You have three hours to complete this exam. The point value of each of the four problems is indicated in brackets, so use your time wisely. In answering the questions, you may quote results that have been derived in the textbook and the homework/exam solutions (but if you do so, please cite explicitly the source of any such quotations).

While working on the exam, you are permitted to consult with your class notes, all class handouts and materials linked to the class website (including problem set and exam solutions) and the course textbook by Ta-Pei Cheng. You may also consult with a second textbook of your choosing. The use of a calculator is encouraged. You may *not* collaborate with anyone or any other source material not listed above during the exam.

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1. [10] In class, I remarked that in a curved spacetime,

$$\eta_{\mu\nu\alpha\beta} \equiv \sqrt{-g}\,\epsilon_{\mu\nu\alpha\beta}$$

is a rank-four pseudotensor, where $\epsilon_{\mu\nu\alpha\beta}$ is the Levi-Civita symbol and $g \equiv \det(g_{\mu\nu})$.

Evaluate $D_\lambda \eta_{\mu\nu\alpha\beta}$, where D_λ is the covariant derivative.

HINT: What is the value of $D_\lambda \eta_{\mu\nu\alpha\beta}$ in the local inertial frame?

2. [40] Consider a spacetime with the (infinitesimal) invariant interval given by

$$ds^2 = -e^{-2ax/c^2}\,c^2dt^2 + dx^2 + dy^2 + dz^2\,, \tag{1}$$

where a is a constant.

(a) Find all the nonzero connection coefficients (i.e., the Christoffel symbols). Employ the Lagrangian method, and extract the relevant connection coefficients from the corresponding geodesic equations.

(b) Given a particle with zero instantaneous velocity at $t = 0$ (*i.e.*, $dx/dt = dy/dt = dz/dt = 0$ at $t = 0$), show that the acceleration, $d^2x/d\tau^2$, (where τ is the *proper time* parameter) at $t = 0$ is a constant given by

$$\left.\frac{d^2x}{d\tau^2}\right|_{t=0} = a\,.$$

Because of this, it may be said that a spacetime with the metric defined via eq. (1) describes a uniform gravitational field in the x direction.

(c) Find the nonzero components of the Ricci tensor. Show that apart from R_{00} and R_{11}, all the other components of the Ricci tensor are zero.

(d) Using the Einstein field equations, evaluate the components T_{00}, T_{11}, T_{22} and T_{33} of the energy-momentum tensor that generates this gravitational field. Check your result by verifying that $D_\mu T^{\mu\nu} = 0$.

3. [20] Consider a nonrotating black hole of mass M governed by the Schwarzschild metric. A doomed rocket ship is on a trajectory that corresponds to a geodesic path headed toward the black hole singularity at $r = 0$. Without loss of generality, assume that the geodesic path lies in the plane corresponding to $\theta = \frac{1}{2}\pi$.

(a) Timelike geodesics in a Schwarzschild geometry are characterized by two constants of motion,

$$E = mc^2 \left(1 - \frac{r_s}{r}\right) \dot{t}\,, \qquad\qquad J \equiv mr^2 \dot{\phi}\,,$$

where $r_s \equiv 2G_N M/c^2$ is the Schwarzschild radius of the black hole, $\dot{t} \equiv dt/d\tau$, $\dot{\phi} \equiv d\phi/d\tau$ and τ is the proper time. Derive an expression for $\dot{r} \equiv dr/d\tau$ for a geodesic path headed toward $r = 0$ (with $\theta = \frac{1}{2}\pi$ fixed).

HINT: Note that $\dot{r} < 0$ since r is *decreasing* with proper time τ.

(b) Find the geodesic path corresponding to the *longest* proper time for an observer who starts at $r = r_s$ and ends up at the black hole singularity $r = 0$. If the mass of the black hole is equal to the mass of our sun, what is the value of this maximal proper time in seconds?

HINT: Using the result of part (a), express the elapsed proper time in the form of an integral, and then determine the values of the constants E and J that maximize the integrand. Employing these values for E and J, integrate the resulting expression, and obtain a formula for the maximal elapsed proper time in terms of r_s. The latter integral will be in the form of a Beta function,

$$B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = \int_0^1 x^{p-1}(1-x)^{q-1}\, dx\,.$$

Finally, recall that for the sun, $r_s = 2.96$ km.

4. [30] Consider a model FLRW universe with $\Omega_{M,0} + \Omega_\Lambda = 1$ and $\Omega_{R,0} = 0$, which closely resembles our present day universe. Using $H_0 = 67.3\ \mathrm{km\,s^{-1}\,Mpc^{-1}} = (14.53 \times 10^9\ \mathrm{years})^{-1}$ for the value of the Hubble parameter today, the vacuum energy density (due to the cosmological constant) is given by $c^2 \rho_\Lambda$, where

$$\rho_\Lambda = 5.83 \times 10^{-27}\ \mathrm{kg\,m^{-3}}\,.$$

(a) Compute the value of the energy density of matter today and show that it is less than the vacuum energy density. Verify that the expansion of the universe today is accelerating.

(b) During the evolution of the universe, the energy density of matter is equal to the vacuum energy density at a moment in time that will be called *matter–vacuum equality*. Determine the age of the universe at matter-vacuum equality. What is the temperature of the cosmic microwave background radiation (CMBR) at matter-vacuum equality What is the cosmological redshift, z, of a photon that is emitted at matter-vacuum equality?

(c) During the evolution of the universe, the expansion of the universe flips from deceleration to acceleration at a point in history. Determine the age of the universe, the temperature of the CMBR and the cosmological redshift corresponding to this moment in time when the expansion of the universe is neither decelerating nor accelerating.