DUE: FRIDAY DECEMBER 4, 2015

FINAL EXAM ALERT: The final exam will be an in-class exam, and will take place in Nat. Sci. Annex 102 from 8–11 am on Tuesday December 8, 2015. The exam will cover the entire course material. While working on the exam, you are permitted to consult with your class notes, all class handouts and materials linked to the class website (including problem set solutions) and the course textbook by Ta-Pei Cheng. You may also consult with a second textbook of your choosing. Please bring a calculator to the exam. You may *not* collaborate with anyone or any other source material not listed above during the exam.

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1. Consider a photon in orbit around a Schwarzschild black hole of mass M. For simplicity, assume that the orbit lies in the equatorial plane (*i.e.* $\theta = \pi/2$ is constant).

(a) Following the derivation given in class for the geodesic of a particle moving in the equatorial plane of a Schwarzschild geometry, show that

$$\bar{E} \equiv \left(1 - \frac{r_s}{r}\right)\dot{t}\,, \qquad\qquad \bar{J} = r^2\dot{\phi}\,,$$

are constants of motion, where $r_s \equiv 2G_N M/c^2$ is the Schwarzschild radius of the black hole, $\dot{t} \equiv dt/d\lambda$ and $\dot{\phi} \equiv d\phi/d\lambda$, and λ is a suitable affine parameter.

(b) Photons travel along paths that satisfy $ds^2 = g_{\mu\nu}dx^\mu dx^\nu = 0$, which yields $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0$. Applying this latter equation to the motion of the photon in the equatorial plane of the Schwarzschild geometry, show that the constants of motion defined in part (a) satisfy

$$\bar{E}^2 = \frac{1}{c^2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{\bar{J}^2}{c^2 r^2}\left(1 - \frac{r_s}{r}\right)\,.$$

(c) Derive the orbit equation of the photon. From this equation, deduce the radius r_0 of a circular orbit.

(d) Consider a stationary observer located at the Schwarzschild coordinates $(r_0, \theta = \frac{1}{2}\pi, \phi)$. Compute the proper time interval (as measured by the stationary observer) for the photon to complete one revolution of the circular orbit of radius r_0. What orbital period does a very distant observer assign to the photon?

(e) Show that the circular orbit obtained in part (c) is unstable.

HINT: Our textbook provides a derivation that makes use of the so-called effective potential. But you can actually prove this result directly as follows. An orbit is unstable if, when perturbed slightly, the perturbation grows as a function of time. In the orbit equation for the photon, put $r = r_0 + \eta$, and deduce an equation for η. Keep only first order terms in η, and solve the resulting equation.

2. Suppose that a galaxy is observed to have a redshift $z = 1$. Assuming a matter-dominated FLRW cosmology, at what fraction t/t_0 of the *present* age of the universe did light leave this galaxy?

3. Consider FLRW cosmological models with curvature parameter $k = 0$.

(a) Suppose that $\Omega_{M,0} = 1$ and $\Omega_{\Lambda,0} = \Omega_{R,0} = 0$, corresponding to a matter-dominated cosmology. Show that the cosmic scale factor is given by

$$a(t) = (\tfrac{3}{2}H_0 t)^{2/3}\,.$$

(b) Consider a model more closely resembling our universe with $\Omega_{M,0} + \Omega_{\Lambda,0} = 1$ and $\Omega_{R,0} = 0$. Assume that $0 < \Omega_{\Lambda,0} < 1$. Find a closed-form expression for $a(t)$ in terms of $\Omega_{M,0}$, $\Omega_{\Lambda,0}$ and H_0, and show that it reduces to the result of part (a) in the limit of $\Omega_{\Lambda,0} \to 0$.

(c) In the model considered in part (b), how large would the ratio $\Omega_{\Lambda,0}/\Omega_{M,0}$ have to be for the universe to be accelerating (i.e., $d^2a/dt^2 > 0$) at the present time?

(d) Using the results of part (b), find an explicit expression for the age of the universe t_0 as a function of H_0 and $\Omega_{\Lambda,0}$. Evaluate t_0 numerically for $H_0^{-1} = 14.53 \times 10^9$ years and $\Omega_{\Lambda,0} = 0.685$.

4. Assume that the universe begins with a big bang at $t = 0$ with a cosmic scale factor of $a(t = 0) = 0$. Subsequently, the universe expands so that $a(t)$ grows with t. Distances are measured by employing the Robertson-Walker metric.

(a) Consider a light source located at a comoving distance $\xi = \xi_H$ at cosmic time $t = 0$. Assume that the light source emits a photon that travels in a radial direction and is detected on Earth at $\xi = 0$ today ($t = t_0$). The proper distance d_H that the photon has traveled is called the *particle horizon*. Obtain an expression for d_H as a function of ξ_H and the curvature parameter k.

(b) Show that the particle horizon today is given by

$$d_H = c\int_0^1 \frac{da}{a\dot{a}}\,,$$

where $\dot{a} \equiv da/dt$. Using this result, express the present distance to the particle horizon in terms of the cosmological parameters by an integral formula analogous to eq. (11.42) on p. 260 of our textbook for the current age of the universe.

(c) Evaluate the formula for the present distance to the particle horizon obtained in part (b) for cosmological parameters $\Omega_0 = 1$, $\Omega_{\Lambda,0} = 0.685$, $\Omega_{R,0} = 9 \times 10^{-5}$, $\Omega_{M,0} = 0.315$, and $H_0 = 67.3$ km s^{-1} Mpc^{-1} (taken from the latest compilation of the Particle Data Group). Express your answer in Gpc.

HINT: Use your favorite software package to numerically evaluate the integral obtained in part (b).

5. The values of the cosmological parameters today, $\Omega_{x,0} \equiv \rho_x(t_0)/\rho_c(t_0)$, where $x = \Lambda$, R, and M refer to the vacuum, radiation and matter, respectively, were given in part (c) of problem 4. The radiation is made up of two components: photons with $\Omega_{\gamma,0} = 5.46 \times 10^{-5}$ and neutrinos. The energy density of photons today, $\rho_\gamma(t_0)$, is determined by the very well measured cosmic microwave background radiation temperature, $T_0 = 2.7255°\text{K}$. The current value of the inverse Hubble parameter is $H_0^{-1} = 14.53 \times 10^9$ years.

(a) Using the values of the cosmological parameters today, find the value of the cosmic scale factor $a(t)$ when $\rho_M = \rho_\gamma$.

(b) What was the temperature of the universe (corresponding to the temperature of the blackbody cosmic photons) when $\rho_M = \rho_\gamma$?

(c) In class, we showed that the Hubble parameter at time t is given by:

$$H^2(t) = H_0^2 \left[\frac{\Omega_{M,0}}{a^3} + \frac{\Omega_{R,0}}{a^4} + \Omega_{\Lambda,0} + \frac{1 - \Omega_0}{a^2} \right] , \tag{1}$$

where $\Omega_0 \equiv \Omega_{M,0} + \Omega_{R,0} + \Omega_{\Lambda,0}$. Note that this is a differential equation that can be used to solve for t as a function of a (or vice versa). Integrate this equation from $t = 0$ (the time of the big bang) to t_{eq}, the time at which $\rho_M = \rho_\gamma$. Explain why you may safely ignore the terms in eq. (1) proportional to $\Omega_{\Lambda,0}$ and $1 - \Omega_0$ when $0 \leq t \leq t_{\text{eq}}$. Then, evaluate the integral and obtain an expression for t_{eq}, under the assumption that $\Omega_{R,0} = \Omega_{\gamma,0}$. Evaluate t_{eq} numerically (in years).

HINT: The upper limit of the a integration is obtained from the result of part (a). [What is the lower limit of the a integration, corresponding to $t = 0$?] The integral you need to evaluate is of the form:

$$\int \frac{x\, dx}{(a + bx)^{1/2}} = \frac{2}{3b^2} \left[(a + bx)^{3/2} - 3a(a + bx)^{1/2} \right] .$$

(d) More accurately, $\Omega_{R,0} = \Omega_{\gamma,0} + \Omega_{\nu,0}$, where $\Omega_{\nu,0}$ is the energy density of cosmic background neutrinos relative to the critical density. In particular, as shown by eq. (10.76) on p. 229 of our textbook, $\Omega_{\nu,0} \simeq 0.68\, \Omega_{\gamma,0}$. Thus, replace $\Omega_{\gamma,0}$ in part (c) with $\Omega_{r,0} = 1.68\, \Omega_{\gamma,0}$ and obtain an improved numerical result for the value of t_{eq} (which now corresponds to the cosmic time elapsed after the big bang at which $\rho_m = \rho_\gamma + \rho_\nu$). That is, t_{eq} corresponds to the age of the universe at "matter–radiation equality." Hence, for $t < t_{\text{eq}}$ the universe was radiation-dominated, while for $t > t_{\text{eq}}$ the universe is matter-dominated.