INSTRUCTIONS: This is a take-home exam. The point value of each problem is indicated in brackets. The exam should be completed and returned to me or to my ISB mailbox at the end of the day on Friday November 14.

While working on the exam, you are permitted to consult with your class notes, Lambourne's textbook and one other relativity textbook and/or mathematics book of your choosing. Please indicate on the exam any additional references that you consult. You may also make use of any material available on the class website. However, you should *not* collaborate with any other source or persons during the exam.

---

1. [10] The metric tensor in curved spacetime is denoted by $g_{\mu\nu}$. The Levi-Civita tensor is denoted by $\epsilon^{\mu\nu\alpha\beta}$. One may be tempted to define the dual metric tensor (analogous to the dual electromagnetic field strength tensor introduced in problem set 1) as follows:

$$\widetilde{g}^{\mu\nu} \equiv \tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta} g_{\alpha\beta}\,.$$

Evaluate $\widetilde{g}^{\mu\nu}$.

2. [30] The line element of special relativity is given by $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$. Transform this line element from the usual $(ct; x, y, z)$ rectangular coordinates to new coordinates $(ct'; x', y', z')$ related by

$$\begin{aligned}
t &= t'\,, \\
x &= x'\cos\omega t' - y'\sin\omega t'\,, \\
y &= x'\sin\omega t' + y'\cos\omega t'\,, \\
z &= z'\,.
\end{aligned}$$

The new coordinates describe a rotating reference frame with angular velocity vector $\vec{\boldsymbol{\omega}} = (0, 0, \omega)$.

(a) Express ds^2 in terms of the new coordinates.

(b) In terms of the new coordinates [*i.e.*, using the invariant line element of part (a)], write down the geodesic equations.

(c) Using the results of part (b), identify the nonvanishing Christoffel symbols.

(d) Using the results of part (b), show that in the non-relativistic limit,

$$\frac{d^2\vec{\boldsymbol{r}}}{dt^2} = -\vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}) - 2\vec{\boldsymbol{\omega}} \times \frac{d\vec{\boldsymbol{r}}}{dt}\,,$$

where $\vec{\boldsymbol{r}} \equiv (x', y', z')$. What is the physical interpretation of the two terms on the right hand side of the above equation?

3. [30] Three observers are standing near each other on the surface of the Earth. Each holds an accurate atomic clock. At time $t = 0$ all the clocks are synchronized. At $t = 0$ the first observer throws her clock straight up. It reaches a maximum height of h and then returns to the first observer at time T as measured by the clock of the second observer, who holds his clock in his hand for the entire time interval. The third observer carries his clock up to a height h and then back down again to its original location, moving with constant speed on each leg of the trip and returning in time T.

(a) Incorporating gravitational time dilation as a consequence of the equivalence principle, show that to order $1/c^2$ the proper time between two spacetime points A and B (at coordinate times t_A and t_B, respectively) is given by

$$\tau_{AB} \simeq \int_{t_A}^{t_B} \left[1 - \frac{1}{c^2}\left(\tfrac{1}{2}\vec{\boldsymbol{v}}^2 - \Phi\right)\right] dt\,, \tag{1}$$

where $\vec{\boldsymbol{v}}$ is the velocity of a particle that moves from point A to point B and Φ is the gravitational potential experienced by the particle in motion.

(b) Calculate the total elapsed time measured on each clock, assuming that the maximum height h is much smaller than the radius of the earth. In your calculation, you may use eq. (1) to account for gravitational effects. Assume non-relativistic trajectories and ignore frictional effects in the motion. Which clock registers the *longest* time?

(c) If the clocks had been carried on the same trajectories (i.e., with the same velocities) but in a horizontal direction, which clock would have the *longest* reading?

4. [30] Consider a spacetime described by the Schwarzschild metric:

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2\,.$$

(a) A clock at fixed (r, θ, ϕ) measures an (infinitesimal) proper time interval, which we shall denote by dT, along its world line. Express dT (as a function of r) in terms of the coordinate time interval dt.

(b) A stationary observer at fixed (t, θ, ϕ) measures an (infinitesimal) radial distance, which we shall denote by dR. Express dR (as a function of r) in terms of the coordinate radial distance dr.

(c) Consider a particle falling radially into the center of the Schwarzschild metric (*i.e.*, falling in radially towards $r = 0$). Assume that the particle initially starts from rest infinitely far away from $r = 0$. Since this is force-free motion, the particle follows a geodesic. Show that the geodesic equation for $dt/d\tau$ (where $s \equiv c\tau$) implies that the quantity

$$E = mc^2\left(1 - \frac{2GM}{c^2 r}\right)\frac{dt}{d\tau}\,, \tag{2}$$

is a constant. We can interpret E as the total conserved energy of the particle. Argue that at $r \to \infty$ (where the initial velocity of the particle is zero), we can set $t = \tau$ and therefore $E = mc^2$ at all points along the particle trajectory. Using eq. (2), deduce a unique expression for $dt/d\tau$ that is valid at all points along the radial geodesic path.

(d) Recall that $ds^2 \equiv c^2 d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu$, from which it follows that

$$g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = c^2 \,.$$

In this problem $g_{\mu\nu}$ is determined from the Schwarzschild line element. Using these results and the result obtained in part (c) for $dt/d\tau$, compute the particle's inward coordinate velocity, $v = dr/dt$, as a function of the coordinate radial distance r. Invert the equation, and integrate from $r = r_0$ to $r = r_s$, where r_0 is some finite coordinate distance such that $r_0 > r_s$ and $r_s \equiv 2GM/c^2$ is called the Schwarzschild radius. Show that the elapsed coordinate time is infinite, independently of the choice of the starting radial coordinate r_0. That is, it takes an infinite coordinate time to reach the Schwarzschild radius.

HINT: For radial motion, θ and ϕ are constants independent of τ. Note that for *inward* radial motion dr/dt is negative.

(e) Compute the velocity dR/dT as measured by a stationary observer at a coordinate radial distance r. Verify that $|dR/dT| \to c$ as $r \to r_s$, where r_s is the Schwarzschild radius defined in part (d).

HINT: Use the results for dT and dR obtained in parts (a) and (b).