1. [10] The metric tensor in curved spacetime is denoted by $g_{\mu\nu}$. The Levi-Civita tensor is denoted by $\epsilon^{\mu\nu\alpha\beta}$. One may be tempted to define the dual metric tensor (analogous to the dual electromagnetic field strength tensor introduced in problem set 1) as follows:

$$\tilde{g}^{\mu\nu} \equiv \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} g_{\alpha\beta}.$$ 

Evaluate $\tilde{g}^{\mu\nu}$.

Since $g_{\alpha\beta} = g_{\beta\alpha}$ and $\epsilon^{\mu\nu\alpha\beta} = -\epsilon^{\mu\nu\beta\alpha}$, it immediately follows that

$$\tilde{g}^{\mu\nu} \equiv \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} g_{\alpha\beta} = 0.$$ 

The formal proof goes as follows.

$$\tilde{g}^{\mu\nu} \equiv \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} g_{\alpha\beta} = -\frac{1}{2} \epsilon^{\mu\nu\beta\alpha} g_{\beta\alpha} = -\frac{1}{2} \epsilon^{\mu\nu\alpha\beta} g_{\alpha\beta} = 0,$$

since a quantity that is equal to its negative must be zero. Note that the second equality above is obtained by using $g_{\alpha\beta} = g_{\beta\alpha}$ and $\epsilon^{\mu\nu\alpha\beta} = -\epsilon^{\mu\nu\beta\alpha}$ and the third equality above is obtained after relabeling $\alpha \rightarrow \beta$ and $\beta \rightarrow \alpha$.

2. [30] The line element of special relativity is given by $ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$. Transform this line element from the usual $(ct; x, y, z)$ rectangular coordinates to new coordinates $(ct'; x', y', z')$ related by

$$t = t', $$

$$x = x' \cos \omega t' - y' \sin \omega t', $$

$$y = x' \sin \omega t' + y' \cos \omega t', $$

$$z = z'. $$

The new coordinates describe a rotating reference frame with angular velocity vector given by $\vec{\omega} = (0, 0, \omega)$.

(a) Express $ds^2$ in terms of the new coordinates.

Differentiating eqs. (1)–(4) yields

$$dt = dt'$$

$$dx = \cos \omega t' \, dx' - \sin \omega t' \, dy' - \omega x' \sin \omega t' \, dt' - \omega y' \cos \omega t' \, dt', $$

$$dy = \sin \omega t' \, dx' + \cos \omega t' \, dy' + \omega x' \cos \omega t' \, dt' - \omega y' \sin \omega t' \, dt', $$

$$dz = dz'. $$

It follows that
\[ dx^2 + dy^2 = [(dx' - \omega y' dt')^2 + (dy' + \omega x' dt')^2] \cos^2 \omega t' \]
\[ + \left( (-dy' - \omega x' dt')^2 + (dx' - \omega y' dt')^2 \right) \sin^2 \omega t' \]
\[ + 2 \left( (dx' - \omega y' dt') (-dy' - \omega x' dt') - (dy' + \omega x' dt') (dx' - \omega y' dt') \right) \sin \omega t' \cos \omega t'. \]
Combining terms yields
\[ dx^2 + dy^2 = dx'^2 + dy'^2 + \omega^2 (x'^2 + y'^2) dt'^2 - 2\omega y' dx' dt' + 2\omega x' dy' dt'. \]

Hence,
\[ ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 = \left[ c^2 - \omega^2 (x^2 + y^2) \right] dt^2 + 2\omega \left[ y' dx' dt' - x' dy' dt' \right] - dx'^2 - dy'^2 - dz'^2. \]

(b) In terms of the new coordinates \( i.e., \) using the invariant line element of part (a), write down the geodesic equations.

Using the result of part (a), and dropping the primes on the coordinates to simplify the typography,
\[ ds^2 \equiv g_{\mu\nu} dx'^\mu dx'^\nu = \left[ c^2 - \omega^2 (x^2 + y^2) \right] dt^2 + 2\omega \left[ y' dx dt - x' dy dt \right] - dx^2 - dy^2 - dz^2. \]

To obtain the corresponding geodesic equations, we consider the Lagrangian \( L = g_{\mu\nu} \ddot{x}^\mu \ddot{x}^\nu, \) where \( \ddot{x}^\mu \equiv dx'^\mu / ds. \) That is,
\[ L = \left[ c^2 - \omega^2 (x^2 + y^2) \right] \dot{t}^2 + 2\omega \dot{y} \dot{x} - 2\omega x \dot{y} - \dot{x}^2 - \dot{y}^2 - \dot{z}^2. \]

Using Lagrange’s equations,
\[ \frac{d}{ds} \left( \frac{\partial L}{\partial \dot{t}} \right) - \frac{\partial L}{\partial t} = 0 \quad \Rightarrow \quad \frac{d}{ds} \left\{ \left[ c^2 - \omega^2 (x^2 + y^2) \right] \dot{t} + \omega y \dot{x} - \omega x \dot{y} \right\} = 0, \]
\[ \frac{d}{ds} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0 \quad \Rightarrow \quad \frac{d}{ds} \left\{ \omega y \dot{x} - \dot{x} \right\} + \omega^2 \dot{x}^2 - \omega y \dot{t} = 0, \]
\[ \frac{d}{ds} \left( \frac{\partial L}{\partial \dot{y}} \right) - \frac{\partial L}{\partial y} = 0 \quad \Rightarrow \quad \frac{d}{ds} \left\{ -\omega x \dot{y} - \dot{y} \right\} + \omega^2 \dot{y}^2 - \omega x \dot{t} = 0, \]
\[ \frac{d}{ds} \left( \frac{\partial L}{\partial \dot{z}} \right) - \frac{\partial L}{\partial z} = 0 \quad \Rightarrow \quad \frac{d}{ds} \left\{ \dot{z} \right\} = 0. \]

Simplifying the above results, we obtain the following four geodesic equations:
\[ \left[ c^2 - \omega^2 (x^2 + y^2) \right] \ddot{t} - 2\omega^2 \dot{t} (\dot{x} \dot{y} + y \ddot{x} - x \ddot{y}) = 0, \]
\[ \ddot{x} - \omega^2 x \ddot{t} - 2\omega y \ddot{t} = 0, \]
\[ \ddot{y} - \omega^2 y \ddot{t} + 2\omega x \ddot{t} = 0, \]
\[ \ddot{z} = 0. \]
Solving eq. (6) for $\ddot{x}$ and eq. (7) for $\ddot{y}$, and inserting these results back into eq. (5) yields $\ddot{t} = 0$. Substituting this result back into eqs. (6) and (7), we obtain
\[
\ddot{x} - \omega^2 x \dot{t}^2 - 2\omega y \dot{t} = 0,
\]
\[
\ddot{y} - \omega^2 y \dot{t}^2 + 2\omega x \dot{t} = 0.
\]
Summarizing our results, the geodesic equations are:
\[
\ddot{t} = 0, \quad (9)
\]
\[
\ddot{x} - \omega^2 x \dot{t}^2 - 2\omega y \dot{t} = 0, \quad (10)
\]
\[
\ddot{y} - \omega^2 y \dot{t}^2 + 2\omega x \dot{t} = 0. \quad (11)
\]
\[
\ddot{z} = 0. \quad (12)
\]

**ALTERNATIVE DERIVATION**

Another strategy for solving this problem is to write down the geodesic equations in terms of the old coordinates, and then rewrite the resulting equations in terms of the new coordinates. In the old coordinates $(t, x, y, z)$, the geodesic equations are trivial:
\[
\ddot{t} = \ddot{x} = \ddot{y} = \ddot{z} = 0. \quad (13)
\]
Taking the second derivative of eqs. (1)–(4) with respect to $s$ and using eq. (13) yields
\[
\dddot{t} = \dddot{x} = \dddot{y} = \dddot{z} = 0, \quad (14)
\]
\[
\dddot{x} \cos \omega t' - \dddot{y} \sin \omega t' = 2\omega \dot{t}'(\dot{x} \sin \omega t' + \dot{y} \cos \omega t') + \omega^2 \dot{t}'^2 (x' \cos \omega t' - y' \sin \omega t'), \quad (15)
\]
\[
\dddot{x} \sin \omega t' + \dddot{y} \cos \omega t' = -2\omega \dot{t}'(\dot{x} \cos \omega t' - \dot{y} \sin \omega t') + \omega^2 \dot{t}'^2 (x' \sin \omega t' + y' \cos \omega t'), \quad (16)
\]
After using eq. (14) to drop additional terms proportional to $\dddot{t}'$ in eqs. (15) and (16). The two latter equations can be further simplified by computing the quantities $(15) \cos \omega t' + (16) \sin \omega t'$ and $-(15) \sin \omega t' + (16) \cos \omega t'$, respectively. The end results are:
\[
\dddot{x}' - \omega^2 x' \dot{t}'^2 - 2\omega y' \dot{t}' = 0,
\]
\[
\dddot{y}' - \omega^2 y' \dot{t}'^2 + 2\omega x' \dot{t}' = 0.
\]
Dropping the prime superscripts, we have recovered eqs. (9)–(12).

(c) Using the results of part (b), identify the nonvanishing Christoffel symbols.

We can now read off from eqs. (9)–(12) the nonzero Christoffel symbols by comparing these equations with the geodesic equation,
\[
\dddot{x}^\mu + \Gamma^\mu_{\alpha \beta} \dot{x}^\alpha \dot{x}^\beta = 0. \quad (17)
\]
Since \(\Gamma_{\alpha\beta}^\mu = \Gamma_{\beta\alpha}^\mu\), we can rewrite eq. (17) as

\[
\ddot{x}^\mu + \sum_\alpha \Gamma^\mu_{\alpha\alpha} (\dot{x}^\alpha)^2 + 2 \sum_{\alpha < \beta} \Gamma^\mu_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = 0.
\]

(18)

where explicit sums over the indices \(\alpha\) and \(\beta\) are employed here to avoid any ambiguity.\(^1\)

Comparing eqs. (9)–(12) with eq. (18), we obtain the following nonzero Christoffel symbols:

\[
\begin{align*}
\Gamma^{1}_{00} &= -\omega^2 x, \\
\Gamma^{1}_{02} &= \Gamma^{1}_{20} = -\omega, \\
\Gamma^{2}_{00} &= -\omega^2 y, \\
\Gamma^{2}_{01} &= \Gamma^{2}_{10} = \omega.
\end{align*}
\]

All other Christoffel symbols are zero.

(d) Using the results of part (b), show that in the non-relativistic limit,

\[
\frac{d^2\vec{r}}{dt^2} = -\vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2\vec{\omega} \times \frac{d\vec{r}}{dt},
\]

where \(\vec{r} \equiv (x', y', z')\). What is the physical interpretation of the two terms on the right hand side of the above equation?

First, we note that in special relativity, we have \(dt = \gamma dt\), where \(ds \equiv c d\tau\) and \(\tau\) is the proper time. Thus, \(\ddot{t} = \frac{d^2t}{ds^2} = c^{-2}\frac{d^2t}{d\tau^2} = 0\) is trivially satisfied. In the non-relativistic limit, \(\gamma \approx 1\) so that \(t \approx \tau\). Hence eqs. (10)–(12) yield

\[
\begin{align*}
\frac{d^2x'}{dt^2} &= \omega^2 x' + 2\omega \frac{dy'}{dt}, \quad (19) \\
\frac{d^2y'}{dt^2} &= \omega^2 y' - 2\omega \frac{dx'}{dt}, \quad (20) \\
\frac{d^2z'}{dt^2} &= 0. \quad (21)
\end{align*}
\]

where we have restored the primes on the coordinates. (Note that \(t' = t\) so there is no need to employ a prime on the time coordinate.) Writing \(\vec{r} = (x', y', z')\) and \(\vec{\omega} = (0, 0, \omega)\), it follows that eqs. (19)–(21) can be rewritten as

\[
\frac{d^2\vec{r}}{dt^2} = -\vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2\vec{\omega} \times \frac{d\vec{r}}{dt}. \quad (22)
\]

\(^1\)In particular, note the factor of 2 that appears in eq. (18). Indeed, we do not have to sum over the indices \(\beta < \alpha\) since we are always free to use \(\Gamma^\mu_{\alpha\beta} = \Gamma^\mu_{\beta\alpha}\) to make sure the first lower index of the Christoffel symbol is smaller in value than the second lower index. This is the origin of the factor of 2 in eq. (18).
For example,
\[
\vec{\omega} \times \vec{r} = \det \begin{pmatrix}
\hat{x} & \hat{y} & \hat{z} \\
0 & 0 & \omega \\
x' & y' & z'
\end{pmatrix} = -\omega y' \hat{x} + \omega x' \hat{y},
\]
and
\[
-\vec{\omega} \times (\vec{\omega} \times \vec{r}) = \det \begin{pmatrix}
\hat{x} & \hat{y} & \hat{z} \\
0 & 0 & \omega \\
\omega y' & -\omega x' & z'
\end{pmatrix} = \omega^2 (x' \hat{x} + y' \hat{y}),
\]
which demonstrates that eqs. (19)–(21) and eq. (22) are equivalent.

The interpretation of the two terms on the right-hand side of eq. (22) is well known. We identify 
\[-\vec{\omega} \times (\vec{\omega} \times \vec{r})\] as the centrifugal acceleration and 
\[-2 \vec{\omega} \times \frac{d\vec{r}}{dt}\] as the Coriolis acceleration, as perceived in the non-inertial rotating reference frame.

3. [30] Three observers are standing near each other on the surface of the Earth. Each holds an accurate atomic clock. At time \(t = 0\) all the clocks are synchronized. At \(t = 0\) the first observer throws her clock straight up. It reaches a maximum height of \(h\) and then returns to the first observer at time \(T\) as measured by the clock of the second observer, who holds his clock in his hand for the entire time interval. The third observer carries his clock up to a height \(h\) and then back down again to its original location, moving with constant speed on each leg of the trip and returning in time \(T\).

(a) Incorporating gravitational time dilation as a consequence of the equivalence principle, show that to order \(1/c^2\) the proper time between two spacetime points \(A\) and \(B\) (at coordinate times \(t_A\) and \(t_B\), respectively) is given by
\[
\tau_{AB} \simeq \int_{t_A}^{t_B} \left[ 1 - \frac{1}{c^2} \left( \frac{1}{2} \vec{\nu}^2 - \Phi \right) \right] dt,
\]
where \(\vec{\nu}\) is the velocity of a particle that moves from point \(A\) to point \(B\) and \(\Phi\) is the gravitational potential experienced by the particle in motion.

In the non-relativistic limit and in the limit of weak gravitational fields, the equivalence principle implies that the Minkowski space metric is modified,
\[
ds^2 = c^2 d\tau^2 = \left( 1 + \frac{2\Phi}{c^2} \right) c^2 dt^2 - d\vec{x}^2,
\]
which is the form of the metric that is used in general relativity.
where $\Phi$ is the gravitational potential, $d\vec{x}^2 \equiv d\vec{x} \cdot d\vec{x} = dx^2 + dy^2 + dz^2$, and $\tau$ is the proper time. Using the chain rule,

$$d\vec{x} = \frac{d\vec{x}}{dt} dt = \vec{v} dt. \quad (25)$$

Both the gravitational potential $\Phi$ and the velocity $\vec{v}$ are functions of $\vec{x}$. Using eq. (25), we can rewrite eq. (24) as

$$d\tau^2 = \left(1 + \frac{2\Phi}{c^2} - \frac{\vec{v}^2}{c^2}\right) dt^2. \quad (26)$$

Note that in the absence of a gravitational field, eq. (26) reduces to the well known special relativistic result, $d\tau = \gamma^{-1} dt$, where $\gamma \equiv (1 - \vec{v}^2/c^2)^{-1/2}$. In the non-relativistic limit we have $|\vec{v}| \ll c$, and in the limit of a weak gravitational field we have $|\Phi/c^2| \ll 1$. Thus, when taking the square root of eq. (26), we can expand to first order in $\vec{v}^2/c^2$ and $\Phi/c^2$ to obtain

$$d\tau = \left(1 + \frac{\Phi}{c^2} - \frac{\vec{v}^2}{2c^2}\right) dt. \quad \text{(b)}$$

Integrating from an initial coordinate time $t_A$ to a final coordinate time $t_B$ yields the elapsed proper time,

$$\tau_{AB} \simeq \int_{t_A}^{t_B} \left[1 - \frac{1}{c^2} \left(\frac{\vec{v}^2}{2} - \Phi\right)\right] dt.$$ 

(b) Calculate the total elapsed time measured on each clock, assuming that the maximum height $h$ is much smaller than the radius of the earth. In your calculation, you may use eq. (23) to account for gravitational effects. Assume non-relativistic trajectories and ignore frictional effects in the motion. Which clock registers the longest time?

The first observer throws her clock straight up. It reaches its maximum height $h$ in coordinate time $\frac{1}{2}T$ and then falls back down again, reaching the earth’s surface ($h = 0$) after a coordinate time $T$ has elapsed. In the original statement of the problem, we denoted by $t = 0$ the initial coordinate time. In what follows, it turns out to be slightly more convenient to denote the coordinate time $t = 0$ corresponding to the time when the clock has reached its maximal height. This means that the clock was initially thrown upwards at time $t = -\frac{1}{2}T$ and has returned to the earth’s surface at time $t = \frac{1}{2}T$. Using the nonrelativistic trajectory, one easily computes $h(t)$ and $v(t)$. For example,

$$v(t) = \frac{dh}{dt} = -gt.$$ 

Integrating this equation with $h(0) = h$ then yields

$$h(t) = h - \frac{1}{2}gt^2. \quad (27)$$

Using the fact that $h(\pm \frac{1}{2}T) = 0$, we can evaluate the maximal height $h$,

$$h = \frac{1}{2}g \left(\pm \frac{1}{2}T\right)^2 = \frac{1}{8}gT^2. \quad (28)$$
The gravitational potential is given by
\[ \Phi(h) = gh(t) = g \left( \frac{1}{8} gT^2 - \frac{1}{2} gt^2 \right) , \]
after employing eqs. (27) and (28). Using eq. (23), the elapsed proper time as seen by the first observer is
\[
\tau = \int_{-T/2}^{T/2} \left\{ \frac{1}{c^2} \left[ 1 - \frac{1}{2} \left( \frac{1}{8} gT^2 - g \left( \frac{1}{8} gT^2 - \frac{1}{2} gt^2 \right) \right) \right] \right\} dt
\]
\[
= T - \frac{g^2}{c^2} \int_{-T/2}^{T/2} (t^2 - \frac{1}{8} T^2) dt
\]
\[
= T - \frac{2g^2}{c^2} \int_{-T/2}^{T/2} (t^2 - \frac{1}{8} T^2) dt
\]
\[
= T \left[ 1 + \frac{1}{24} \left( \frac{gT}{c} \right)^2 \right]. \quad (29)
\]
Note that in the penultimate step, we used the fact that the integrand is an even function of \( t \) (i.e., it is invariant under the transformation \( t \to -t \)). As a result, the value of the integral is unchanged if we integrate from \( t = 0 \) to \( t = \frac{1}{2} T \) and multiply the result by two.

For the second observer who holds his clock at rest, \( \tau = T \). For the third observer, \( v(t) \) is constant for \( -\frac{1}{2} T \leq t \leq 0 \) and then changes sign for \( 0 \leq t \leq \frac{1}{2} T \). Since it takes a time \( \frac{1}{2} T \) to cover a distance \( h \), the corresponding velocities are \( v(t) = \pm \frac{2h}{T} = \pm \frac{1}{4} gT \), respectively (after using eq. (28) to express \( h \) in terms of \( T \)). The height of the clock as a function of \( t \) is
\[
h(t) = h - \frac{1}{4} gT|t|, \quad (30)
\]
and the gravitational potential is
\[ \Phi(h) = gh(t) = g \left( \frac{1}{8} gT^2 - \frac{1}{4} gT|t| \right) , \]
after employing eqs. (27) and (30). Using eq. (23), the elapsed proper time as seen by the third observer is
\[
\tau = \int_{-T/2}^{T/2} \left\{ \frac{1}{c^2} \left[ 1 - \frac{1}{2} \left( \frac{1}{8} gT^2 - g \left( \frac{1}{8} gT^2 - \frac{1}{4} gT|t| \right) \right) \right] \right\} dt
\]
\[
= T + \frac{2g^2T}{c^2} \int_{-T/2}^{T/2} \left( \frac{3}{32} T - \frac{1}{4} t \right) dt
\]
\[
= T \left[ 1 + \frac{1}{32} \left( \frac{gT}{c} \right)^2 \right]. \quad (31)
\]
The clock that registers the longest proper time is the clock of observer 1.
In summary, the elapsed proper times are:

\[
\tau = \begin{cases} 
T \left[ 1 + \frac{1}{24} \left( \frac{gT}{c} \right)^2 \right], & \text{for observer 1,} \\
T, & \text{for observer 2,} \\
T \left[ 1 + \frac{1}{32} \left( \frac{gT}{c} \right)^2 \right], & \text{for observer 3.} 
\end{cases}
\]

The significance of this result is as follows. The clock that registers the longest proper time is the clock of observer 1. This case corresponds to the extremal value of the proper time interval. That is, among all possible paths that start and end at \(h = 0\), the trajectory that corresponds to the extremum of the proper time belongs to the “free particle” moving in curved spacetime. Here, the effects of gravity have been incorporated in the metric that describes the curved spacetime. Indeed, the clock of observer 1 is the only clock that experiences no non-gravitational forces. The clocks of observers 2 and 3 must experience a non-gravitational force to explain their motion in the gravitational field. For example, observer 2 who holds his clock must exert a force equal to the weight of the clock.

(c) If the clocks had been carried on the same trajectories (i.e., with the same velocities) but in a horizontal direction, which clock would have the longest reading?

If the clocks had been carried on the same trajectories as in part (b) but in a horizontal direction, then the gravitational field plays no role since the gravitational potential is a constant, which can be set to zero without loss of generality. In this case, eq. (23) yields

\[
\tau = T - \frac{1}{2c^2} \int_{T/2}^{T/2} \vec{v}(t)^2 dt.
\]

Hence, the clock of observer 2, which is at rest, would register the longest reading. That is \(\tau = T\) if \(\vec{v} = 0\) and \(\tau < T\) if \(\vec{v} \neq 0\).

4. [30] Consider a spacetime described by the Schwarzschild metric:

\[
\begin{align*}
ds^2 &= \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta \, d\phi^2. 
\end{align*}
\] (32)

(a) A clock at fixed \((r, \theta, \phi)\) measures an (infinitesimal) proper time interval, which we shall denote by \(dT\), along its world line. Express \(dT\) (as a function of \(r\)) in terms of the coordinate time interval \(dt\).

For fixed \(r, \theta\) and \(\phi\), we have

\[
ds^2 = c^2 dT^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 \quad \Rightarrow \quad dT = \left(1 - \frac{2GM}{c^2 r}\right)^{1/2} dt.
\]
(b) A stationary observer at fixed \((t, \theta, \phi)\) measures an (infinitesimal) radial distance, which we shall denote by \(dR\). Express \(dR\) (as a function of \(r\)) in terms of the coordinate radial distance \(dr\).

For fixed \(t, \theta\) and \(\phi\), we have

\[
ds^2 = dR^2 = \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 \implies dR = \left(1 - \frac{2GM}{c^2 r}\right)^{-1/2} dr.
\]

(c) Consider a particle falling radially into the center of the Schwarzschild metric (i.e., falling in radially towards \(r = 0\)). Assume that the particle initially starts from rest infinitely far away from \(r = 0\). Since this is force-free motion, the particle follows a geodesic. Show that the geodesic equation for \(\frac{dt}{d\tau}\) (where \(s \equiv c\tau\)) implies that the quantity

\[
E = mc^2 \left(1 - \frac{2GM}{c^2 r}\right) \frac{dt}{d\tau},
\]

is a constant. We can interpret \(E\) as the total conserved energy of the particle. Argue that at \(r \to \infty\) (where the initial velocity of the particle is zero), we can set \(t = \tau\) and therefore \(E = mc^2\) at all points along the particle trajectory. Using eq. (33), deduce a unique expression for \(\frac{dt}{d\tau}\) that is valid at all points along the radial geodesic path.

The falling particle follows a geodesic. To obtain the geodesic equation, we employ the Lagrangian \(L = g_{\mu \nu} \dot{x}^\mu \dot{x}^\nu\), where \(\dot{x}^\mu \equiv dx^\mu / ds\). Using the metric specified in eq. (32),

\[
L = \left(1 - \frac{2GM}{c^2 r}\right) c^2 \dot{t}^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} r^2 - r^2 \dot{\theta}^2 - r^2 \sin^2 \theta \dot{\phi}^2.
\]

Lagrange’s equation corresponding to the parameter \(t\) is given by

\[
\frac{d}{ds} \left( \frac{\partial L}{\partial \dot{t}} \right) - \frac{\partial L}{\partial t} = 0 \implies \frac{d}{ds} \left[ \left(1 - \frac{2GM}{c^2 r}\right) \dot{t} \right] = 0.
\]

This means that the quantity

\[
\kappa \equiv \left(1 - \frac{2GM}{c^2 r}\right) \frac{dt}{ds},
\]

is a constant along the geodesic. If we write \(ds = c d\tau\), where \(\tau\) is the proper time, and define a new constant \(E \equiv \kappa mc^3\), then it follows that

\[
E = mc^2 \left(1 - \frac{2GM}{c^2 r}\right) \frac{dt}{d\tau}
\]

is a constant along the geodesic.

Consider a point along the geodesic corresponding to \(r \to \infty\). Infinitely far from the origin of the coordinate system, the gravitational potential approaches zero, so the Schwarzschild metric reduces to the Minkowski metric. In this case, we can identify \(d\tau = \gamma^{-1} dt\), where
\[ \gamma \equiv (1 - v^2/c^2)^{-1/2}. \] Moreover, we note that the particle is initially at rest (at \( r \to \infty \)), in which case \( dt/d\tau = 1 \). Hence, infinitely far from the origin, we see from eq. (34) that \( E = mc^2 \). Since \( E \) is a constant along the geodesic path, it follows that \( E = mc^2 \) for all points along the geodesic. Hence, it follows from eq. (34) that for all points along the geodesic,

\[ \frac{dt}{d\tau} = \left( 1 - \frac{2GM}{c^2 r} \right)^{-1}. \]  

(d) Recall that \( ds^2 \equiv c^2 d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu \), from which it follows that

\[ g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = c^2. \]  

In this problem \( g_{\mu\nu} \) is determined from the Schwarzschild line element. Using these results and the result obtained in part (c) for \( dt/d\tau \), compute the particle’s inward coordinate velocity, \( v = dr/dt \), as a function of the coordinate radial distance \( r \). Invert the equation, and integrate from \( r = r_0 \) to \( r = r_s \), where \( r_0 \) is some finite coordinate distance such that \( r_0 > r_s \) and \( r_s \equiv 2GM/c^2 \) is called the Schwarzschild radius. Show that the elapsed coordinate time is infinite, independently of the choice of the starting radial coordinate \( r_0 \). That is, it takes an infinite coordinate time to reach the Schwarzschild radius.

Writing out eq. (36) in full for the Schwarzschild metric,

\[ c^2 \left( 1 - \frac{2GM}{c^2 r} \right) \left( \frac{dt}{d\tau} \right)^2 - \left( 1 - \frac{2GM}{c^2 r} \right)^{-1} \left( \frac{dr}{d\tau} \right)^2 - r^2 \left( \frac{d\theta}{d\tau} \right)^2 - r^2 \sin^2 \theta \left( \frac{d\phi}{d\tau} \right)^2 = c^2. \]

For a radial trajectory, \( \theta \) and \( \phi \) are constants independent of \( \tau \), so that \( d\theta/d\tau = d\phi/d\tau = 0 \). Hence,

\[ \left( 1 - \frac{2GM}{c^2 r} \right) \frac{dt}{d\tau} \right)^2 - \frac{1}{c^2} \left( 1 - \frac{2GM}{c^2 r} \right)^{-1} \left( \frac{dr}{d\tau} \right)^2 = 1. \]

Using the chain rule, \( dr/d\tau = (dr/dt)(dt/d\tau) \), it follows that

\[ \left( \frac{dt}{d\tau} \right)^2 \left[ \left( 1 - \frac{2GM}{c^2 r} \right) - \frac{1}{c^2} \left( 1 - \frac{2GM}{c^2 r} \right)^{-1} \left( \frac{dr}{dt} \right)^2 \right] = 1. \]

Inserting the result for \( dt/d\tau \) obtained in eq. (35),

\[ \left( 1 - \frac{2GM}{c^2 r} \right) - \frac{1}{c^2} \left( 1 - \frac{2GM}{c^2 r} \right)^{-1} \left( \frac{dr}{dt} \right)^2 = \left( 1 - \frac{2GM}{c^2 r} \right)^2. \]

Dividing both sides of the equation by a factor of \( 1 - 2GM/(c^2r) \) yields

\[ 1 - \frac{1}{c^2} \left( 1 - \frac{2GM}{c^2 r} \right)^{-2} \left( \frac{dr}{dt} \right)^2 = 1 - \frac{2GM}{c^2 r}. \]

\(^2\)Note that eq. (35) differs from \( dt/dT \) of part (a), since the spacetime path with fixed \( r, \theta \) and \( \phi \) is not a geodesic.
It is now a simple matter to solve for \((dr/dt)^2\),
\[
\left(\frac{dr}{dt}\right)^2 = \frac{2GM}{r} \left(1 - \frac{2GM}{c^2r}\right)^2.
\]
Taking the square root of this equation, and noting that for inward radial motion \(dr/dt\) is negative, we end up with
\[
\frac{dr}{dt} = -\left(\frac{2GM}{r}\right)^{1/2} \left(1 - \frac{2GM}{c^2r}\right).
\]
Inverting eq. (37),
\[
\frac{dt}{dr} = -\left(\frac{r}{2GM}\right)^{1/2} \left(1 - \frac{2GM}{c^2r}\right)^{-1}.
\]
Integrating from \(r = r_0\) to \(r = r_s \equiv 2GM/c^2\) (under the assumption that \(r_0 > r_s\)),
\[
t = -\frac{1}{\sqrt{2GM}} \int_{r_0}^{r_s} \frac{\sqrt{r} dr}{1 - r_s/r}.
\]
Here \(t\) is the elapsed coordinate time it take for a particle to move from \(r_0\) to \(r_s \equiv 2GM/c^2\). We can rewrite the above integral as
\[
t = \frac{1}{\sqrt{2GM}} \int_{r_s}^{r_0} \frac{r^{3/2} dr}{r - r_s}.
\]
This integral is logarithmically divergent, independently of the value of \(r_0\). Indeed, in the vicinity of \(r \sim r_s\),
\[
t \sim \left(\frac{r_s^3}{2GM}\right)^{1/2} \int_{r_s}^{r_s} \frac{1}{r - r_s} \sim -\left(\frac{r_s^3}{2GM}\right)^{1/2} \lim_{r \rightarrow r_s} \ln(r - r_s) = \infty.
\]
This means that it takes an infinite amount of time (where time is the coordinate time measured by a stationary observer at infinity) to reach the point \(r = r_s\) along the geodesic.

(e) Compute the velocity \(dR/dT\) as measured by a stationary observer at a coordinate radial distance \(r\). Verify that \(|dR/dT| \rightarrow c\) as \(r \rightarrow r_s\), where \(r_s\) is the Schwarzschild radius defined in part (d).

Using the results for \(dT\) and \(dR\) obtained in parts (a) and (b),
\[
\frac{dR}{dT} = \left(1 - \frac{2GM}{c^2r}\right)^{-1} \frac{dr}{dt}.
\]
Using eq. (37) for \(dr/dt\),
\[
\frac{dR}{dT} = -\left(\frac{2GM}{r}\right)^{1/2} = -c \left(\frac{r_s}{r}\right)^{1/2}.
\]
Indeed, the magnitude of the velocity \(|dR/dT| \rightarrow c\) as \(r \rightarrow r_s\).