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APPENDIX A

UNITS OF MEASUREMENT

The following units, abbreviations and prefixes are from the
Système International d'Unités (designated SI in all Languages.)

Prefixes.

Abbreviations		
Prefix	Multiplication factor	Symbol
tera	10^{12}	T
giga	10^9	G
mega	10^6	M
kilo	10^3	K
hecto	10^2	h
deka	10	da
deci	10^{-1}	d
centi	10^{-2}	c
milli	10^{-3}	m
micro	10^{-6}	μ
nano	10^{-9}	n
pico	10^{-12}	p

Basic Units.

Basic units of measurement		
Unit	Name	Symbol
Length	meter	m
Mass	kilogram	kg
Time	second	s
Electric current	ampere	A
Temperature	degree Kelvin	$^{\circ}$ K
Luminous intensity	candela	cd

Supplementary units		
Unit	Name	Symbol
Plane angle	radian	rad
Solid angle	steradian	sr

DERIVED UNITS		
Name	Units	Symbol
Area	square meter	m ²
Volume	cubic meter	m ³
Frequency	hertz	Hz (s ⁻¹)
Density	kilogram per cubic meter	kg/m ³
Velocity	meter per second	m/s
Angular velocity	radian per second	rad/s
Acceleration	meter per second squared	m/s ²
Angular acceleration	radian per second squared	rad/s ²
Force	newton	N (kg · m/s ²)
Pressure	newton per square meter	N/m ²
Kinematic viscosity	square meter per second	m ² /s
Dynamic viscosity	newton second per square meter	N · s/m ²
Work, energy, quantity of heat	joule	J (N · m)
Power	watt	W (J/s)
Electric charge	coulomb	C (A · s)
Voltage, Potential difference	volt	V (W/A)
Electromotive force	volt	V (W/A)
Electric force field	volt per meter	V/m
Electric resistance	ohm	Ω (V/A)
Electric capacitance	farad	F (A · s/V)
Magnetic flux	weber	Wb (V · s)
Inductance	henry	H (V · s/A)
Magnetic flux density	tesla	T (Wb/m ²)
Magnetic field strength	ampere per meter	A/m
Magnetomotive force	ampere	A

Physical constants.

$$4 \arctan 1 = \pi = 3.14159\,26535\,89793\,23846\,2643 \dots$$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e = 2.71828\,18284\,59045\,23536\,0287 \dots$$

$$\text{Euler's constant } \gamma = 0.57721\,56649\,01532\,86060\,6512 \dots$$

$$\gamma = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \log n\right)$$

$$\text{speed of light in vacuum} = 2.997925(10)^8 \, m \, s^{-1}$$

$$\text{electron charge} = 1.60210(10)^{-19} \, C$$

$$\text{Avogadro's constant} = 6.02252(10)^{23} \, mol^{-1}$$

$$\text{Plank's constant} = 6.6256(10)^{-34} \, J \, s$$

$$\text{Universal gas constant} = 8.3143 \, J \, K^{-1} \, mol^{-1} = 8314.3 \, J \, Kg^{-1} \, K^{-1}$$

$$\text{Boltzmann constant} = 1.38054(10)^{-23} \, J \, K^{-1}$$

$$\text{Stefan-Boltzmann constant} = 5.6697(10)^{-8} \, W \, m^{-2} \, K^{-4}$$

$$\text{Gravitational constant} = 6.67(10)^{-11} \, N \, m^2 \, kg^{-2}$$

APPENDIX B

CHRISTOFFEL SYMBOLS OF SECOND KIND

1. Cylindrical coordinates $(r, \theta, z) = (x^1, x^2, x^3)$

$$\begin{array}{lll} x = r \cos \theta & r \geq 0 & h_1 = 1 \\ y = r \sin \theta & 0 \leq \theta \leq 2\pi & h_2 = r \\ z = z & -\infty < z < \infty & h_3 = 1 \end{array}$$

The coordinate curves are formed by the intersection of the coordinate surfaces

$$\begin{array}{ll} x^2 + y^2 = r^2, & \text{Cylinders} \\ y/x = \tan \theta & \text{Planes} \\ z = \text{Constant} & \text{Planes.} \end{array}$$

$$\left\{ \begin{array}{c} 1 \\ 2 \end{array} \right\} = -r \qquad \left\{ \begin{array}{c} 2 \\ 1 \end{array} \right\} = \left\{ \begin{array}{c} 2 \\ 2 \end{array} \right\} = \frac{1}{r}$$

2. Spherical coordinates $(\rho, \theta, \phi) = (x^1, x^2, x^3)$

$$\begin{array}{lll} x = \rho \sin \theta \cos \phi & \rho \geq 0 & h_1 = 1 \\ y = \rho \sin \theta \sin \phi & 0 \leq \theta \leq \pi & h_2 = \rho \\ z = \rho \cos \theta & 0 \leq \phi \leq 2\pi & h_3 = \rho \sin \theta \end{array}$$

The coordinate curves are formed by the intersection of the coordinate surfaces

$$\begin{array}{ll} x^2 + y^2 + z^2 = \rho^2 & \text{Spheres} \\ x^2 + y^2 = \tan^2 \theta z & \text{Cones} \\ y = x \tan \phi & \text{Planes.} \end{array}$$

$$\begin{array}{ll} \left\{ \begin{array}{c} 1 \\ 2 \end{array} \right\} = -\rho & \left\{ \begin{array}{c} 2 \\ 1 \end{array} \right\} = \left\{ \begin{array}{c} 2 \\ 2 \end{array} \right\} = \frac{1}{\rho} \\ \left\{ \begin{array}{c} 1 \\ 3 \end{array} \right\} = -\rho \sin^2 \theta & \left\{ \begin{array}{c} 3 \\ 1 \end{array} \right\} = \left\{ \begin{array}{c} 3 \\ 3 \end{array} \right\} = \frac{1}{\rho} \\ \left\{ \begin{array}{c} 2 \\ 3 \end{array} \right\} = -\sin \theta \cos \theta & \left\{ \begin{array}{c} 3 \\ 2 \end{array} \right\} = \left\{ \begin{array}{c} 3 \\ 2 \end{array} \right\} = \cot \theta \end{array}$$

3. Parabolic cylindrical coordinates $(\xi, \eta, z) = (x^1, x^2, x^3)$

$$\begin{array}{lll} x = \xi\eta & -\infty < \xi < \infty & h_1 = \sqrt{\xi^2 + \eta^2} \\ y = \frac{1}{2}(\xi^2 - \eta^2) & -\infty < z < \infty & h_2 = \sqrt{\xi^2 + \eta^2} \\ z = z & \eta \geq 0 & h_3 = 1 \end{array}$$

The coordinate curves are formed by the intersection of the coordinate surfaces

$$\begin{array}{ll} x^2 = -2\xi^2(y - \frac{\xi^2}{2}) & \text{Parabolic cylinders} \\ x^2 = 2\eta^2(y + \frac{\eta^2}{2}) & \text{Parabolic cylinders} \\ z = \text{Constant} & \text{Planes.} \end{array}$$

$$\begin{array}{ll} \left\{ \begin{array}{c} 1 \\ 1 \end{array} \right\} = \frac{\xi}{\xi^2 + \eta^2} & \left\{ \begin{array}{c} 1 \\ 2 \end{array} \right\} = \frac{-\xi}{\xi^2 + \eta^2} \\ \left\{ \begin{array}{c} 2 \\ 2 \end{array} \right\} = \frac{\eta}{\xi^2 + \eta^2} & \left\{ \begin{array}{c} 1 \\ 1 \end{array} \right\} = \left\{ \begin{array}{c} 1 \\ 2 \end{array} \right\} = \frac{\eta}{\xi^2 + \eta^2} \\ \left\{ \begin{array}{c} 2 \\ 1 \end{array} \right\} = \frac{-\eta}{\xi^2 + \eta^2} & \left\{ \begin{array}{c} 2 \\ 1 \end{array} \right\} = \left\{ \begin{array}{c} 2 \\ 1 \end{array} \right\} = \frac{\xi}{\xi^2 + \eta^2} \end{array}$$

4. Parabolic coordinates $(\xi, \eta, \phi) = (x^1, x^2, x^3)$

$$\begin{array}{lll} x = \xi\eta \cos \phi & \xi \geq 0 & h_1 = \sqrt{\xi^2 + \eta^2} \\ y = \xi\eta \sin \phi & \eta \geq 0 & h_2 = \sqrt{\xi^2 + \eta^2} \\ z = \frac{1}{2}(\xi^2 - \eta^2) & 0 < \phi < 2\pi & h_3 = \xi\eta \end{array}$$

The coordinate curves are formed by the intersection of the coordinate surfaces

$$\begin{array}{ll} x^2 + y^2 = -2\xi^2(z - \frac{\xi^2}{2}) & \text{Paraboloids} \\ x^2 + y^2 = 2\eta^2(z + \frac{\eta^2}{2}) & \text{Paraboloids} \\ y = x \tan \phi & \text{Planes.} \end{array}$$

$$\begin{array}{ll} \left\{ \begin{array}{c} 1 \\ 1 \end{array} \right\} = \frac{\xi}{\xi^2 + \eta^2} & \left\{ \begin{array}{c} 1 \\ 3 \end{array} \right\} = \frac{-\xi\eta^2}{\xi^2 + \eta^2} \\ \left\{ \begin{array}{c} 2 \\ 2 \end{array} \right\} = \frac{\eta}{\xi^2 + \eta^2} & \left\{ \begin{array}{c} 1 \\ 2 \end{array} \right\} = \left\{ \begin{array}{c} 1 \\ 2 \end{array} \right\} = \frac{\eta}{\xi^2 + \eta^2} \\ \left\{ \begin{array}{c} 1 \\ 2 \end{array} \right\} = \frac{-\xi}{\xi^2 + \eta^2} & \left\{ \begin{array}{c} 2 \\ 1 \end{array} \right\} = \left\{ \begin{array}{c} 2 \\ 1 \end{array} \right\} = \frac{\xi}{\xi^2 + \eta^2} \\ \left\{ \begin{array}{c} 2 \\ 1 \end{array} \right\} = \frac{-\eta}{\xi^2 + \eta^2} & \left\{ \begin{array}{c} 3 \\ 3 \end{array} \right\} = \left\{ \begin{array}{c} 3 \\ 2 \end{array} \right\} = \frac{1}{\eta} \\ \left\{ \begin{array}{c} 2 \\ 3 \end{array} \right\} = \frac{-\eta\xi^2}{\xi^2 + \eta^2} & \left\{ \begin{array}{c} 3 \\ 1 \end{array} \right\} = \left\{ \begin{array}{c} 3 \\ 1 \end{array} \right\} = \frac{1}{\xi} \end{array}$$

5. Elliptic cylindrical coordinates $(\xi, \eta, z) = (x^1, x^2, x^3)$

$$\begin{aligned} x &= \cosh \xi \cos \eta & \xi &\geq 0 & h_1 &= \sqrt{\sinh^2 \xi + \sin^2 \eta} \\ y &= \sinh \xi \sin \eta & 0 &\leq \eta \leq 2\pi & h_2 &= \sqrt{\sinh^2 \xi + \sin^2 \eta} \\ z &= z & -\infty &< z < \infty & h_3 &= 1 \end{aligned}$$

The coordinate curves are formed by the intersection of the coordinate surfaces

$$\begin{aligned} \frac{x^2}{\cosh^2 \xi} + \frac{y^2}{\sinh^2 \xi} &= 1 & \text{Elliptic cylinders} \\ \frac{x^2}{\cos^2 \eta} - \frac{y^2}{\sin^2 \eta} &= 1 & \text{Hyperbolic cylinders} \\ z &= \text{Constant} & \text{Planes.} \end{aligned}$$

$$\begin{aligned} \left\{ \begin{array}{c} 1 \\ 1 \end{array} \right\} &= \frac{\sinh \xi \cosh \xi}{\sinh^2 \xi + \sin^2 \eta} & \left\{ \begin{array}{c} 2 \\ 2 \end{array} \right\} &= \frac{\sin \eta \cos \eta}{\sinh^2 \xi + \sin^2 \eta} \\ \left\{ \begin{array}{c} 1 \\ 2 \end{array} \right\} &= \frac{-\sinh \xi \cosh \xi}{\sinh^2 \xi + \sin^2 \eta} & \left\{ \begin{array}{c} 2 \\ 1 \end{array} \right\} &= \frac{-\sin \eta \cos \eta}{\sinh^2 \xi + \sin^2 \eta} \\ \left\{ \begin{array}{c} 1 \\ 1 \end{array} \right\} &= \left\{ \begin{array}{c} 1 \\ 2 \end{array} \right\} = \frac{\sin \eta \cos \eta}{\sinh^2 \xi + \sin^2 \eta} & \left\{ \begin{array}{c} 2 \\ 1 \end{array} \right\} &= \left\{ \begin{array}{c} 2 \\ 2 \end{array} \right\} = \frac{\sinh \xi \cosh \xi}{\sinh^2 \xi + \sin^2 \eta} \end{aligned}$$

6. Elliptic coordinates $(\xi, \eta, \phi) = (x^1, x^2, x^3)$

$$\begin{aligned} x &= \sqrt{(1-\eta^2)(\xi^2-1)} \cos \phi & 1 &\leq \xi < \infty & h_1 &= \sqrt{\frac{\xi^2-\eta^2}{\xi^2-1}} \\ y &= \sqrt{(1-\eta^2)(\xi^2-1)} \sin \phi & -1 &\leq \eta \leq 1 & h_2 &= \sqrt{\frac{\xi^2-\eta^2}{1-\eta^2}} \\ z &= \xi \eta & 0 &\leq \phi < 2\pi & h_3 &= \sqrt{(1-\eta^2)(\xi^2-1)} \end{aligned}$$

The coordinate curves are formed by the intersection of the coordinate surfaces

$$\begin{aligned} \frac{x^2}{\xi^2-1} + \frac{y^2}{\xi^2-1} + \frac{z^2}{\xi^2} &= 1 & \text{Prolate ellipsoid} \\ \frac{z^2}{\eta^2} - \frac{x^2}{1-\eta^2} - \frac{y^2}{1-\eta^2} &= 1 & \text{Two-sheeted hyperboloid} \\ y &= x \tan \phi & \text{Planes} \end{aligned}$$

$$\begin{aligned} \left\{ \begin{array}{c} 1 \\ 1 \end{array} \right\} &= -\frac{\xi}{-1+\xi^2} + \frac{\xi}{\xi^2-\eta^2} & \left\{ \begin{array}{c} 2 \\ 3 \end{array} \right\} &= \frac{(-1+\xi^2)\eta(1-\eta^2)}{\xi^2-\eta^2} \\ \left\{ \begin{array}{c} 2 \\ 2 \end{array} \right\} &= \frac{\eta}{1-\eta^2} - \frac{\eta}{\xi^2-\eta^2} & \left\{ \begin{array}{c} 1 \\ 1 \end{array} \right\} &= -\frac{\eta}{\xi^2-\eta^2} \\ \left\{ \begin{array}{c} 1 \\ 2 \end{array} \right\} &= -\frac{\xi(-1+\xi^2)}{(1-\eta^2)(\xi^2-\eta^2)} & \left\{ \begin{array}{c} 2 \\ 2 \end{array} \right\} &= \frac{\xi}{\xi^2-\eta^2} \\ \left\{ \begin{array}{c} 1 \\ 3 \end{array} \right\} &= -\frac{\xi(-1+\xi^2)(1-\eta^2)}{\xi^2-\eta^2} & \left\{ \begin{array}{c} 3 \\ 3 \end{array} \right\} &= \frac{\xi}{-1+\xi^2} \\ \left\{ \begin{array}{c} 2 \\ 1 \end{array} \right\} &= \frac{\eta(1-\eta^2)}{(-1+\xi^2)(\xi^2-\eta^2)} & \left\{ \begin{array}{c} 3 \\ 2 \end{array} \right\} &= -\frac{\eta}{1-\eta^2} \end{aligned}$$

7. **Bipolar coordinates** $(u, v, z) = (x^1, x^2, x^3)$

$$\begin{aligned} x &= \frac{a \sinh v}{\cosh v - \cos u}, & 0 \leq u < 2\pi & & h_1^2 &= h_2^2 \\ y &= \frac{a \sin u}{\cosh v - \cos u}, & -\infty < v < \infty & & h_2^2 &= \frac{a^2}{(\cosh v - \cos u)^2} \\ z &= z & -\infty < z < \infty & & h_3^2 &= 1 \end{aligned}$$

The coordinate curves are formed by the intersection of the coordinate surfaces

$$\begin{aligned} (x - a \coth v)^2 + y^2 &= \frac{a^2}{\sinh^2 v} && \text{Cylinders} \\ x^2 + (y - a \cot u)^2 &= \frac{a^2}{\sin^2 u} && \text{Cylinders} \\ z &= \text{Constant} && \text{Planes.} \end{aligned}$$

$$\begin{aligned} \left\{ \begin{array}{c} 1 \\ 11 \end{array} \right\} &= \frac{\sin u}{\cos u - \cosh v} & \left\{ \begin{array}{c} 2 \\ 11 \end{array} \right\} &= \frac{\sinh v}{-\cos u + \cosh v} \\ \left\{ \begin{array}{c} 2 \\ 22 \end{array} \right\} &= \frac{\sinh v}{\cos u - \cosh v} & \left\{ \begin{array}{c} 1 \\ 12 \end{array} \right\} &= \frac{\sinh v}{\cos u - \cosh v} \\ \left\{ \begin{array}{c} 1 \\ 22 \end{array} \right\} &= \frac{\sin u}{-\cos u + \cosh v} & \left\{ \begin{array}{c} 2 \\ 21 \end{array} \right\} &= \frac{\sin u}{\cos u - \cosh v} \end{aligned}$$

8. **Conical coordinates** $(u, v, w) = (x^1, x^2, x^3)$

$$\begin{aligned} x &= \frac{uvw}{ab}, & b^2 > v^2 > a^2 > w^2, & \quad u \geq 0 & & h_1^2 = 1 \\ y &= \frac{u}{a} \sqrt{\frac{(v^2 - a^2)(w^2 - a^2)}{a^2 - b^2}} & & & & h_2^2 = \frac{u^2(v^2 - w^2)}{(v^2 - a^2)(b^2 - v^2)} \\ z &= \frac{v}{b} \sqrt{\frac{(v^2 - b^2)(w^2 - b^2)}{b^2 - a^2}} & & & & h_3^2 = \frac{u^2(v^2 - w^2)}{(w^2 - a^2)(w^2 - b^2)} \end{aligned}$$

The coordinate curves are formed by the intersection of the coordinate surfaces

$$\begin{aligned} x^2 + y^2 + z^2 &= u^2 && \text{Spheres} \\ \frac{x^2}{v^2} + \frac{y^2}{v^2 - a^2} + \frac{z^2}{v^2 - b^2} &= 0, && \text{Cones} \\ \frac{x^2}{w^2} + \frac{y^2}{w^2 - a^2} + \frac{z^2}{w^2 - b^2} &= 0, && \text{Cones.} \end{aligned}$$

$$\begin{aligned} \left\{ \begin{array}{c} 2 \\ 22 \end{array} \right\} &= \frac{v}{b^2 - v^2} - \frac{v}{-a^2 + v^2} + \frac{v}{v^2 - w^2} & \left\{ \begin{array}{c} 3 \\ 22 \end{array} \right\} &= \frac{w(-a^2 + w^2)(-b^2 + w^2)}{(b^2 - v^2)(-a^2 + v^2)(v^2 - w^2)} \\ \left\{ \begin{array}{c} 3 \\ 33 \end{array} \right\} &= -\frac{w}{v^2 - w^2} - \frac{w}{-a^2 + w^2} - \frac{w}{-b^2 + w^2} & \left\{ \begin{array}{c} 2 \\ 21 \end{array} \right\} &= \frac{1}{u} \\ \left\{ \begin{array}{c} 1 \\ 22 \end{array} \right\} &= -\frac{u(v^2 - w^2)}{(b^2 - v^2)(-a^2 + v^2)} & \left\{ \begin{array}{c} 2 \\ 23 \end{array} \right\} &= -\frac{w}{v^2 - w^2} \\ \left\{ \begin{array}{c} 1 \\ 33 \end{array} \right\} &= -\frac{u(v^2 - w^2)}{(-a^2 + w^2)(-b^2 + w^2)} & \left\{ \begin{array}{c} 3 \\ 31 \end{array} \right\} &= \frac{1}{u} \\ \left\{ \begin{array}{c} 2 \\ 33 \end{array} \right\} &= -\frac{v(b^2 - v^2)(-a^2 + v^2)}{(v^2 - w^2)(-a^2 + w^2)(-b^2 + w^2)} & \left\{ \begin{array}{c} 3 \\ 32 \end{array} \right\} &= \frac{v}{v^2 - w^2} \end{aligned}$$

9. **Prolate spheroidal coordinates** $(u, v, \phi) = (x^1, x^2, x^3)$

$$\begin{aligned} x &= a \sinh u \sin v \cos \phi, & u &\geq 0 & h_1^2 &= h_2^2 \\ y &= a \sinh u \sin v \sin \phi, & 0 &\leq v \leq \pi & h_2^2 &= a^2(\sinh^2 u + \sin^2 v) \\ z &= a \cosh u \cos v, & 0 &\leq \phi < 2\pi & h_3^2 &= a^2 \sinh^2 u \sin^2 v \end{aligned}$$

The coordinate curves are formed by the intersection of the coordinate surfaces

$$\begin{aligned} \frac{x^2}{(a \sinh u)^2} + \frac{y^2}{(a \sinh u)^2} + \frac{z^2}{(a \cosh u)^2} &= 1, & \text{Prolate ellipsoids} \\ \frac{x^2}{(a \cos v)^2} - \frac{y^2}{(a \sin v)^2} - \frac{z^2}{(a \cos v)^2} &= 1, & \text{Two-sheeted hyperboloid} \\ y &= x \tan \phi, & \text{Planes.} \end{aligned}$$

$$\begin{aligned} \left\{ \begin{array}{c} 1 \\ 1 \end{array} \right\} &= \frac{\cosh u \sinh u}{\sin^2 v + \sinh^2 u} & \left\{ \begin{array}{c} 2 \\ 3 \end{array} \right\} &= -\frac{\cos v \sin v \sinh^2 u}{\sin^2 v + \sinh^2 u} \\ \left\{ \begin{array}{c} 2 \\ 2 \end{array} \right\} &= \frac{\cos v \sin v}{\sin^2 v + \sinh^2 u} & \left\{ \begin{array}{c} 1 \\ 1 \end{array} \right\} &= \frac{\cos v \sin v}{\sin^2 v + \sinh^2 u} \\ \left\{ \begin{array}{c} 1 \\ 2 \end{array} \right\} &= -\frac{\cosh u \sinh u}{\sin^2 v + \sinh^2 u} & \left\{ \begin{array}{c} 2 \\ 2 \end{array} \right\} &= \frac{\cosh u \sinh u}{\sin^2 v + \sinh^2 u} \\ \left\{ \begin{array}{c} 1 \\ 3 \end{array} \right\} &= -\frac{\sin^2 v \cosh u \sinh u}{\sin^2 v + \sinh^2 u} & \left\{ \begin{array}{c} 3 \\ 3 \end{array} \right\} &= \frac{\cosh u}{\sinh u} \\ \left\{ \begin{array}{c} 2 \\ 1 \end{array} \right\} &= -\frac{\cos v \sin v}{\sin^2 v + \sinh^2 u} & \left\{ \begin{array}{c} 3 \\ 1 \end{array} \right\} &= \frac{\cos v}{\sin v} \end{aligned}$$

10. **Oblate spheroidal coordinates** $(\xi, \eta, \phi) = (x^1, x^2, x^3)$

$$\begin{aligned} x &= a \cosh \xi \cos \eta \cos \phi, & \xi &\geq 0 & h_1^2 &= h_2^2 \\ y &= a \cosh \xi \cos \eta \sin \phi, & -\frac{\pi}{2} &\leq \eta \leq \frac{\pi}{2} & h_2^2 &= a^2(\sinh^2 \xi + \sin^2 \eta) \\ z &= a \sinh \xi \sin \eta, & 0 &\leq \phi \leq 2\pi & h_3^2 &= a^2 \cosh^2 \xi \cos^2 \eta \end{aligned}$$

The coordinate curves are formed by the intersection of the coordinate surfaces

$$\begin{aligned} \frac{x^2}{(a \cosh \xi)^2} + \frac{y^2}{(a \cosh \xi)^2} + \frac{z^2}{(a \sinh \xi)^2} &= 1, & \text{Oblate ellipsoids} \\ \frac{x^2}{(a \cos \eta)^2} + \frac{y^2}{(a \cos \eta)^2} - \frac{z^2}{(a \sin \eta)^2} &= 1, & \text{One-sheet hyperboloids} \\ y &= x \tan \phi, & \text{Planes.} \end{aligned}$$

$$\begin{aligned} \left\{ \begin{array}{c} 1 \\ 1 \end{array} \right\} &= \frac{\cosh \xi \sinh \xi}{\sin^2 \eta + \sinh^2 \xi} & \left\{ \begin{array}{c} 2 \\ 3 \end{array} \right\} &= \frac{\cos \eta \sin \eta \cosh^2 \xi}{\sin^2 \eta + \sinh^2 \xi} \\ \left\{ \begin{array}{c} 2 \\ 2 \end{array} \right\} &= \frac{\cos \eta \sin \eta}{\sin^2 \eta + \sinh^2 \xi} & \left\{ \begin{array}{c} 1 \\ 1 \end{array} \right\} &= \frac{\cos \eta \sin \eta}{\sin^2 \eta + \sinh^2 \xi} \\ \left\{ \begin{array}{c} 1 \\ 2 \end{array} \right\} &= -\frac{\cosh \xi \sinh \xi}{\sin^2 \eta + \sinh^2 \xi} & \left\{ \begin{array}{c} 2 \\ 2 \end{array} \right\} &= \frac{\cosh \xi \sinh \xi}{\sin^2 \eta + \sinh^2 \xi} \\ \left\{ \begin{array}{c} 1 \\ 3 \end{array} \right\} &= -\frac{\cos^2 \eta \cosh \xi \sinh \xi}{\sin^2 \eta + \sinh^2 \xi} & \left\{ \begin{array}{c} 3 \\ 3 \end{array} \right\} &= \frac{\sinh \xi}{\cosh \xi} \\ \left\{ \begin{array}{c} 2 \\ 1 \end{array} \right\} &= -\frac{\cos \eta \sin \eta}{\sin^2 \eta + \sinh^2 \xi} & \left\{ \begin{array}{c} 3 \\ 1 \end{array} \right\} &= -\frac{\sin \eta}{\cos \eta} \end{aligned}$$

11. Toroidal coordinates $(u, v, \phi) = (x^1, x^2, x^3)$

$$\begin{aligned}
x &= \frac{a \sinh v \cos \phi}{\cosh v - \cos u}, & 0 \leq u < 2\pi & & h_1^2 &= h_2^2 \\
y &= \frac{a \sinh v \sin \phi}{\cosh v - \cos u}, & -\infty < v < \infty & & h_2^2 &= \frac{a^2}{(\cosh v - \cos u)^2} \\
z &= \frac{a \sin u}{\cosh v - \cos u}, & 0 \leq \phi < 2\pi & & h_3^2 &= \frac{a^2 \sinh^2 v}{(\cosh v - \cos u)^2}
\end{aligned}$$

The coordinate curves are formed by the intersection of the coordinate surfaces

$$\begin{aligned}
x^2 + y^2 + \left(z - \frac{a \cos u}{\sin u}\right)^2 &= \frac{a^2}{\sin^2 u}, & \text{Spheres} \\
\left(\sqrt{x^2 + y^2} - a \frac{\cosh v}{\sinh v}\right)^2 + z^2 &= \frac{a^2}{\sinh^2 v}, & \text{Tores} \\
y &= x \tan \phi, & \text{planes}
\end{aligned}$$

$$\begin{aligned}
\begin{Bmatrix} 1 \\ 11 \end{Bmatrix} &= \frac{\sin u}{\cos u - \cosh v} & \begin{Bmatrix} 2 \\ 33 \end{Bmatrix} &= -\frac{\sinh v (\cos u \cosh v - 1)}{\cos u - \cosh v} \\
\begin{Bmatrix} 2 \\ 22 \end{Bmatrix} &= \frac{\sinh v}{\cos u - \cosh v} & \begin{Bmatrix} 1 \\ 12 \end{Bmatrix} &= \frac{\sinh v}{\cos u - \cosh v} \\
\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} &= \frac{\sin u}{-\cos u + \cosh v} & \begin{Bmatrix} 2 \\ 21 \end{Bmatrix} &= \frac{\sin u}{\cos u - \cosh v} \\
\begin{Bmatrix} 1 \\ 33 \end{Bmatrix} &= \frac{\sin u \sinh v^2}{-\cos u + \cosh v} & \begin{Bmatrix} 3 \\ 31 \end{Bmatrix} &= \frac{\sin u}{\cos u - \cosh v} \\
\begin{Bmatrix} 2 \\ 11 \end{Bmatrix} &= \frac{\sinh v}{-\cos u + \cosh v} & \begin{Bmatrix} 3 \\ 32 \end{Bmatrix} &= \frac{\cos u \cosh v - 1}{\cos u \sinh v - \cosh v \sinh v}
\end{aligned}$$

12. Confocal ellipsoidal coordinates $(u, v, w) = (x^1, x^2, x^3)$

$$\begin{aligned} x^2 &= \frac{(a^2 - u)(a^2 - v)(a^2 - w)}{(a^2 - b^2)(a^2 - c^2)}, & u < c^2 < b^2 < a^2 \\ y^2 &= \frac{(b^2 - u)(b^2 - v)(b^2 - w)}{(b^2 - a^2)(b^2 - c^2)}, & c^2 < v < b^2 < a^2 \\ z^2 &= \frac{(c^2 - u)(c^2 - v)(c^2 - w)}{(c^2 - a^2)(c^2 - b^2)}, & c^2 < b^2 < v < a^2 \end{aligned}$$

$$\begin{aligned} h_1^2 &= \frac{(u - v)(u - w)}{4(a^2 - u)(b^2 - u)(c^2 - u)} \\ h_2^2 &= \frac{(v - u)(v - w)}{4(a^2 - v)(b^2 - v)(c^2 - v)} \\ h_3^2 &= \frac{(w - u)(w - v)}{4(a^2 - w)(b^2 - w)(c^2 - w)} \end{aligned}$$

$$\begin{aligned} \left\{ \begin{matrix} 1 \\ 11 \end{matrix} \right\} &= \frac{1}{2(a^2 - u)} + \frac{1}{2(b^2 - u)} + \frac{1}{2(c^2 - u)} + \frac{1}{2(u - v)} + \frac{1}{2(u - w)} \\ \left\{ \begin{matrix} 2 \\ 22 \end{matrix} \right\} &= \frac{1}{2(a^2 - v)} + \frac{1}{2(b^2 - v)} + \frac{1}{2(c^2 - v)} + \frac{1}{2(-u + v)} + \frac{1}{2(v - w)} \\ \left\{ \begin{matrix} 3 \\ 33 \end{matrix} \right\} &= \frac{1}{2(a^2 - w)} + \frac{1}{2(b^2 - w)} + \frac{1}{2(c^2 - w)} + \frac{1}{2(-u + w)} + \frac{1}{2(-v + w)} \end{aligned}$$

$$\begin{aligned} \left\{ \begin{matrix} 1 \\ 22 \end{matrix} \right\} &= \frac{(a^2 - u)(b^2 - u)(c^2 - u)(v - w)}{2(a^2 - v)(b^2 - v)(c^2 - v)(u - v)(u - w)} & \left\{ \begin{matrix} 1 \\ 12 \end{matrix} \right\} &= \frac{-1}{2(u - v)} \\ \left\{ \begin{matrix} 1 \\ 33 \end{matrix} \right\} &= \frac{(a^2 - u)(b^2 - u)(c^2 - u)(-v + w)}{2(u - v)(a^2 - w)(b^2 - w)(c^2 - w)(u - w)} & \left\{ \begin{matrix} 1 \\ 13 \end{matrix} \right\} &= \frac{-1}{2(u - w)} \\ \left\{ \begin{matrix} 2 \\ 11 \end{matrix} \right\} &= \frac{(a^2 - v)(b^2 - v)(c^2 - v)(u - w)}{2(a^2 - u)(b^2 - u)(c^2 - u)(-u + v)(v - w)} & \left\{ \begin{matrix} 2 \\ 21 \end{matrix} \right\} &= \frac{-1}{2(-u + v)} \\ \left\{ \begin{matrix} 2 \\ 33 \end{matrix} \right\} &= \frac{(a^2 - v)(b^2 - v)(c^2 - v)(-u + w)}{2(-u + v)(a^2 - w)(b^2 - w)(c^2 - w)(v - w)} & \left\{ \begin{matrix} 2 \\ 23 \end{matrix} \right\} &= \frac{-1}{2(v - w)} \\ \left\{ \begin{matrix} 3 \\ 11 \end{matrix} \right\} &= \frac{(u - v)(a^2 - w)(b^2 - w)(c^2 - w)}{2(a^2 - u)(b^2 - u)(c^2 - u)(-u + w)(-v + w)} & \left\{ \begin{matrix} 3 \\ 31 \end{matrix} \right\} &= \frac{-1}{2(-u + w)} \\ \left\{ \begin{matrix} 3 \\ 22 \end{matrix} \right\} &= \frac{(-u + v)(a^2 - w)(b^2 - w)(c^2 - w)}{2(a^2 - v)(b^2 - v)(c^2 - v)(-u + w)(-v + w)} & \left\{ \begin{matrix} 3 \\ 32 \end{matrix} \right\} &= \frac{-1}{2(-v + w)} \end{aligned}$$

APPENDIX C

VECTOR IDENTITIES

The following identities assume that $\vec{A}, \vec{B}, \vec{C}, \vec{D}$ are differentiable vector functions of position while f, f_1, f_2 are differentiable scalar functions of position.

1.	$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$
2.	$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$
3.	$(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C})$
4.	$\vec{A} \times (\vec{B} \times \vec{C}) + \vec{B} \times (\vec{C} \times \vec{A}) + \vec{C} \times (\vec{A} \times \vec{B}) = \vec{0}$
5.	$(\vec{A} \times \vec{B}) \times (\vec{C} \times \vec{D}) = \vec{B}(\vec{A} \cdot \vec{C} \times \vec{D}) - \vec{A}(\vec{B} \cdot \vec{C} \times \vec{D})$ $= \vec{C}(\vec{A} \cdot \vec{B} \times \vec{C}) - \vec{D}(\vec{A} \cdot \vec{B} \times \vec{C})$
6.	$(\vec{A} \times \vec{B}) \cdot (\vec{B} \times \vec{C}) \times (\vec{C} \times \vec{A}) = (\vec{A} \cdot \vec{B} \times \vec{C})^2$
7.	$\nabla(f_1 + f_2) = \nabla f_1 + \nabla f_2$
8.	$\nabla \cdot (\vec{A} + \vec{B}) = \nabla \cdot \vec{A} + \nabla \cdot \vec{B}$
9.	$\nabla \times (\vec{A} + \vec{B}) = \nabla \times \vec{A} + \nabla \times \vec{B}$
10.	$\nabla(f\vec{A}) = (\nabla f) \cdot \vec{A} + f\nabla \cdot \vec{A}$
11.	$\nabla(f_1 f_2) = f_1 \nabla f_2 + f_2 \nabla f_1$
12.	$\nabla \times (f\vec{A}) = (\nabla f) \times \vec{A} + f(\nabla \times \vec{A})$
13.	$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$
14.	$(\vec{A} \cdot \nabla)\vec{A} = \nabla \left(\frac{ \vec{A} ^2}{2} \right) - \vec{A} \times (\nabla \times \vec{A})$
15.	$\nabla(\vec{A} \cdot \vec{B}) = (\vec{B} \cdot \nabla)\vec{A} + (\vec{A} \cdot \nabla)\vec{B} + \vec{B} \times (\nabla \times \vec{A}) + \vec{A} \times (\nabla \times \vec{B})$
16.	$\nabla \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \nabla)\vec{A} - \vec{B}(\nabla \cdot \vec{A}) - (\vec{A} \cdot \nabla)\vec{B} + \vec{A}(\nabla \cdot \vec{B})$
17.	$\nabla \cdot (\nabla f) = \nabla^2 f$
18.	$\nabla \times (\nabla f) = \vec{0}$
19.	$\nabla \cdot (\nabla \times \vec{A}) = 0$
20.	$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$