MIDTERM ALERT: The midterm exam will be a take-home exam. The exam will be handed out in class on Thursday November 13. Completed exams should be returned to me by the end of the day on Friday November 14. The exam will be based on the material listed in the first seven topics of the Physics 171 Course Outline (and on the material covered on the first three problem sets). While working on the exam, you are permitted to consult with your class notes, any material provided on the class website, Lambourne's textbook and one other relativity textbook of your choosing. However, you should *not* collaborate with anyone else during the exam.

DUE: THURSDAY NOVEMBER 13, 2014

1. Consider the connection coefficients $\Gamma^{\beta}{}_{\mu\nu}$ prior to imposing the condition that $\Gamma^{\beta}{}_{\mu\nu}$ is symmetric under the interchange of its lower two indices. The torsion tensor, $T^{\beta}{}_{\mu\nu}$, is defined as

$$T^{\beta}{}_{\mu\nu} \equiv \Gamma^{\beta}{}_{\mu\nu} - \Gamma^{\beta}{}_{\nu\mu} \,.$$

(a) In class, I derived the transformation law for the connection coefficients under a general coordinate transformation. Using this result, derive the corresponding transformation law for the torsion tensor. Is it correct to call $T^{\beta}{}_{\mu\nu}$ a tensor?

(b) Prove that if the torsion tensor vanishes in any local inertial frame, then $T^{\beta}{}_{\mu\nu} = 0$ at all spacetime points.

2. (a) Show that raising and lowering of indices commutes with covariant differentiation; *e.g.*, $\nabla_\alpha A_\mu = \nabla_\alpha (g_{\mu\nu} A^\nu) = g_{\mu\nu} \nabla_\alpha A^\nu$.

(b) Suppose that A_μ is a covariant vector and $F_{\mu\nu}$ is an antisymmetric tensor. Prove that:

(i) $\nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu \,,$

(ii) $\nabla_\rho F_{\mu\nu} + \nabla_\nu F_{\rho\mu} + \nabla_\mu F_{\nu\rho} = \partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} + \partial_\mu F_{\nu\rho} \,.$

(c) Maxwell's equations in Minkowski space are given in eq. (2.102) and (2.103) on p. 75 of Lambourne. Using the principle of general covariance and the results of part (b), find the appropriate generalization of Maxwell's equations in curved spacetime.

(d) How should the equation for current conservation ($\partial_\mu J^\mu = 0$) be generalized in curved spacetime? [*EXTRA CREDIT:* Prove that this result is a consequence of Maxwell's equations in curved spacetime.]

3. (a) Suppose that the metric $g_{\mu\nu}$ is diagonal. Prove the following result for four-dimensional spacetime:

$$\Gamma^{\mu}{}_{\mu\nu} = \frac{1}{2g}\frac{\partial g}{\partial x^{\nu}} = \frac{1}{\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial x^{\nu}}\,,$$

where $g \equiv \det g_{\mu\nu}$. Note the implicit sum over the index μ.

(b) The result of part (a) is actually valid for an arbitrary choice of metric. Using this result, show that if A^{μ} is a contravariant vector and $F^{\mu\nu}$ is an antisymmetric tensor, then:

(i) $\nabla_{\nu} A^{\nu} = \dfrac{1}{\sqrt{-g}}\,\partial_{\nu}(\sqrt{-g}\,A^{\nu})\,,$

(ii) $\nabla_{\nu} F^{\mu\nu} = \dfrac{1}{\sqrt{-g}}\,\partial_{\nu}(\sqrt{-g}\,F^{\mu\nu})\,,$

where $\partial_{\nu} \equiv \partial/\partial x^{\nu}$.

(c) *EXTRA CREDIT:* Prove the result of part (a) without assuming any special form for the metric.

4. Consider a three-dimensional spacetime with a metric that is given by:

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right)c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\phi^2\,.$$

(a) From the corresponding Lagrangian $L = g_{\mu\nu} q^{\mu} q^{\nu}$, where $q^{\mu} \equiv dx^{\mu}/ds$, write down the Euler-Lagrange equations which (as shown in class) are equivalent to the geodesic equations

$$\frac{dq^{\mu}}{ds} + \Gamma^{\mu}{}_{\alpha\beta}\, q^{\alpha} q^{\beta} = 0\,.$$

Use this result to work out the non-vanishing connection coefficients.

(b) Check the connection coefficients obtained in part (a) by calculating them directly from the formula for $\Gamma^{\mu}{}_{\alpha\beta}$ in terms of the derivatives of the metric tensor.

5. The metric for the two-dimensional surface of a sphere of radius 1 is given by

$$ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2\,.$$

(a) Using the geodesic equations, show that all lines of longitude, corresponding to constant azimuthal angle ϕ on the surface of a sphere, are geodesics.

(b) *EXTRA CREDIT:* Solve the geodesic equations for the most general geodesic on the surface of a sphere.