DUE: FRIDAY DECEMBER 12, 2014

FINAL EXAM ALERT: The final exam will be an in-class exam, and will take place in Nat. Sci. Annex 103 from 12–3 pm on Monday December 15, 2003. The exam will cover the entire course material. While working on the exam, you are permitted to consult with your class notes, all class handouts and materials linked to the class website (including problem set solutions) and the course textbook by Lambourne. You may also consult with a second textbook of your choosing. Please bring a calculator to the exam. You may *not* collaborate with anyone or any other source material not listed above during the exam.

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1. Once across the event horizon of a black hole, what is the *longest* proper time the observer can spend before reaching the singularity of the black hole?

2. Suppose that a galaxy is observed to have a redshift $z = 1$. Assuming a matter-dominated FRW cosmology, at what fraction t/t_0 of the *present* age of the universe did light leave this galaxy?

3. Lambourne considers three simple cosmological models with curvature parameter $k = 0$ in Section 8.33 of his textbook. For example, in a model with $\Omega_{m,0} = 1$ and $\Omega_{\Lambda,0} = \Omega_{r,0} = 0$, Lambourne obtains $R(t) = R(t_0)(\frac{3}{2}H_0t)^{2/3}$ for the cosmic scale factor as a function of time t.

(a) Consider a model more closely resembling our universe with $\Omega_{m,0} + \Omega_{\Lambda,0} = 1$ and $\Omega_{r,0} = 0$. Assume that $0 < \Omega_{\Lambda,0} < 1$. Find a closed-form expression for $R(t)$ in terms of $\Omega_{m,0}$, $\Omega_{\Lambda,0}$ and H_0, and show that it reduces to the expected result in the limit of $\Omega_{\Lambda,0} \to 0$.

(b) In the model considered in part (a), how large would the ratio $\Omega_{\Lambda,0}/\Omega_{m,0}$ have to be for the universe to be accelerating (i.e., $d^2R/dt^2 > 0$) at the present time?

(c) Using the results of part (a), find an explicit expression for the age of the universe t_0 as a function of H_0 and $\Omega_{\Lambda,0}$. Evaluate t_0 numerically for $H_0^{-1} = 14.53 \times 10^9$ years and $\Omega_{\Lambda,0} = 0.685$.

4. (a) Express the present distance to the particle horizon in terms of the cosmological parameters by an integral formula analogous to eq. (8.68) of Lambourne for the current age of the universe (there is a typographical error in this latter equation which is corrected below):

$$t_0 = \frac{1}{H_0}\int_0^1 \frac{dx}{x\sqrt{\Omega_{\Lambda,0} + (1-\Omega_0)x^{-2} + \Omega_{m,0}x^{-3} + \Omega_{r,0}x^{-4}}}\,.$$

(b) Evaluate the formula for the present distance to the particle horizon obtained in part (a) above for cosmological parameters $\Omega_0 = 1$, $\Omega_{\Lambda,0} = 0.685$, $\Omega_{r,0} = 9 \times 10^{-5}$, $\Omega_{m,0} = 0.315$, and $H_0 = 67.3$ km s^{-1} Mpc^{-1} (taken from the latest compilation of the Particle Data Group). Express your answer in Gpc.

HINT: Use your favorite software package to numerically evaluate the integral obtained in part (a).

5. The values of the cosmological parameters today, $\Omega_{x,0} \equiv \rho_x(t_0)/\rho_c(t_0)$, where $x = \Lambda$, r, and m refer to the vacuum, radiation and matter, respectively, were given in part (b) of problem 4, and t_0 refers to the present day. The radiation is made up of two components: photons with $\Omega_{\gamma,0} = 5.46 \times 10^{-5}$ and neutrinos. The energy density of photons, $\rho_\gamma(t_0)$, is determined by the very well measured cosmic microwave background radiation temperature, $T_0 = 2.7255^\circ$K. The current value of the inverse Hubble parameter is $H_0^{-1} = 14.53 \times 10^9$ years.

(a) Using the values of the cosmological parameters today, find the relative value of the cosmic scale factor, $a(t) \equiv R(t)/R(t_0)$, when $\rho_m = \rho_\gamma$.

(b) What was the temperature of the universe (corresponding to the temperature of the blackbody cosmic photons) when $\rho_m = \rho_\gamma$?

(c) In class, we showed that the Hubble parameter at time t is given by:

$$H^2(t) = H_0^2 \left[\frac{\Omega_{m,0}}{a^3} + \frac{\Omega_{r,0}}{a^4} + \Omega_{\Lambda,0} + \frac{1 - \Omega_0}{a^2} \right] , \tag{1}$$

where $\Omega_0 \equiv \Omega_{m,0} + \Omega_{r,0} + \Omega_{\Lambda,0}$, and $a \equiv R(t)/R(t_0)$. Note that this is a differential equation, which can be used to solve for t as a function of a (or vice versa). Integrate this equation from $t = 0$ (the time of the big bang) to $t_{\rm eq}$, the time at which $\rho_m = \rho_\gamma$. Explain why you may safely ignore the terms in eq. (1) proportional to $\Omega_{\Lambda,0}$ and $1 - \Omega_0$ when $0 \le t \le t_{\rm eq}$. Then, evaluate the integral and obtain an expression for $t_{\rm eq}$, under the assumption that $\Omega_{r,0} = \Omega_{\gamma,0}$. Evaluate $t_{\rm eq}$ numerically (in years).

HINT: The upper limit of the a integration is obtained from the result of part (a). [What is the lower limit of the a integration, corresponding to $t = 0$?] The integral you need to evaluate is of the form:

$$\int \frac{x\,dx}{(a + bx)^{1/2}} = \frac{2}{3b^2} \left[(a + bx)^{3/2} - 3a(a + bx)^{1/2} \right] .$$

(d) More accurately, $\Omega_{r,0} = \Omega_{\gamma,0} + \Omega_{\nu,0}$, where $\Omega_{\nu,0}$ is the energy density of cosmic background neutrinos relative to the critical density. One can show that $\Omega_{\nu,0} \simeq 0.68\,\Omega_{\gamma,0}$. Thus, replace $\Omega_{\gamma,0}$ in part (c) with $\Omega_{r,0} = 1.68\,\Omega_{\gamma,0}$ and obtain an improved numerical result for the value of $t_{\rm eq}$ (which now corresponds to the cosmic time elapsed after the big bang at which $\rho_m = \rho_\gamma + \rho_\nu$). That is, $t_{\rm eq}$ corresponds to the age of the universe at "matter–radiation equality." Hence, for $t < t_{\rm eq}$ the universe was radiation-dominated, while for $t > t_{\rm eq}$ the universe is matter-dominated.