DUE: TUESDAY DECEMBER 2, 2014

1. In class, I proved that for any covariant vector V_μ,

$$(\nabla_\beta \nabla_\alpha - \nabla_\alpha \nabla_\beta) V_\mu = R^\rho{}_{\mu\alpha\beta} V_\rho \,. \tag{1}$$

(a) Show that for a contravariant vector W^μ,

$$(\nabla_\beta \nabla_\alpha - \nabla_\alpha \nabla_\beta) W^\mu = -R^\mu{}_{\nu\alpha\beta} W^\nu \,. \tag{2}$$

(b) Given a contravariant second rank tensor $T^{\mu\nu}$ and a covariant vector V^μ, the contravariant vector defined by $W^\mu \equiv T^{\mu\nu} V_\nu$ satisfies eq. (2). Using eqs. (1)–(2) and Leibnitz's rule, show that

$$(\nabla_\beta \nabla_\alpha - \nabla_\alpha \nabla_\beta) T^{\mu\nu} = -R^\mu{}_{\rho\alpha\beta} T^{\rho\nu} - R^\nu{}_{\rho\alpha\beta} T^{\mu\rho} \,. \tag{3}$$

(c) Noting the forms of eqs. (2) and (3), deduce without further computation a formula for $(\nabla_\beta \nabla_\alpha - \nabla_\alpha \nabla_\beta) T_{\mu\nu}$, where $T_{\mu\nu}$ is a covariant second rank tensor, based on the form of eq. (1). Do the same for $(\nabla_\beta \nabla_\alpha - \nabla_\alpha \nabla_\beta) T^\mu{}_\nu$, where $T^\mu{}_\nu$ is a mixed second rank tensor.

(d) Using the properties of the Riemann curvature tensor, prove that the Ricci tensor is symmetric, i.e., $R_{\mu\nu} = R_{\nu\mu}$, where $R_{\mu\nu} \equiv g^{\alpha\beta} R_{\alpha\mu\nu\beta}$. Show that if any other index pair is summed using the inverse metric, the result is either zero or a multiple of the Ricci tensor.

2. Dust is a fluid without internal stress or pressure. Its energy-momentum tensor is $T^{\mu\nu} = \rho u^\mu u^\nu$, where ρ is a scalar quantity (which may depend on x^μ) and $u^\mu \equiv dx^\mu/d\tau$ is the velocity four-vector. Show that $\nabla_\nu T^{\mu\nu} = 0$ implies that the dust particles follow geodesics.

HINT: You will need to invoke the identity $g_{\alpha\beta} u^\alpha u^\beta = c^2$. Taking the covariant derivative of this relation will also yield a useful identity.

3. In class, we derived the Schwarzschild metric as a static spherically symmetric solution to the vacuum Einstein equations, $R_{\mu\nu} = 0$.

(a) Assume that there is a non-zero cosmological constant. By solving the modified Einstein equations, $R_{\mu\nu} = \Lambda g_{\mu\nu}$, determine the appropriate modification to the Schwarzschild metric.

(b) Using the metric obtained in part (a) for the case of $\Lambda \neq 0$, determine the orbit equation for a test particle in orbit around a spherically symmetric star of mass M.

(c) Using the new orbit equation derived in part (b), compute the perihelion advance of Mercury, assuming that the orbit is nearly circular, i.e., the eccentricity $|e| \ll 1$. Mercury makes 415 revolutions per century, has an eccentricity $e = 0.2056$ and a semi-major axis $a = 5.791 \times 10^{10}$ m. Using the observed data, set an upper bound on the value of Λ.

4. The Schwarzschild metric is given by:

$$ds^2 \equiv c^2 d\tau^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2 \,.$$

(a) Derive the geodesic equation for $\dot{\theta}$ (where $\dot{\theta} \equiv d\theta/d\tau$).

(b) Using the result of part (a), show that all orbits in the Schwarzschild geometry are planar.

HINT: Show that it is always possible to choose coordinates so that at $\tau = 0$, we have $\theta = \pi/2$ and $\dot{\theta} = 0$.

5. Consider a photon in orbit in a Schwarzschild geometry. For simplicity, assume that the orbit lies in the equatorial plane (*i.e.* $\theta = \pi/2$ is constant).

(a) Show that the geodesic equations imply that:

$$\bar{E}^2 = \frac{1}{c^2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{\bar{J}^2}{c^2 r^2}\left(1 - \frac{2GM}{rc^2}\right),$$

where $\bar{E}$ and $\bar{J}$ are constants of the motion and λ is an affine parameter.

(b) Define the effective potential:

$$V_{\text{eff}} = \frac{\bar{J}^2}{c^2 r^2}\left(1 - \frac{2GM}{rc^2}\right).$$

In class, we discussed how the effective potential yields information about the orbits of massive particles. Employing similar considerations, show that for photons there exists an unstable circular orbit of radius $\frac{3}{2} r_s$, where $r_s \equiv 2GM/c^2$ is the Schwarzschild radius.

[*HINT:* Make sure that you check for minima and maxima of V_{eff}.]

(c) Compute the proper time for the photon to complete one revolution of the circular orbit as measured by an observer stationed at $r = \frac{3}{2} r_s$.

(d) What orbital period does a very distant observer assign to the photon?

(e) The instability of the orbit can be exhibited directly. Show, by perturbing the geodesic in the equatorial plane, that the circular orbit of the photon at $r = \frac{3}{2} r_s$ is unstable.

HINT: In the orbit equation for the photon, put $r = \frac{3}{2} r_s + \eta$, and deduce an equation for η. Keep only first order terms in η, and solve the resulting equation.