DUE: TUESDAY OCTOBER 14, 2014

1. Suppose that $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$ are proper Euclidean three-vectors, whereas $\vec{\boldsymbol{c}}$ and $\vec{\boldsymbol{d}}$ are improper three-vectors (also called pseudovectors). Show that

(a) $\vec{\boldsymbol{a}}\boldsymbol{\cdot}\vec{\boldsymbol{b}}$ and $\vec{\boldsymbol{c}}\boldsymbol{\cdot}\vec{\boldsymbol{d}}$ are both scalar quantities, whereas $\vec{\boldsymbol{a}}\boldsymbol{\cdot}\vec{\boldsymbol{c}}$ is a pseudoscalar.

(b) $\vec{\boldsymbol{a}}\times\vec{\boldsymbol{b}}$ and $\vec{\boldsymbol{c}}\times\vec{\boldsymbol{d}}$ are both pseudovectors, whereas $\vec{\boldsymbol{a}}\times\vec{\boldsymbol{c}}$ is a proper vector.

HINT: In solving part (b), recall that the definition of the determinant of an arbitrary 3×3 matrix can be written as:

$$\epsilon_{ijk}R_{i\ell}R_{jm}R_{kn}=\epsilon_{\ell mn}\det R\,. \tag{1}$$

One can obtain a useful identity by multiplying both sides of eq. (1) by $R^{-1}_{\ell p}$ and summing over ℓ. Then, if R is an orthogonal matrix, show that $R^{-1}_{\ell p}=R_{p\ell}$, and verify that the end result (after an appropriate relabeling of indices) is: $\epsilon_{ijk}R_{jm}R_{kn}=\epsilon_{\ell mn}(\det R)R_{i\ell}\,.$

2. Consider a tensor T_{ij} in three dimensional Euclidean space. Under an arbitrary rotation of the three-dimensional coordinate space, the tensor is transformed. Show that:

(a) if T_{ij} is symmetric and traceless, then the transformed tensor is traceless and symmetric.

(b) if T_{ij} is antisymmetric, then the transformed tensor is antisymmetric.

State the analogous result for a tensor $T^{\mu\nu}$ in four-dimensional Minkowski space. Define carefully what you mean by a traceless tensor in this case.

3. The four-velocity vector is given by $U^\mu=dx^\mu/d\tau=(\gamma c\,;\,\gamma\vec{\boldsymbol{v}})$, where $\gamma\equiv(1-v^2/c^2)^{-1/2}$ and τ is the proper time. The four-momentum vector is $P^\mu=mU^\mu=(E/c\,;\,\vec{\boldsymbol{p}})$, where m is the mass, $E=\gamma mc^2$ and $\vec{\boldsymbol{p}}=\gamma m\vec{\boldsymbol{v}}$.

(a) Determine the components of the four-acceleration vector, $A^\mu=dU^\mu/d\tau=(A^0\,;\,\vec{\boldsymbol{A}})$ in terms of the three-velocity $\vec{\boldsymbol{v}}=d\vec{\boldsymbol{x}}/dt$ and the three-acceleration $\vec{\boldsymbol{a}}=d\vec{\boldsymbol{v}}/dt$.

HINT: Recall that $d\tau=\gamma^{-1}dt$. Using the chain rule, show that $c^2\,d\gamma/d\tau=\gamma^4\,\vec{\boldsymbol{v}}\boldsymbol{\cdot}\vec{\boldsymbol{a}}$.

(b) Show that $U^\mu A_\mu=0$.

(c) The relativistic generalization of Newton's second law is $F^\mu=mA^\mu$ where F^μ is the four-force vector. Using this result, show that the three-force $\vec{\boldsymbol{f}}=d\vec{\boldsymbol{p}}/dt$ is proportional to the three-acceleration $\vec{\boldsymbol{a}}$ in two special cases: (i) linear motion where $\vec{\boldsymbol{a}}$ is parallel to $\vec{\boldsymbol{v}}$; and (ii) circular motion where $\vec{\boldsymbol{a}}$ is perpendicular to $\vec{\boldsymbol{v}}$. In cases (i) and (ii) where $\vec{\boldsymbol{f}}\propto\vec{\boldsymbol{a}}$, determine the corresponding proportionality constant.

4. In electrodynamics, one can introduce a scalar potential Φ and a vector potential $\vec{\boldsymbol{A}}$ as follows (using SI units):[1]

$$\vec{\boldsymbol{E}} = -\vec{\boldsymbol{\nabla}}\Phi - \frac{\partial \vec{\boldsymbol{A}}}{\partial t}\,, \qquad\qquad \vec{\boldsymbol{B}} = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}\,.$$

Define the four-vector potential $A^\mu = (\Phi/c\,;\,\vec{\boldsymbol{A}})$.

(a) Verify that the electromagnetic field strength tensor $F^{\mu\nu}$ can be written in terms of the four-vector potential as

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu\,.$$

(b) Two of the four Maxwell's equations can be expressed in relativistic notation as:

$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu\,, \tag{2}$$

where $J^\nu = (\rho c\,;\,\vec{\boldsymbol{J}})$. Noting that $F^{\mu\nu}$ is an antisymmetric tensor, show that current conservation, $\partial_\nu J^\nu = 0$, is automatically satisfied.

(c) Define the dual electromagnetic field strength tensor by:

$$\widetilde{F}^{\mu\nu} \equiv \tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}\,.$$

Explicitly work out the components of $\widetilde{F}^{\mu\nu}$. A *duality* transformation is an abstract transformation that changes $F^{\mu\nu}$ into $\widetilde{F}^{\mu\nu}$. How do the $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ fields change under this transformation?

(d) Using the result of part (a) [in which all indices are lowered], show that

$$\partial_\mu \widetilde{F}^{\mu\nu} = 0\,, \tag{3}$$

where $\widetilde{F}^{\mu\nu}$ is defined in part (c). By explicitly working out the components of eq. (3), show that one obtains the other two Maxwell equations.

REMARK: If $J^\mu = 0$ (no external currents), note the symmetry of Maxwell's equations [eqs. (2) and (3)] under the duality transformation.

5. Supernova 1987A was observed to occur at a distance of 170,000 light years from earth. Neutrinos and photons emitted from the supernova were observed by detectors on earth. The neutrinos (on average) had an energy of 10 MeV. Assume that the neutrinos have a rest mass-energy of $mc^2 = 1$ eV (whereas photons are massless).

(a) What is the elapsed time for one of the neutrinos to travel from the supernova to earth (as measured in the neutrino's frame)?

(b) Suppose the neutrino and photon were initially emitted from the supernova at the same time. How much later than the photon will the neutrino arrive at earth?

[1] In this problem $\vec{\boldsymbol{A}}$ is completely unrelated to the space components of the four-acceleration defined in problem 2.