You have three hours to complete this exam. The point value of each of the three problems is indicated in brackets, so use your time wisely. In answering the questions, you may quote results that have been derived in the textbook and the homework/exam solutions (but if you do so, please cite explicitly the source of any such quotations).

While working on the exam, you are permitted to consult with your class notes, all class handouts and materials linked to the class website (including problem set and exam solutions) and the course textbook by Lambourne. You may also consult with a second textbook of your choosing. You may *not* collaborate with anyone or any other source material not listed above during the exam.

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1. [10] Assume that $f(x)$ is a well behaved scalar function of the spacetime coordinate x^μ. Evaluate:

$$(\nabla_\beta \nabla_\alpha - \nabla_\alpha \nabla_\beta) f(x) \,.$$

2. [30] Consider the following line element in a two-dimensional spacetime:

$$ds^2 = e^{f(r)}\, c^2 dt^2 - dr^2 \,,$$

where $f(r)$ is a function only of r (*i.e.*, f is does not depend explicitly on t).

(a) Compute the non-vanishing Christoffel symbols.

HINT: Chain rule. Need I say more?

(b) Compute the Ricci tensor, $R_{\mu\nu}$. Show that it is proportional to $g_{\mu\nu}$, where the coefficient multiplying $g_{\mu\nu}$ depends on the first and second derivatives (with respect to r) of the function f.

(c) Using the result of part (b), find the most general form for $f(r)$ such that the spacetime under consideration is flat by solving a simple differential equation. The solution to this equation will depend on two constants. Choose one constant such that the metric reduces to the standard Minkowski metric at $r = 0$. What is the physical interpretation of the second constant?

3. [30] Consider a spacecraft in a circular orbit around a Schwarzschild black hole. As usual, we denote the Schwarzschild coordinates by $(ct\,;\, r, \theta, \phi)$ and assume that the orbit is in the plane where $\theta = \pi/2$. We denote the two conserved quantities E and J by

$$E \equiv mc^2 \left(1 - \frac{r_s}{r}\right) \frac{dt}{d\tau} \,, \qquad J \equiv mr^2 \frac{d\phi}{d\tau} \,, \tag{1}$$

where $r_s \equiv 2GM/c^2$ and τ is the proper time.

(a) Using the Lagrangian method, write down the geodesic equations of motion for the variable r of the Schwarzschild metric. Noting that θ is constant and r is independent of τ for a circular orbit, derive a simple equation for $\dot{t}/\dot{\phi}$ (where $\dot{t} \equiv dt/d\tau$ and $\dot{\phi} \equiv d\phi/d\tau$) and then use eq. (1) to rewrite this equation in the following form,

$$\frac{E}{cJ} = \left(\tfrac{1}{2} r_s r\right)^{-1/2} \left(1 - \frac{r_s}{r}\right) . \tag{2}$$

(b) Obtain expressions for $dt/d\phi$ and $d\tau/d\phi$ in terms of r and r_s.

HINT: To derive an expression for $dt/d\phi$, use the chain rule, $dt/d\phi = (dt/d\tau)/(d\phi/d\tau) = \dot{t}/\dot{\phi}$, where $\dot{t}/\dot{\phi}$ is obtained in part (a). To derive the expression for $d\tau/d\phi = 1/\dot{\phi} = (1/\dot{t})dt/d\phi$, you will need to express $\dot{t}$ in terms of r and r_s as follows. Recall that for a timelike geodesic, $g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = c^2$. Using this equation, derive a second relation between E and J for a circular orbit in the $\theta = \frac{1}{2}\pi$ plane. Then, using eq. (2) to eliminate J and eq. (1) to express E in terms of $\dot{t}$, show that $\dot{t} = [1 - \frac{3}{2}(r_s/r)]^{-1/2}$.

(c) Consider two twin astronauts, Alice and Bob, in the orbiting spacecraft at orbital radius $r = 2r_s$. At coordinate time $t = 0$, Alice leaves the spacecraft and uses a rocket-pack to maintain a fixed position at radial distance $r = 2r_s$ and at fixed $\theta = \pi/2$ and $\phi = 0$. Bob stays behind inside the orbiting spacecraft. After one orbital revolution, Alice returns to the orbiting spacecraft. Determine which twin has aged more when Alice and Bob reunite.

HINT: Using the results of part (b), first determine the value of the coordinate time t when the twins reunite by integrating $dt/d\phi$ from $\phi = 0$ to 2π. Then, compute the elapsed proper times for Alice and Bob during their respective journeys.

4. [30] Consider a closed ($k = +1$) FRW universe containing non-relativistic matter density ρ_m, zero radiation density $\rho_r = 0$, and vacuum energy density ρ_Λ corresponding to a positive cosmological constant.

(a) Show that for a given value of Λ, there is a critical value of ρ_m for which the cosmic scale factor does not change with time. Find this critical value of ρ_m.

HINT: To determine the conditions for a static universe you should employ the Friedmann equations with a non-zero cosmological constant and set $\dot{R} = \ddot{R} = 0$.

(b) Assuming that the conditions of part (a) are satisfied, show that Λ depends on the value of the scale factor R, and find the explicit relation between them.

(c) For $k = 1$, the Robertson-Walker metric can be written in the form

$$ds^2 = c^2 dt^2 - R^2\left[d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta d\phi^2)\right] , \tag{3}$$

where $0 \leq \psi \leq \pi$, $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. Using this form of the metric, compute the spacial volume of the universe in terms of Λ.

HINT: First compute the volume as a function of R and then use the result of part (b). Recall that $\int_0^\pi \sin^2\psi d\psi = \frac{1}{2}\pi$.