'^{\,\alpha}} + \Gamma'^{\,\mu}_{\alpha\beta} V'^{\,\beta} &= \frac{\partial x^\beta}{\partial x'^{\,\alpha}} \frac{\partial x'^{\,\mu}}{\partial x^\rho} \frac{\partial V^\rho}{\partial x^\beta} + \frac{\partial x^\beta}{\partial x'^{\,\alpha}} \frac{\partial^2 x'^{\,\mu}}{\partial x^\beta \partial x^\rho} V^\rho \\ &\quad + \left(\frac{\partial x'^{\,\mu}}{\partial x^\beta} \frac{\partial x^\rho}{\partial x'^{\,\beta}} \frac{\partial x^\tau}{\partial x'^{\,\alpha}} \Gamma^\beta_{\tau\rho} - \frac{\partial x^\tau}{\partial x'^{\,\alpha}} \frac{\partial^2 x'^{\,\mu}}{\partial x^\tau \partial x^\rho} \frac{\partial x^\rho}{\partial x'^{\,\beta}} \right) \frac{\partial x'^{\,\beta}}{\partial x^\nu} V^\nu \,. \end{aligned} \tag{28}$$

We now simplify the second line of eq. (28) with the help of the chain rule,

$$\frac{\partial x^\rho}{\partial x'^{\,\beta}} \frac{\partial x'^{\,\beta}}{\partial x^\nu} = \frac{\partial x^\rho}{\partial x^\nu} = \delta^\rho_\nu \,.$$

Using the resulting Kronecker delta to sum over ν, we obtain

$$\frac{\partial V'^{\,\mu}}{\partial x'^{\,\alpha}} + \Gamma'^{\,\mu}_{\alpha\beta} V'^{\,\beta} = \frac{\partial x^\beta}{\partial x'^{\,\alpha}} \frac{\partial x'^{\,\mu}}{\partial x^\rho} \frac{\partial V^\rho}{\partial x^\beta} + \frac{\partial x^\beta}{\partial x'^{\,\alpha}} \frac{\partial^2 x'^{\,\mu}}{\partial x^\beta \partial x^\rho} V^\rho + \left(\frac{\partial x'^{\,\mu}}{\partial x^\beta} \frac{\partial x^\tau}{\partial x'^{\,\alpha}} \Gamma^\beta_{\tau\rho} - \frac{\partial x^\tau}{\partial x'^{\,\alpha}} \frac{\partial^2 x'^{\,\mu}}{\partial x^\tau \partial x^\rho} \right) V^\rho \,. \tag{29}$$

We now relabel the indices on the second line of eq. (29) as follows: $\tau \to \beta$ and $\beta \to \tau$, which yields

$$\begin{aligned} \frac{\partial V'^{\,\mu}}{\partial x'^{\,\alpha}} + \Gamma'^{\,\mu}_{\alpha\beta} V'^{\,\beta} &= \frac{\partial x^\beta}{\partial x'^{\,\alpha}} \frac{\partial x'^{\,\mu}}{\partial x^\rho} \frac{\partial V^\rho}{\partial x^\beta} + \frac{\partial x^\beta}{\partial x'^{\,\alpha}} \frac{\partial^2 x'^{\,\mu}}{\partial x^\beta \partial x^\rho} V^\rho + \left(\frac{\partial x'^{\,\mu}}{\partial x^\tau} \frac{\partial x^\beta}{\partial x'^{\,\alpha}} \Gamma^\tau_{\beta\rho} - \frac{\partial x^\beta}{\partial x'^{\,\alpha}} \frac{\partial^2 x'^{\,\mu}}{\partial x^\beta \partial x^\rho} \right) V^\rho \\ &= \frac{\partial x^\beta}{\partial x'^{\,\alpha}} \frac{\partial x'^{\,\mu}}{\partial x^\rho} \frac{\partial V^\rho}{\partial x^\beta} + \frac{\partial x'^{\,\mu}}{\partial x^\tau} \frac{\partial x^\beta}{\partial x'^{\,\alpha}} \Gamma^\tau_{\beta\rho} V^\rho \,. \end{aligned} \tag{30}$$

Note the cancellation of the two terms which were associated with the inhomogeneous terms of the transformation laws of $\partial V^\mu/\partial x^\alpha$ and $\Gamma^\mu_{\alpha\beta}$, respectively! Finally, relabeling $\rho \to \tau$ in the first term on the last line of eq. (30) yields

$$\frac{\partial V'^{\,\mu}}{\partial x'^{\,\alpha}} + \Gamma'^{\,\mu}_{\alpha\beta} V'^{\,\beta} = \frac{\partial x^\beta}{\partial x'^{\,\alpha}} \frac{\partial x'^{\,\mu}}{\partial x^\tau} \left(\frac{\partial V^\tau}{\partial x^\beta} + \Gamma^\tau_{\beta\rho} V^\rho \right) \,. \tag{31}$$

Using the definition of the covariant derivative [cf. eq. (8)], we can rewrite eq. (31) as

$$D'_\alpha V^\mu = \frac{\partial x^\beta}{\partial x'^{\,\alpha}}\,\frac{\partial x'^{\,\mu}}{\partial x^\tau}\,D_\beta V^\tau\,. \tag{32}$$

But this is the transformation law for a second rank mixed tensor. As advertised, the inhomogeneous terms in the transformation laws for $\partial V^\mu/\partial x^\alpha$ and $\Gamma^\mu_{\alpha\beta}$ have canceled out! Of course, this result was expected in light of the discussion following eq. (10).

4. Two applications of the transformation law for the connection coefficients

1. Define the *torsion tensor* by

$$Q^\mu_{\alpha\beta} \equiv \Gamma^\mu_{\alpha\beta} - \Gamma^\mu_{\beta\alpha}\,.$$

That is, the torsion tensor is the antisymmetric part of the connection coefficients, i.e. it satisfies $Q^\mu_{\alpha\beta} = -Q^\mu_{\beta\alpha}$. To justify calling $Q^\mu_{\alpha\beta}$ a tensor, we shall now prove that its transformation properties are indeed tensorial. Using eq. (17),

$$Q'^{\,\sigma}_{\alpha\mu} = \Gamma'^{\,\sigma}_{\alpha\mu} - \Gamma'^{\,\sigma}_{\mu\alpha} = \frac{\partial x'^{\,\sigma}}{\partial x^\beta}\,\frac{\partial x^\rho}{\partial x'^{\,\mu}}\,\frac{\partial x^\tau}{\partial x'^{\,\alpha}}\,\Gamma^\beta_{\tau\rho} - \frac{\partial x'^{\,\sigma}}{\partial x^\beta}\,\frac{\partial x^\rho}{\partial x'^{\,\alpha}}\,\frac{\partial x^\tau}{\partial x'^{\,\mu}}\,\Gamma^\beta_{\tau\rho}\,. \tag{33}$$

Note that the inhomogeneous term in eq. (17) has dropped out due to[5]

$$\frac{\partial^2 x^\beta}{\partial x'^{\,\alpha}\partial x'^{\,\mu}} = \frac{\partial^2 x^\beta}{\partial x'^{\,\mu}\partial x'^{\,\alpha}}\,.$$

If we now relabel the dummy indices, $\beta \to \tau$ and $\tau \to \beta$, in the last term of eq. (33), then we obtain

$$Q'^{\,\sigma}_{\alpha\mu} = \Gamma'^{\,\sigma}_{\alpha\mu} - \Gamma'^{\,\sigma}_{\mu\alpha} = \frac{\partial x'^{\,\sigma}}{\partial x^\beta}\,\frac{\partial x^\rho}{\partial x'^{\,\mu}}\,\frac{\partial x^\tau}{\partial x'^{\,\alpha}}(\Gamma^\beta_{\tau\rho} - \Gamma^\beta_{\rho\tau}) = \frac{\partial x'^{\,\sigma}}{\partial x^\beta}\,\frac{\partial x^\rho}{\partial x'^{\,\mu}}\,\frac{\partial x^\tau}{\partial x'^{\,\alpha}}\,Q^\beta_{\tau\rho}\,.$$

That is, we have shown that under a general coordinate transformation, the torsion tensor transforms as

$$Q'^{\,\sigma}_{\alpha\mu} = \frac{\partial x'^{\,\sigma}}{\partial x^\beta}\,\frac{\partial x^\rho}{\partial x'^{\,\mu}}\,\frac{\partial x^\tau}{\partial x'^{\,\alpha}}\,Q^\beta_{\tau\rho}\,, \tag{34}$$

which is the correct transformation law for a mixed third rank tensor with two covariant indices and one contravariant index.

2. Consider the transformation law of the connection coefficients given in eq. (17), which is rewritten below,

$$\Gamma'^{\,\sigma}_{\alpha\mu} = \frac{\partial x'^{\,\sigma}}{\partial x^\beta}\,\frac{\partial x^\rho}{\partial x'^{\,\mu}}\,\frac{\partial x^\tau}{\partial x'^{\,\alpha}}\,\Gamma^\beta_{\tau\rho} + \frac{\partial x'^{\,\sigma}}{\partial x^\beta}\,\frac{\partial^2 x^\beta}{\partial x'^{\,\alpha}\partial x'^{\,\mu}}\,. \tag{35}$$

[5]By assumption, the general coordinate transformation $x' = x'(x)$ is assumed to be smooth and non-singular, which implies that it is permissible to interchange the order of the partial differentiation.

We shall evaluate eq. (35) at a point P in the spacetime such that the unprimed coordinates correspond to the local inertial frame (i.e. the freely falling frame) at point P. In the local inertial frame, the basis vectors $\vec{\boldsymbol{e}}_\mu$ can be chosen to be orthonormal with respect to the Minkowski metric $\eta_{\mu\nu}$ in a neighborhood of P. In particular, the basis vectors remain orthonormal to first order in a Taylor series about the point P. These requirements are equivalent to the following two conditions,

$$\vec{\boldsymbol{e}}_\mu \cdot \vec{\boldsymbol{e}}_\nu \big|_P = g_{\mu\nu}\big|_P = \eta_{\mu\nu}\,, \qquad\qquad \left(\frac{\partial \vec{\boldsymbol{e}}_\mu}{\partial x^\alpha}\right)_P = 0\,. \tag{36}$$

This means that locally in the neighborhood of P, the reference frame is indistinguishable from an inertial frame. In light of eq. (4), it then follows that

$$\Gamma^\mu_{\alpha\beta}\big|_P = 0\,. \tag{37}$$

Let us now transform the coordinates x in the neighborhood of P from those used in defining the local inertial frame to an arbitrary coordinate system (non necessarily inertial). The new coordinates are $x' = x'(x)$. Then using eq. (35), it follows that in the primed coordinate system, the connection coefficients in the neighborhood of P are given by

$$\Gamma'^{\sigma}_{\alpha\mu}\big|_P = \frac{\partial x'^{\sigma}}{\partial x^\beta}\frac{\partial^2 x^\beta}{\partial x'^{\alpha}\partial x'^{\mu}}\bigg|_P\,.$$

Thus, it follows that

$$\Gamma'^{\sigma}_{\alpha\mu}\big|_P = \Gamma'^{\sigma}_{\mu\alpha}\big|_P\,, \tag{38}$$

due to the fact that interchanging α and μ does not change the value of $\partial^2 x^\beta/\partial x'^{\alpha}\partial x'^{\mu}$ (as noted in footnote 5). But, the equivalence principle states that in the neighborhood of *any* point P it is possible to transform the coordinates to the local inertial frame (where the connection coefficients vanish). Thus, the above argument can be repeated for *any* point in the spacetime to obtain eq. (38) at any point P. We can therefore conclude that

$$\Gamma^\mu_{\alpha\beta} = \Gamma^\mu_{\beta\alpha}\,, \tag{39}$$

at all points in the spacetime independently of the choice of coordinates. In particular, this means that the torsion tensor is identically zero. This is the motivation for choosing the connection coefficients that are symmetric under the interchange of its lower two indices. As shown in class, under the latter assumption, we can derive an expression for the connection coefficients in terms of the metric,

$$\Gamma^\mu_{\alpha\beta} = \tfrac{1}{2} g^{\mu\nu}\left(\frac{\partial g_{\nu\beta}}{\partial x^\alpha} + \frac{\partial g_{\nu\alpha}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu}\right)\,.$$

The discussion above seems to imply that eq. (39) is a consequence of the equivalence principle. However, it turns out that definition of the local inertial frame via eq. (36) is too strong. In most textbooks, the local inertial frame is defined to satisfy the following conditions,

$$g_{\mu\nu}\big|_P = \eta_{\mu\nu}\,, \qquad\qquad \left(\frac{\partial g_{\mu\nu}}{\partial x^\alpha}\right)_P = 0\,. \tag{40}$$

Although eq. (36) implies eq. (40), it is possible to satisfy eq. (40) without the connection coefficients vanishing at the point P [in which case, the second condition of eq. (36) would be violated]. However, Einstein's general relativity is based on the assumption that eq. (39) is satisfied, in which case the local inertial frame does satisfy eqs. (36) and (37).