INSTRUCTIONS: This is a take-home exam. The point value of each problem is indicated in brackets. Please return the completed exam to me or my ISB mailbox no later than 12 noon on Monday November 9.

While working on the exam, you are permitted to consult with your class notes, Ta-Pei Cheng's textbook and one other relativity textbook and/or mathematics book of your choosing. Please indicate on the exam any additional references that you consulted. You may also make use of any material available on the class website. However, you should *not* collaborate with any other persons or consult other source materials during the exam.

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1. [10] The Ricci tensor in curved spacetime is denoted by $R_{\mu\nu}$. The Levi-Civita tensor is denoted by $\epsilon^{\mu\nu\alpha\beta}$. One may be tempted to introduce the dual Ricci tensor density (analogous to the dual electromagnetic field strength tensor introduced in eq. (12.36) of our textbook) as follows:

$$\widetilde{R}^{\mu\nu} \equiv \tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta}R_{\alpha\beta}\,.$$

Evaluate $\widetilde{R}^{\mu\nu}$.

2. [30] The line element of special relativity, $ds^2 = -c^2d\tau^2 = -c^2dt^2 + d\vec{\boldsymbol{x}}^{\,2}$, can be used to prove that $d\tau = dt/\gamma$, where $\gamma = (1 - v^2/c^2)^{-1/2}$. Consider the modification of the Minkowski line element due to the presence of a weak gravitational potential $\Phi(\vec{\boldsymbol{x}})$ in the non-relativistic approximation, i.e. $v \ll c$ and $|\Phi/c^2| \ll 1$. This modification incorporates gravitational time dilation as a consequence of the equivalence principle.

(a) Show that to order $1/c^2$,

$$d\tau \simeq \left[1 - \frac{1}{c^2}\left(\tfrac{1}{2}\vec{\boldsymbol{v}}^{\,2} - \Phi\right)\right]dt\,. \tag{1}$$

(b) Consider a satellite in a circular orbit of radius $r = R+h$ around the Earth, where R is the radius of the earth and h is the height of the orbit above the Earth's surface. Using Newtonian mechanics, compute the velocity of the satellite as a function of r.

HINT: Your result should also depend on the gravitational constant G_N and the mass of the earth M.

(c) A standard clock on the satellite is compared with an identical clock on Earth. Using the results of parts (a) and (b) and neglecting the rotation of the Earth, compute the ratio of the rate of the orbiting clock to that of the clock on Earth, employing the same approximations used in part (a). Which clock runs faster?

HINT: The answer to which clock runs faster depends on the value of r. Do *not* assume that $h \ll R$.

3. [20] Consider a time-like geodesic in a curved spacetime, $x^\mu(\tau)$, that is parameterized by the proper time τ. The tangent vector at any proper time τ along the geodesic is the velocity four-vector,

$$u^\mu(\tau) = \frac{dx^\mu}{d\tau}\,.$$

(a) Prove that $u^\mu(\tau)$ is parallel-transported along the geodesic. That is, for any two proper times, τ_1 and τ_2, the four-vector $u^\mu(\tau_1)$ when parallel-transported along the geodesic from τ_1 to τ_2, yields the four-vector $u^\mu(\tau_2)$.

HINT: Write down the equation that describes a vector parallel-transported along a curve $x^\mu(\tau)$ and compare it with the geodesic equation.

(b) Following the above hint, consider the equation satisfied by $u^\mu(\tau)$. Evaluate this equation in the local inertial frame (LIF). What is the nature of the motion in the LIF?

4. [40] Consider a three-dimensional spacetime with a metric that is given by[1]

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\theta^2\,.$$

(a) The velocity vector is defined by

$$u^\beta = \frac{dx^\beta}{d\tau}\,, \tag{2}$$

where $x^\beta = (ct\,;\, r\,,\, \theta)$ labels the spacetime coordinates and τ is the proper time. Evaluate the components of the velocity vector of a stationary observer who remains at a fixed value of (r, θ) in a spacetime whose metric is given in eq. (2).

(b) In a curved spacetime, the acceleration vector is defined by

$$a^\beta = u^\alpha D_\alpha u^\beta\,, \tag{3}$$

where D_α is the covariant derivative operator. Prove that in a local inertial frame, eq. (3) reduces to the definition of the acceleration vector in special relativity.

(c) Consider a stationary observer who remains at a fixed value of (r, θ) in a spacetime whose metric is given in eq. (2). Using eq. (3) and the results of part (a), evaluate the components of the acceleration vector of the stationary observer. In what direction does the acceleration point?

(d) The length of the acceleration vector obtained in part (c) is a scalar with respect to general coordinate transformations. Evaluate this scalar as a function of the spacetime coordinates. Explain the behavior of this scalar in the limits of $r \to \infty$ and $r \to 2GM/c^2$.

(e) [*EXTRA CREDIT*] Repeat parts (c) and (d) for a four-dimensional spacetime governed by the Schwarzschild metric, assuming that the stationary observer remains at a fixed value of (r, θ, ϕ). How are the results of parts (c) and (d) modified?

[1] This metric was treated in problem 3 of Problem Set 3.