1. (a) Prove that in the local inertial frame, $D_\alpha g_{\mu\nu} = 0$, where D_α is the covariant derivative operator.

(b) Using the result of part (a), explain why $D_\alpha g_{\mu\nu} = 0$ is valid in an arbitrary reference frame.

(c) Starting from the expression for the covariant derivative of a contravariant vector,

$$D_\mu V^\nu = \frac{\partial V^\nu}{\partial x^\mu} + \Gamma^\nu_{\mu\alpha} V^\alpha \,,$$

derive an expression for the covariant derivative of a covariant vector, $V_\nu \equiv g_{\nu\beta} V^\beta$.

HINT: Be sure to make use of the result of part (b) above.

2. Let $x^\mu(\tau)$ represent a timelike curve in spacetime, where τ is the proper time as measured by an observer traveling along the curve. The four-vector velocity $u^\mu \equiv dx^\mu/d\tau$ is tangent to the curve at any point τ along the curve.

(a) If $g_{\mu\nu}$ is the metric of spacetime, compute the magnitude of the vector u^μ. Compare your result with the one obtained in flat Minkowski spacetime.

HINT: The magnitude of a timelike vector v^μ is given by $(-g_{\mu\nu} v^\mu v^\nu)^{1/2}$.

(b) Consider a contravariant timelike vector v^μ at a point P on the curve $x^\mu(\tau)$. Move the vector v^μ from the point P to an arbitrary point Q on the curve via parallel transport. Prove that the magnitude of the vector v^μ at the point Q is equal to the magnitude of v^μ at the point P.

(c) Suppose that the curve $x^\mu(\tau)$ is a geodesic and $u^\mu \equiv dx^\mu/d\tau$ is the corresponding tangent vector. At the point P on the geodesic curve, let $v^\mu = u^\mu$. Now, parallel transport the vector v^μ along the geodesic curve to an arbitrary point Q. Show that $v^\mu = u^\mu$ at the point Q.

NOTE: The result of part (c) implies that a vector tangent to a geodesic at a given point will always remain tangent to the geodesic curve when parallel transported along the geodesic.

3. (a) Consider a spacetime governed by a diagonal metric $g_{\mu\mu}$. Show that the corresponding connection coefficients satisfy the following conditions:

$$\Gamma^\mu_{\alpha\beta} = 0 \ \text{ if } \mu,\ \alpha \text{ and } \beta \text{ are all different.} \tag{1}$$

(b) Consider a Schwarzschild geometry with a metric that is given by:

$$ds^2 = -\left(1 - \frac{2G_N M}{c^2 r}\right) c^2 dt^2 + \left(1 - \frac{2G_N M}{c^2 r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,.$$

From the corresponding Lagrangian $L = g_{\mu\nu}q^\mu q^\nu$, where $q^\mu \equiv dx^\mu/ds$, write down the Euler-Lagrange equations which (as shown in class) are equivalent to the geodesic equations,

$$\frac{dq^\mu}{ds} + \Gamma^\mu_{\alpha\beta}\, q^\alpha q^\beta = 0\,.$$

Use this result to work out the non-vanishing connection coefficients. Check that your results for the connection coefficients satisfy the conditions specified in eq. (1).

4. Consider the following line element in a two-dimensional spacetime:

$$ds^2 = -e^{f(r)}\, c^2 dt^2 + dr^2\,,$$

where $f(r)$ is a function only of r (*i.e.*, f is does not depend explicitly on t).

(a) Compute the non-vanishing connection coefficients.

(b) Compute the Ricci tensor, $R_{\mu\nu}$. Show that it is proportional to $g_{\mu\nu}$, where the coefficient multiplying $g_{\mu\nu}$ depends on the first and second derivatives (with respect to r) of the function f.

(c) Using the result of part (b), find the most general form for $f(r)$ such that the spacetime under consideration is flat by solving a simple differential equation. The solution to this equation will depend on two constants. Choose one constant such that the metric reduces to the standard Minkowski metric at $r = 0$. What is the physical interpretation of the second constant?

5. (a) Assume that $f(x)$ is a well behaved scalar function of the spacetime coordinate x^μ. Evaluate:

$$(D_\beta D_\alpha - D_\alpha D_\beta) f(x)\,,$$

where D_α is the covariant derivative operator.

(b) In class, I proved that for any covariant vector V_μ,

$$(D_\beta D_\alpha - D_\alpha D_\beta) V_\mu = R^\rho{}_{\mu\alpha\beta} V_\rho\,, \tag{2}$$

where $R^\rho{}_{\mu\alpha\beta}$ is the Riemann curvature tensor. Show that for a contravariant vector W^μ,

$$(D_\beta D_\alpha - D_\alpha D_\beta) W^\mu = -R^\mu{}_{\nu\alpha\beta} W^\nu\,. \tag{3}$$

(c) Given a contravariant second rank tensor $T^{\mu\nu}$ and a covariant vector V^μ, the contravariant vector defined by $W^\mu \equiv T^{\mu\nu}V_\nu$ satisfies eq. (3). Using eqs. (2)–(3) and Leibniz's rule, show that

$$(D_\beta D_\alpha - D_\alpha D_\beta) T^{\mu\nu} = -R^\mu{}_{\rho\alpha\beta} T^{\rho\nu} - R^\nu{}_{\rho\alpha\beta} T^{\mu\rho}\,. \tag{4}$$

(d) Noting the forms of eqs. (3) and (4), deduce without further computation a formula for $(D_\beta D_\alpha - D_\alpha D_\beta) T_{\mu\nu}$, where $T_{\mu\nu}$ is a covariant second rank tensor, based on the form of eq. (2). Do the same for $(D_\beta D_\alpha - D_\alpha D_\beta) T^\mu{}_\nu$, where $T^\mu{}_\nu$ is a mixed second rank tensor.