Chapter 1
Special Theory of Relativity

1.1 Physics before 1905

At the end of the 19th c entury there was a common belief that mechanics was a final physical theory. In this framework, mechanics would be able to explain all known physical phenomena. One of tasks undertaken by physicists of that period was looking for a maximally simple description of physical phenomena using a minimal number of logically independent postulates.

Principle of Relativity

Principle of Galileo

The principle of relativity, formulated by Galileo, was commonly accepted as such fundamental postulate. It states that in absence of interaction (with other bodies) a physical body remains in uniform motion – it moves with constant speed along a straight line. This statement cannot be true in any frame of reference. For instance, it does not hold in rotating systems where the trajectory of such a body is a curve. The class of reference frames in which Galileo's principle does apply is called *inertial frames of reference* or *inertial reference systems*. Such frames move in relation to each other with constant velocity. The principle of Galileo was then reformulated as so-called *principle of relativity* which states that *the laws of classical physics hold in all inertial frames of reference*.

Inertial frame of reference

The mathematical transformation of coordinates associated with two different inertial systems is called *classical transformation* or *Galileo's transformation*. It establishes relations between the positions and velocities in these inertial systems. We denote by t, $\boldsymbol{x}$ time and position vector of the material point in the reference frame S and by t', $\boldsymbol{x}'$ its time and position vector in the frame S' that remains in motion with the velocity $\boldsymbol{V}$ with respect to S. The classical transformation has the

Classical transformation

form:

$$t' = t, \tag{1.1}$$

$$\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{V}t. \tag{1.2}$$

The transformation (1.1) is *the postulate* – one of the foundations of Newtonian physics. It constitutes the definition of *simultaneity*. of events. On the other hand (1.2) has origin in classical concept of addition of positions. The classical transformation implies the velocity addition formula

Classical concept of simultaneity

Classical addition of velocities

$$\boldsymbol{v}' = \boldsymbol{v} - \boldsymbol{V} \tag{1.3}$$

where $\boldsymbol{V}$ is velocity of S' with respect to S and $\boldsymbol{v}' \equiv \frac{d\boldsymbol{x}'}{dt'}$, $\boldsymbol{v} \equiv \frac{d\boldsymbol{x}}{dt}$ are velocities of the material point in these frames.

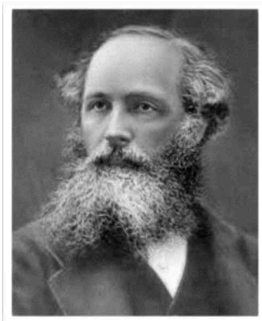


Figure 1.1: James Clerk Maxwell (1831-1879)

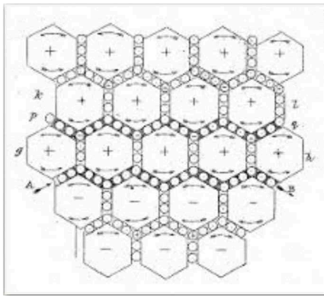


Figure 1.2: Picture from Maxwell's work *On Physical Lines of Force*.

Maxwell's electromagnetic theory

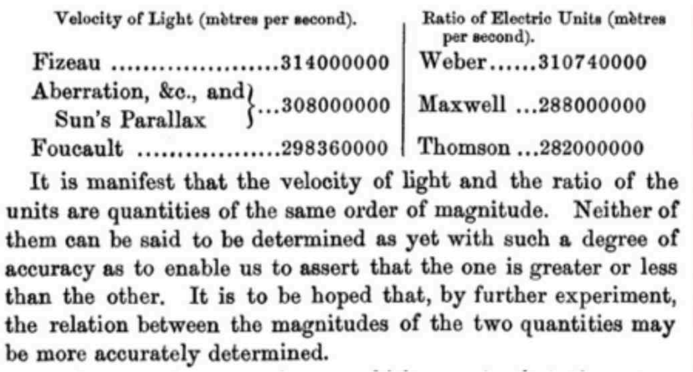
In 1854 James Clerk Maxwell (1831-1879) began work on formulation of the theory of electromagnetic phenomenas. Yet in this year he wrote a letter to his friend William Thomson, announcing that he "intends to attack electromagnetism". The first work entitled *On Faraday's Lines of Force* (1855) was unsuccessful and presented barely *qualitative* hydrodynamical approach to electromagnetic material medium. In 1861, he published a dissertation *On Physical Lines of Force*, in which he tried to build a mechanical model of electromagnetic field.

This dissertation contains some elements of electromagnetic field theory, subsequently developed in Maxwell's next work entitled *A Dynamical Theory of the Electromagnetic Field* (1864-1865). In order to replace the Faraday's idea of polarisation of medium Maxwell has introduced the concept of electric displacement and medium variation called by him *displacement current*.

Both Maxwell's works contain a very important conjecture. Namely, Maxwell has noticed that the ratio of electric and magnetic units in field equations has dimension of velocity. Moreover, he has also noticed that the numerical value of this characteristic speed coincided with the experimental value of the speed of light. He assumed that there can exist periodic transverse displacement waves in the medium and calculated the speed of their propagation. He got the value which was very close to the value of speed of light. The results of his theoretical study of electricity and magnetism led him to the *hypothesis that light is an electromagnetic wave*.

Observation that a velocity parameter has value that coincides with the light speed

Hypothesis that light is an electromagnetic wave



Velocity of Light (mètres per second).		Ratio of Electric Units (mètres per second).	
Fizeau	314000000	Weber	310740000
Aberration, &c., and Sun's Parallax	308000000	Maxwell	288000000
Foucault	298360000	Thomson	282000000

It is manifest that the velocity of light and the ratio of the units are quantities of the same order of magnitude. Neither of them can be said to be determined as yet with such a degree of accuracy as to enable us to assert that the one is greater or less than the other. It is to be hoped that, by further experiment, the relation between the magnitudes of the two quantities may be more accurately determined.

Figure 1.3: The value of speed of light and ratio of electric and magnetic units.

In 1873 he published *Treatise on Electricity and Magnetism* in which he has presented the final form of electromagnetic field equations. In contrast to his previous works this publication does not contain any mechanical models of the electromagnetic field. The field equations were obtained considering the hypothesis of existence of *displacement current* and *electronic state* i.e. such a state that variation in the magnetic field creates *solenoidal* electric field.

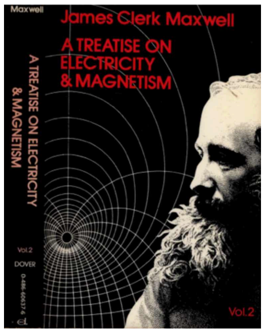


Figure 1.4: Treatise on Electricity and Magnetism

The form of Maxwell's equations presented in the treatise is significantly different from their present form. In particular, Maxwell did not use the vector calculus. In particular, some of the equations were written using the quaternion notation. The currently used, vector form of Maxwell's equation was proposed by Oliver Heaviside (1850-1925), which has significantly contributed to the popularisation of Maxwell's theory. Let us stress that Maxwell was not aware of the fundamental character of the theory he created. In particular, until his last days he was convinced that the foundations of electromagnetism are based on the concept of a material medium. Such a medium would enable propagation of electromagnetic waves. This medium was termed *aether*. The constant parameter having dimension of velocity that appears in Maxwell's equations was interpreted as the speed of propagation of electromagnetic waves in the aether.

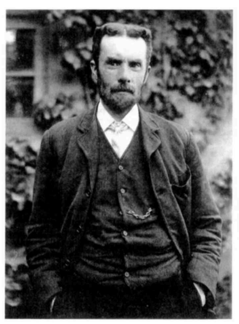


Figure 1.5: Oliver Heaviside (1850-1925).

Problems with electromagnetism

It turned out quickly that Maxwell's theory of electromagnetic phenomena, although very consistent with experiments, leads to some fundamental problems:

1. A problem with construction of mechanical model of aether;

2. A problem with the lack of covariance of Maxwell's equations under the classical transformation.

Luminiferous aether

The existence of a substance (medium) that supports the propagation of electromagnetic oscillations was expected in Maxwell's times.[1] It was, to some extend, natural expectation. All known wave phenomena, such as sound propagation, possessed mechanical interpretation. Many renowned physicists, as for instance *Augustus Cauchy, George FitzGerald, Goeorge Green, Oliver Heaviside, Herman Helmholtz, Gustav Kirchhoff, Joseph Larmor, Hendrik Lorentz, James Mac Cullagh, James Clerk Maxwell, Arnold Sommerfeld, George Stokes, William Tomson (Kelvin)*, published papers concerning the construction of mechanical model of the aether. The fundamental question that they pretend to answer was if the aether drifts together with moving objects.

[1] A. Einstein, L. Infeld, *The Evolution of Physics*

Does the aether drift together with objects?

- Let us assume that *the aether drifts* and consider a laboratory S' with transparent walls and with the source of light at its center. If such a laboratory moves with a constant velocity $\boldsymbol{V}$ with respect to external observer S, then provided that the laws of classical mechanics are valid, the speed of light observed in S would be given as a length of the vector obtained adding velocity of the light in the aether and velocity of the aether with respect to S.

  It means that observed light speed should *depend on the motion of its source*. This conclusion has no confirmation in experiments. The further development of the theory and new experimental results forced physicists to conclude that light moves in empty space with the same speed c – independently on its frequency and motion of its source (so-called " c – principle"). It is important to notice that if the speed of light would depend on the motion of its source, then the images of tight binary stairs would be much more involved than they actually are.

  Yes: light speed depends on motion of its source

- Another possibility is that *the aether cannot be carried* by moving bodies. In such a case its rest frame would be unique. Consequently, there would be possible "absolute motion" i.e. motion with respect to the aether. In such a case Galileo's principle would not be true anymore. The form of physical laws in the aether reference frame would be different to their form in other reference frames. Thus each observer would be able to determine whether he is or not in motion. It would mean thet the measured speed of light would depend on *velocity of the observer with respect to the aether*. The problem of measurement of the speed of light in dependence on the velocity of the observer was considered in 1887 by Albert Abraham Michaelson and Edward Morley. The results of their work has been presented in the paper *On relative motion of Earth and relative luminiferous aether* published in *American Journal of Science* **34**, 333 (1887). The authors

  No: Galileo's principle is invalid

  No: the light speed depends on motion

determined that the *speed of light does not depend on motion of the observer* (in this case the Earth).

All assumptions about aether leads to incompatibility with experimental data

- Both these assumptions lead to incompatibilities with experimental data. Physicists of that period tried to maintain aether theories postulating that the aether can be carried only partially, however, without success in confrontation with experimental data.

Thus physicists were forced to abandon the concept of ether and looking for a new theory that would be compatible with *experimental facts*:

1. *All laws of nature are strictly the same in two inertial reference frames. There is no way to detect absolute uniform motion.*

2. *The speed of light in a vacuum is c, and its value does not depend on motion of a light source or motion of a detector.*

Covariance of Maxwell's equations

Incompatibility between Maxwell's equations and the classical transformation

Let us note that addition of velocities (that follows from the classical transformation) stays in *conflict* with the constancy of light speed. There was more problems associated with the classical transformation.

According to experimental data the laws of electromagnetism are valid in distinct inertial reference frames. Mathematically it means that Maxwell's equations have exactly the same *form* in each two different inertial frames. In other words, when doing a series of experiments in two laboratories S and S' that move uniformly with respect to each other one would deduce from tese experiments exactly the same Maxwell's equations. Sine both reference frames are related by classical transformation one would expect that equations in S' can be obtained from equations in S just applying the classical transformation. It turns out that the classical transformation spoils the form of Maxwell's equations. The resulting set of equations has not form of Maxwell's equations – it contains some new terms! Thus we see that *the classical transformation*:

- *contradicts the constancy of the speed of light,*

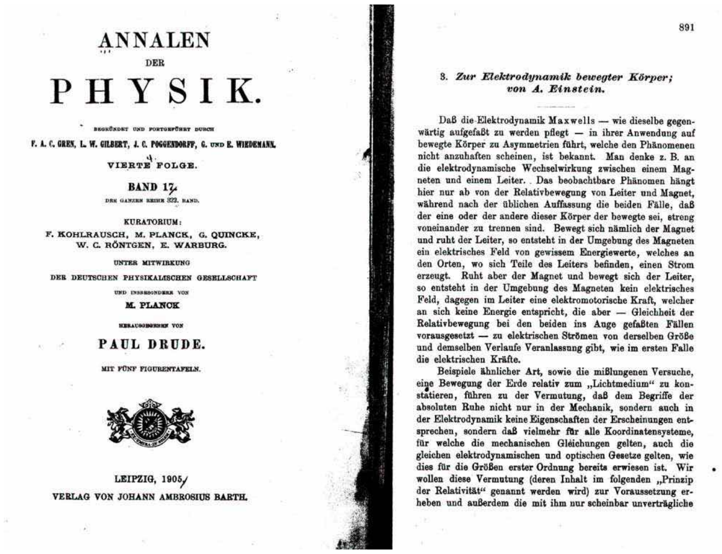
- *does not allow for covariance of Maxwell's equations.*

Classical transformation must be rejected!

In such a case the only reasonable way out is *rejection* the form of the classical transformation. This rises the question of the form of "correct" transformation. Such transformations must preserve covariance of Maxwell's equations and constant character of light speed. The last means that for any speed v the sum (more precisely – its composition "$\vdash$") of this speed with the speed of light c must be equal to c.

1.2 Special Theory of Relativity

A special theory of relativity was proposed by Einstein as an attempt to get rid of the unsatisfactory situation that led to the confrontation of electromagnetic phenomena with classical mechanics. Einstein's work published at June 30, 1905 in *Annalem der Physik* constituted breakthrough in our understanding of space and time.


ANNALEN
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BEGRÜNDET UND FORTGEFÜHRT DURCH
F. A. C. GREN, L. W. GILBERT, J. C. POGGENDORFF, G. UND E. WIEDEMANN.

VIERTE FOLGE.

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DER GANZEN REIHE 322. BAND.

KURATORIUM:
F. KOHLRAUSCH, M. PLANCK, G. QUINCKE,
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UNTER MITWIRKUNG
DER DEUTSCHEN PHYSIKALISCHEN GESELLSCHAFT
UND INSBESONDERE VON
M. PLANCK

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MIT FÜNF FIGURENTAFELN.

LEIPZIG, 1905.
VERLAG VON JOHANN AMBROSIUS BARTH.

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3. *Zur Elektrodynamik bewegter Körper;*
von A. Einstein.

Daß die Elektrodynamik Maxwells — wie dieselbe gegenwärtig aufgefaßt zu werden pflegt — in ihrer Anwendung auf bewegte Körper zu Asymmetrien führt, welche den Phänomenen nicht anzuhaften scheinen, ist bekannt. Man denke z. B. an die elektrodynamische Wechselwirkung zwischen einem Magneten und einem Leiter. Das beobachtbare Phänomen hängt hier nur ab von der Relativbewegung von Leiter und Magnet, während nach der üblichen Auffassung die beiden Fälle, daß der eine oder der andere dieser Körper der bewegte sei, streng voneinander zu trennen sind. Bewegt sich nämlich der Magnet und ruht der Leiter, so entsteht in der Umgebung des Magneten ein elektrisches Feld von gewissem Energiewerte, welches an den Orten, wo sich Teile des Leiters befinden, einen Strom erzeugt. Ruht aber der Magnet und bewegt sich der Leiter, so entsteht in der Umgebung des Magneten kein elektrisches Feld, dagegen im Leiter eine elektromotorische Kraft, welcher an sich keine Energie entspricht, die aber — Gleichheit der Relativbewegung bei den beiden ins Auge gefaßten Fällen vorausgesetzt — zu elektrischen Strömen von derselben Größe und demselben Verlaufe Veranlassung gibt, wie im ersten Falle die elektrischen Kräfte.

Beispiele ähnlicher Art, sowie die mißlungenen Versuche, eine Bewegung der Erde relativ zum „Lichtmedium" zu konstatieren, führen zu der Vermutung, daß dem Begriffe der absoluten Ruhe nicht nur in der Mechanik, sondern auch in der Elektrodynamik keine Eigenschaften der Erscheinungen entsprechen, sondern daß vielmehr für alle Koordinatensysteme, für welche die mechanischen Gleichungen gelten, auch die gleichen elektrodynamischen und optischen Gesetze gelten, wie dies für die Größen erster Ordnung bereits erwiesen ist. Wir wollen diese Vermutung (deren Inhalt im folgenden „Prinzip der Relativität" genannt werden wird) zur Voraussetzung erheben und außerdem die mit ihm nur scheinbar unverträgliche

Figure 1.6: On the Electrodynamics of Moving Bodies.

The work itself has a profound connection with electrodynamics as suggested by its title *On the Electrodynamics of Moving Bodies*. Einstein has assumed two postulates:

Einstein's postulates

1. *The laws of physics are the same in all inertial reference frames.*

2. *The speed of light in vacuum has the same value c in all inertial reference frames.*

The transformation of coordinates in two different inertial frames is not a classical transformation. It has such a form that provides constancy of light speed in all inertial reference frames. It is called *Lorentz transformation*.

Simultaneity

New concept of simultaneity

The postulate that the speed of light takes the same value in all reference frames is explicit *resignation from the concept of absolute simultaneity* (1.1). The assumption that $c' = c$ and $t' = t$ leads to logical contradiction. The resignation from the absolute character of time means that time should be treated not as an external parameter but rather as a *coordinate*. In this approach, the concept of simultaneity *depends on the observer*. Each observer has his own time coordinate. Collections of events having equal value of time coordinate in a given reference frame form *surfaces of simultaneity*. There are following relativistic effects that have their origin in non-existence of absolute simultaneity:

Relativistic effects that originate in lack of absolute simultaneity

- Lorentz–FitzGerald contraction,
- dilation of time.

Spacetime

An important feature of the Special Theory of Relativity is the fact that it can be expressed in geometric language. A geometrisation of the theory allows for deeper insight into its structure. In order to geometrize the Special Theory of Relativity one starts with a set of basic objects called *events* which are counterparts of *points* in Euclid's geometry.

Definition.[2] A collection of all events that carry information about "when" and "where" independently on what "happened" is called *spacetime*.

[2] Definition introduced by Andrzej Mariusz Trautman (born January 4, 1933) a Polish mathematical physicist who has made contributions to classical gravitation in general and to general relativity in particular. Trautman and Ivor Robinson discovered a family of exact solutions of the Einstein field equation, the Robinson-Trautman gravitational waves.

Thus spacetime is a set of labels. A mathematical model of spacetime is obtained imposing some structure on this set. In the case of Special Theory of Relativity a spacetime has structure of *four-dimensional affine space*, *i.e.* the homogeneous space with operation of translation. This spacetime is called *Minkowski spacetime*.

Affine space structure of Minkowski spacetime

Affine space as a model of spacetime

The model of Minkowski spacetime is a four-dimensional affine space (M, V^4) with scalar product where M stands for set of points (events) and V^4 ia a four dimensional real vector space. We shall use letters $p, q, r, \ldots \in M$ for denoting events and $x, y, z, \ldots \in V^4$ for denoting vectors – called here *four-vectors*.

Figure 1.7: Hermann Minkowski (1864-1909).

In similarity to the Euclidean affine space we have translation operation "+" (translation) which maps events on events

$$M \times V^4 \ni (p, x) \to p + x \in M \tag{1.4}$$

and has properties:

1.
$$(p + x) + y = p + (x + y), \qquad \forall p \in M \quad \text{and} \quad \forall x, y \in V^4;$$

2.
$$p + 0 = p \qquad \forall p \in M;$$

3.
$$\forall p \in M \qquad \exists!\, x \in V^4 : \qquad q = p + x.$$

If one distinguishes an event $p_0 \in M$ and fixes basis $\mathbf{e}_\alpha$ in V^4, then each point $p \in M$ can be uniquely represented as a translation of p_0 by four-vector $x \in V^4$

Cartesian coordinates

$$p = p_0 + x = p_0 + \sum_{\alpha=0}^{3} x^\alpha(p)\mathbf{e}_\alpha \tag{1.5}$$

where coefficients $\{x^\alpha(p)\}_{\alpha=0,1,2,3}$ are *Cartesian coordinates* of the point p in the affine basis $(p_0, \mathbf{e}_\alpha)$.

We shall use index notation for expressions containing sums and assume that greek indices run over a set of numbers

Notation

$$\alpha, \beta, \ldots = \{0, 1, 2, 3\}$$

wheres latin indices (spatial indices) run over

$$i, j, \ldots = \{1, 2, 3\}.$$

The index $\alpha = 0$ labels temporal coordinate x^0, temporal vector $\mathbf{e}_0$ etc. Thus the four vector x in (1.5) reads

$$\begin{aligned} x^\alpha(p)\mathbf{e}_\alpha &\equiv x^0(p)\mathbf{e}_0 + x^1(p)\mathbf{e}_1 + x^2(p)\mathbf{e}_2 + x^3(p)\mathbf{e}_3 \\ &\equiv x^0(p)\mathbf{e}_0 + x^i(p)\mathbf{e}_i \end{aligned} \tag{1.6}$$

In order to introduce metric structure in Minkowski spacetime we choose a *metric tensor* whose components form a symmetric nonsingular matrix

$$g_{\alpha\beta} := \boldsymbol{g}(\mathbf{e}_\alpha, \mathbf{e}_\beta). \tag{1.7}$$

The expression $\boldsymbol{g}(x, y)$ represents a scalar product of two four-vectors at p

$$\boldsymbol{g}(x, y) = \boldsymbol{g}(x^\alpha \mathbf{e}_\alpha, y^\beta \mathbf{e}_\beta) = x^\alpha y^\beta \boldsymbol{g}(\mathbf{e}_\alpha, \mathbf{e}_\beta) = x^\alpha y^\beta g_{\alpha\beta} \in \mathbb{R}.$$

In *Cartesian* coordinates the metric tensor has components

$$g_{\alpha\beta} \equiv \eta_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \equiv \mathrm{diag}(1, -1, -1, -1). \tag{1.8}$$

Note that another equivalent choice has the form

$$\eta_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1).$$

Curvilinear coordinates can be introduced as diffeomorphic transformation

$$x'^\alpha = x'^\alpha(x^0, \ldots, x^3), \qquad \alpha = 0, \ldots, 3 \tag{1.9}$$

i.e. an invertible transformation which is at least of $\mathcal{C}^1$ class. It is required that the inverse transformation is also at least of $\mathcal{C}^1$ class.

Metric tensor is a mapping

$$\boldsymbol{g} : (x, y) \mapsto \boldsymbol{g}(x, y) \in \mathbb{R}$$

which is

1. *bilinear*
$$\begin{aligned} \boldsymbol{g}(ax + by, z) =& a\boldsymbol{g}(x, z) \\ &+ b\boldsymbol{g}(y, z) \end{aligned}$$
$$\forall\, x, y, z \in V^4 \quad \text{and} \quad \forall\, a, b \in \mathbb{R}$$
2. *symmetric*
$$\boldsymbol{g}(x, y) = \boldsymbol{g}(y, x) \quad \forall\, x, y \in V^4$$
3. *nondegenerate*
$$\boldsymbol{g}(x, y) = 0 \quad \forall y \quad \text{then } x = 0$$

Tangent space

At any point of Minkowski spacetime there can be introduced a set of smooth curves. The vectors tangent to these curves at p form tangent space. It is denoted by

$$T_pM := \{v : \quad v \text{ is tangent to } M \text{ at } p\}.$$

Let $\{\mathbf{e}_\alpha(p)\}_{\alpha=0,\dots,3}$ be a basis of tangent space T_pM. Any other basis at p, denoted by $\{\mathbf{e}'_\alpha(p)\}_{\alpha=0,\dots,3}$, is related to the previous one by linear transformation

Transformation of basis vectors

$$\mathbf{e}'_\alpha(p) = \frac{\partial x^\beta}{\partial x'^\alpha}(p)\mathbf{e}_\beta(p). \tag{1.10}$$

where x'^α are "new" coordinates. Four-vectors are geometric objects which means that they do not depend on coordinates. Their components however do depend. Taking $v = v'^\alpha \mathbf{e}'_\alpha(p) = v^\beta \mathbf{e}_\beta(p)$ we can conclude that components of four-vector transform as

$$v'^\alpha = \frac{\partial x'^\alpha}{\partial x^\beta}(p)v^\beta. \tag{1.11}$$

In particular, one can choose Cartesian coordinates x^α in Minkowski spacetime. Thus $\mathbf{e}_\alpha(p) = \mathbf{e}_\alpha$ where vectors $\mathbf{e}_\alpha \in V^4$ and $\mathbf{e}_\alpha(p) \in T_pM$.

Minkowski diagrams

Minkowski diagrams constitute a very useful tool in analysis of problems in Special Theory of Relativity. They are sections of spacetime (two- or three- dimensional charts), see Fig.1.8. Each event in Minkowski spacetime is represented by collection of numbers (x^0, x^1, x^2, x^3). Time coordinate is defined as $x^0 := ct$ where c is the speed of light. Some of numbers $\{x^\alpha\}$ can be fixed for whole diagram. In such a case we usually ignore the presence of irrelevant coordinates writing components of four-vector as it were two–component or three–component vectors i.e. (x^0, x^1), (x^0, x^1, x^2). A history of material objects in Minkowski spacetime is represented by curves, sheets or hyper sheets – depending on dimensionality of these object. They are called: *world lines* for points, *world sheets* for one dimensional objects and *world volumes* for two and three dimensional objects.

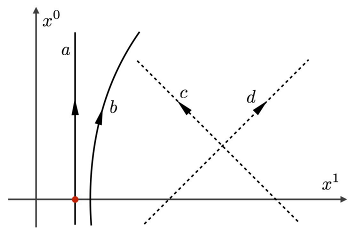


Figure 1.8: Worldlines of point-like particles in reference frame S: (a) particle at rest, (b) accelerated particles, (c,d) fotons.

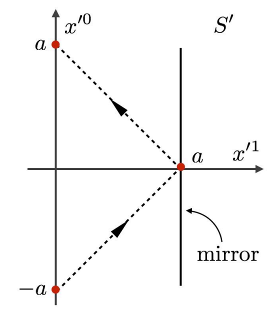


Figure 1.9: Construction of axis x'^1. The event $(0, a)$ belongs to the axis x'^1 and it is given by reflection of the light beam emitted at the origin at $x'^0 = -a$.

Typical Minkowski diagrams contain world lines and world sheets. A constant character of the speed of light is represented on *any two dimensional diagram* by straight lines that form angles $\frac{\pi}{4}$ with its axes. These axes are defined as follows:

- x^0 – a world line of the inertial observer,
- x^1 – a set of events simultaneous with the event $(0, 0)$.

Let S and S' be two different inertial observers. We shall construct axes of a diagram associated with the observer S' on a diagram S. Thus we consider an observer S' as being in uniform motion along the x^1 axis. Its velocity is denoted by V. The x'^0 axis is just a world line of the observer S' so it corresponds to a straight line that form an angle ϕ with the x^0 axis. This angle, shown in Figure 1.10(a), is associated with relative velocity

$$\tan\phi = \beta \equiv \frac{V}{c}.$$

In order to determine the x'^1 axis one has to identify at least one event *simultaneous* with $(x^0, x^1) = (0, 0) = (x'^0, x'^1)$, where the second equality is an assumption which does not lead to loss of generality. Such an event can be determined with the help of *thought experiment* (*gedankenexperiment*). Let us consider a mirror in S' in a distance $x'^1 = a$ from the origin, see Figure 1.10(b). A constant character of the light speed leads to the conclusion that the light impulse (a photon) emitted from $(x'^0, x'^1) = (-a, 0)$ in direction of the mirror reaches it at $(x'^0, x'^1) = (0, a)$ and after reflection it returns to the observer S' at $(x'^0, x'^1) = (a, 0)$. The event correspondinf with reflection of a photon from the mirror is represented by intersection of the light rays. Such rays form $\frac{\pi}{4}$ angles with x^0 and x^1 axes on the Minkowski diagram S. The intersection of light beams corresponds with an event which is *simultaneous* with $(0, 0)$ event. The coordinates of this event in S are such that $x^0 \neq 0$. It means that x^1 and x'^1 axes form a certain angle. These axes are *simultaneity lines* associated with different inertial observers. The fact that x^1 and x'^1 axes do not coincide reflects the fact that the concept of absolute simultaneity does not exist in special relativity. With help of elementary (Euclidean!) geometry applied to triangles on the diagram we conclude that the angle that form x^1 and x'^1 axes has value ϕ *i.e.* it has the same value as the angle formed by x'^0 and x'^0 axes.

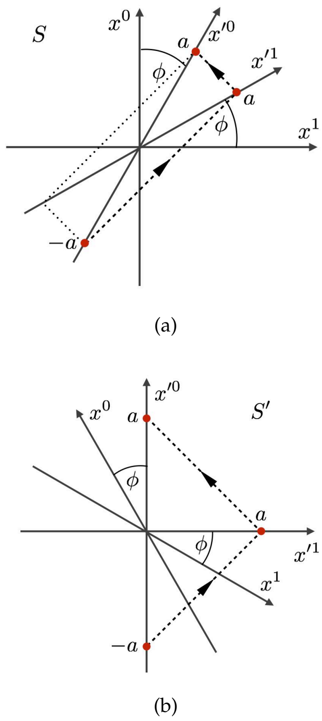


Figure 1.10: (a) Construction of the x'^1 axis at diagram of the observer S. (b) The same situation seen in the reference frame S'.

Geometry of spacetime

Spacetime geometry is *not Euclidean*. A fundamental concept which lies at the base of Euclidean geometry is the concept of distance between points. This quantity is given as the length of the vector connecting two points. Let $\Delta\boldsymbol{r}$ be a vector which connects two points in Euclidean space. Its "square" is given by expression

Finite line element in $\mathbb{E}^3$

$$\Delta l^2 \equiv \Delta\boldsymbol{r} \cdot \Delta\boldsymbol{r} = (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2. \tag{1.12}$$

The value of quantity (1.12) does not depend on the choice of the reference frame *i.e.* it is *invariant* under rotations, translations and reflections. Such invariance is a consequence of *symmetries* of Euclidean space.

In the case of Minkowski spacetime there also exists invariant expression associated with four-vector Δx that connects events A and B such that $B = A + \Delta x$ where $\Delta x = \Delta x^\mu \mathbf{e}_\mu$ and $\Delta x^\mu := x^\mu_B - x^\mu_A$. This expression is called *spacetime interval* and it has the form

Spacetime interval

$$\boxed{\Delta s^2 \equiv (\Delta x^0)^2 - (\Delta x^1)^2 - (\Delta x^2)^2 - (\Delta x^3)^2.} \tag{1.13}$$

Spacetime interval (1.13) has larger number of symmetries then spatial interval (1.12). It is onvariant under rotations, translations and reflections (now extended to time reflections). There is also another symmetry associated with the constant character of the speed of light. It is described by continuous group of transformations.

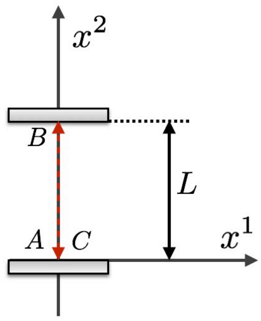


Figure 1.11: Two mirrors an a ray beam in the rest frame of mirrors.

We consider another *thought experiment* which allows us to show the invariance of expression (1.12) under transformations that connects two inertial reference frames. In this experiment we consider two parallel mirrors separated by distance $\Delta x^2 = L$ in certain reference frame S. A light impulse that travels along the axis x^2 is reflected periodically by the mirrors, see Figure 1.11. Its world line is shown in Figure 1.12. In Figure 1.13 we show its spatial trajectory which forms a polygonal chain in the reference frame S'. The reference frame S' has velocity $-V$ (where $V > 0$) in direction of axis x^1 in the reference frame S. We shall denote A, B, C as events that represent consecutive reflections of the light impulse. Differences of coordinates of events A and C have values:

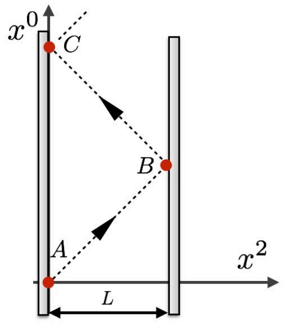


Figure 1.12: A ray beam and two mirrors in the rest frame of mirrors.

- in the inertial frame S :

$$\Delta x^0 \equiv c\Delta t = 2L, \qquad \Delta x^1 = \Delta x^2 = \Delta x^3 = 0$$

- in the inertial frame S' :

$$\Delta x'^0 \equiv c\,\Delta t' = 2\sqrt{L^2 + \left(\frac{\Delta x'^1}{2}\right)^2}, \quad \Delta x'^1 = \frac{V}{c}\Delta x'^0, \quad \Delta x'^2 = \Delta x'^3 = 0$$

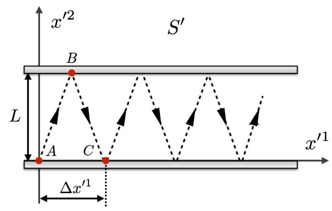


Figure 1.13: A ray beam and two mirrors in the inertial reference frame which moves with relation to the mirrors.

where $2\sqrt{L^2 + \left(\frac{\Delta x'^1}{2}\right)^2}$ is *the route traveled* by the light impulse in S' whereas $\Delta x'^1$ is a relative dislocation of the reference frames during the interval of time in which the light impulse returns to the mirror. It has been already set, according to the second postulate, that the speed of light in S' has the value c. It follows that values of spacetime intervals calculated in coordinates S and S' are perfectly the same

$$\Delta s^2 = 4L^2 = \Delta s'^2.$$

Note, that although our argument is not a general proof, it strongly suggests the expression (1.12) as a good "candidate" for invariant quantity in Minkowski spacetime.

If any two events A and B are infinitesimally close[3], $B \to A$, than the interval associated with such events is also an infinitesimal expression

$$ds^2 = \lim_{B \to A} \Delta s^2 \equiv (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \qquad (1.14)$$
$$= (dx^0)^2 - d\boldsymbol{x} \cdot d\boldsymbol{x}$$

[3] In this section we denote by dx^μ infinitesimal differences of coordinates and not linear basis forms in cotangent space T^*pM. For instance, dx^1 is an infinitesimal difference of coordinate x^1 and not a form dx^1 such that $\langle dx^1, \boldsymbol{\partial}_\alpha \rangle = \delta^1_\alpha$.

and it is called *line element* of Minkowski spacetime.

The invariant character of ds^2 is shown in few steps.

Invariance of the interval

1. The universal character of light speed means that if the line element vanishes in S, *i.e.* $ds^2 = 0$ then it must also vanish in S' so $ds'^2 = 0$. This statement must be also true for finite expression Δs^2 which is the interval between two events: emission of the spherical electromagnetic wave and any event at the front of this wave.
2. Two infinitesimal quantities *of the same order* must be *proportional*.
3. The proportionality coefficient must be some function of absolute value of relative velocity of reference frames S i S'. It leads to expressions
$$ds'^2 = a(|\boldsymbol{V}|)ds^2, \qquad ds^2 = a(|-\boldsymbol{V}|)ds'^2. \qquad (1.15)$$
4. Since $|-\boldsymbol{V}| = |\boldsymbol{V}|$, then combining both formulas (1.15) one gets $a(|\boldsymbol{V}|)^2 = 1$. Clearly, the coefficient does not depend on $|\boldsymbol{V}|$ *i.e.* $a = \pm 1$. Moreover, both expressions ds^2 and ds'^2 must *coincide* in the limit $V \to \mathbf{0}$. It allows to eliminate the case with the minus sign.

Causal structure of Minkowski spacetime

Let us consider the four-vector $\Delta x = \Delta x^\mu \mathbf{e}_\mu$ that connects events A and B: $B = A + \Delta x$. The spacetime interval Δs^2 associated with these events has interpretation of "square" of this four-vector i.e. scalar product with itself involving adequate metric tensor

Square of the four-vector

$$\boldsymbol{g}(\Delta x, \Delta x) = \boldsymbol{g}(\Delta x^\mu \mathbf{e}_\mu, \Delta x^\nu \mathbf{e}_\nu) = \boldsymbol{g}(\mathbf{e}_\mu, \mathbf{e}_\nu)\Delta x^\mu \Delta x^\nu =$$
$$= \eta_{\mu\nu}\Delta x^\mu \Delta x^\nu = \Delta s^2.$$

There are four-vectors of three kinds:

Classification of four-vectors

- *null* for $\Delta s^2 = 0$,
- *time like* for $\Delta s^2 > 0$,
- *space like* for $\Delta s^2 < 0$.

Notice that in alternative convention $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ the four-vector with $\Delta s^2 > 0$ is *space like* and four-vector with $\Delta s^2 < 0$ is *time like*.

The appearance of three types of four-vectors reflects the fact geometry of Minkowski spacetime is essentially different from the geometry of Euclidean space. Any two events can be connected by four-vectors: a null one, a temporal one or a spatial one.

The invariance of the interval tightly related to the universal character of the speed of light implies that *the relation between each two events has universal character i.e.* it does not depend on the choice of inertial reference frame. In particular, for any two events separated by *space like* four-vector there exists certain inertial reference frame in which those events are *simultaneous*. Similarly, there exist an inertial reference frame in which two events separated by *time like* four-vector have *the same value of spatial coordiantes* in certain inertial reference frame.

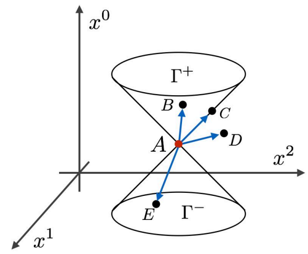


Figure 1.14: The light cone of the event A. The surfaces Γ^+ and Γ^- are, respectively, the future and the past light cones of A. Events $B, C \in \Gamma^+$ and E are causally connected with A. Event D belongs to elsewhere of A.

Causal relations between events can be geometrically illustrated with the help of *light cone structure*, see Figure 1.14. We *choose* certain event A and assume that it coincides with the apex of a light cone. A side surface Γ of the cone contains all events that are connected with A by null four-vectors. The surface Γ^+, which is characterized by condition $\Delta x^0 > 0$, is called *the future light cone* and the surface Γ^- for $\Delta x^0 < 0$ is called *the past light cone*. The interior of the light cone is a set of events that are separated from A by time like four-vectors. The events inside the future light cone can be reached from A by world lines (region of dependence of event A). Similarly, all events that belong to the interior of past light cone reach A along world lines (region of dependence of event A).

The region outside of the light cone contains all events separated from A by space like four-vectors. This is so-called *elsewhere region*. Events remaining outside the light cone are neither achievable from A (cannot depend on A) nor can make influence on A. It follows from this elementary analysis that Minkowski spacetime (spacetime of Special Theory of Relativity) has *causal structure*.

Calibration of axes

Let us consider a class of inertial reference frames with synchronized clocks. A hyperbola

$$(x^0)^2 - (x^1)^2 = a^2. \tag{1.16}$$

is a collection of events which have numerically the same value of temporal coordinate (each one in respective inertial reference frame). Similarly a hyperbola

$$(x^0)^2 - (x^1)^2 = -b^2 \tag{1.17}$$

represent a set of events having the same value of spatial coordinate in any inertial reference frame.

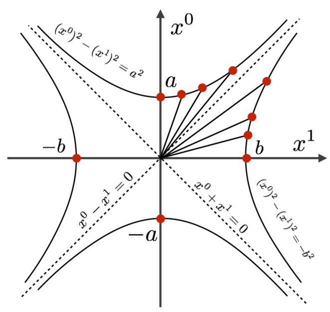
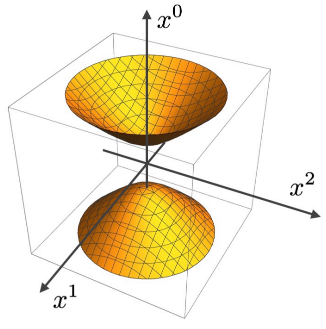


Figure 1.15: Invariant hyperbolae in 1+1 dimensions.

Left hand sides of equations (1.16) i (1.17) are spacetime intervals of two events, one localized at the origin of the reference frame $(0, 0)$ and the other one given by (x^0, x^1) in the reference frame S. Due to the invariance of the interval the values a^2, $(-b^2)$ can be expressed in S' by identical combinations of new coordinates $(x'^0)^2 - (x'^1)^2 = a^2$ and $(x'^0)^2 - (x'^1)^2 = -b^2$. Hyperbolae are curves that are invariant under change of inertial frame.

They have a direct application to axes calibration of the reference frame S' drawn on the diagram S. The hyperbola (1.16) is a set of events possessing the property that clocks of the inertial observers with common origins shows identical times. In other words, if the set of synchronized clocks coincide at $(0, 0)$ and these clocks have different velocities then events labeled by identical marks of all clocks form a hyperbola (1.16).

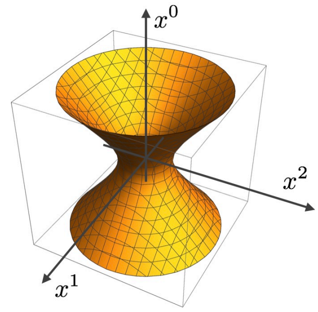
In (2+1)-dimensions such hyperbolas are substituted by hyperboloids. The spacetime hyperboloids are counterparts of spheres in Euclidean space. Such hyperboloids are shown in Figure 1.16 and Figure 1.17.



(a)

Figure 1.16: Hyperboloid

$$(x^0)^2 - (x^1)^2 - (x^2)^2 = a^2$$



(a)

Figure 1.17: Hyperboloid

$$(x^0)^2 - (x^1)^2 - (x^2)^2 = -b^2$$

Lorentz-FitzGerald contraction

Lorentz-FitzGerald contraction does not have dynamic character. It is a purely *kinematic* effect which is a direct consequence of failure of absolute simultaneity. To discuss this effect we shall consider a *thought experiment* in w (1+1) dimensions.

We consider a stiff rod which has length L in his own rest frame S. Let S' be another inertial frame that drifts with velocity V in S along the x^1 axis. A history of the rod form a world sheet in Minkowski spacetime. Both inertial observers S and S' slice the spacetime with their own simultaneity surfaces i.e. with x^1 and x'^1 axes. It means that their perform *cross-sections* of the world sheet of the rod. This situation is sketched in Figure 1.18 and Figure 1.19.

A difference of spatial coordinates Δx^1, $(\Delta x'^1)$ of the extreme points of the cross-section has interpretation of *length of the rod* L, (L') in reference frame S, (S'). One can assume without loss of generality that $(x^0, x^1) = (0, 0) = (x'^0, x'^1)$. x^0 and x'^0 axes form an angle ϕ on a diagram S. Similarly, x^1 and x'^1 axes form angle with the same value. This angle is equal to dimensionless relative velocity $\tan\phi = \beta$.

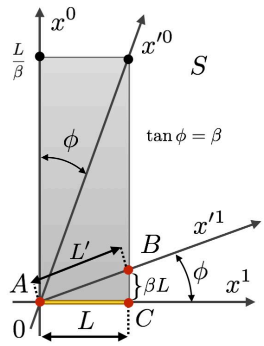


Figure 1.18: The rod at rest in the reference frame S and observer S' in motion.

Let $(x^0, x^1) = (0, 0)$ be the coordinates of the end of the rod in S (event A) and $(x^0, x^1) = (0, L)$ be coordinates of its second end (event C). Events A and B are (by assumption) simultaneous in S' and they have coordinates $(x'^0, x'^1) = (0, 0)$ and $(x'^0, x'^1) = (0, L')$, where length L' of the rod can be obtained from the interval Δs^2_{AB}. The ratio of coordinates x^0_B/x^1_B in the reference frame S is given by the angle

$$\tan\phi = \beta = \frac{\beta L}{L} = \frac{\Delta x^0_{AB}}{\Delta x^1_{AB}} = \frac{x^0_B}{x^1_B}.$$

that form x^1 and x'^1 axes. Invariance of the interval leads to equality

$$\begin{aligned} \Delta s'^2_{AB} &= \Delta s^2_{AB} \\ 0^2 - L'^2 &= (\beta L)^2 - L^2 \end{aligned}$$

which gives

$$L' = \sqrt{1-\beta^2}L \equiv \frac{1}{\gamma}L,$$

where $0 \le \beta < 1$. It means that the rod in motion in S' is *shorter* than rod at rest in S.[4]

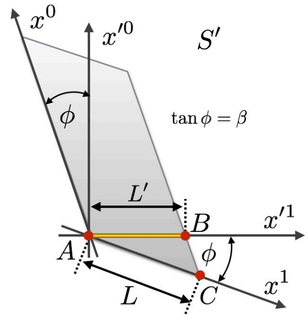


Figure 1.19: The rod in uniform motion in the reference frame S' in which the rod remains in motion with constant velocity.

[4] It is worth to notice that, the observer S can *qualitatively* justify the result obtained by S'. From its point of view events A and B corresponding with endpoints of the rod are not simultaneous. Taking into account that the observer S' drifts in S, (moves during the measure process) the observer S will interpret the divergence of the result obtained by S' with its own as the result of dislocation of S'. From the point of view of S, the dislocation of S' during the measure process has a value $\Delta L = \beta^2 L$,and so the result would be $(1-\beta^2)L$. This is *quantitatively* inconsistent with the correct result $L' = \sqrt{1-\beta^2}L$.

Time dilation

We consider two inertial frames S and S' that move with relative velocity $\boldsymbol{V}$. One can always choose the x^1 and x'^1 axes in direction of the velocity vector. We assume that S' drifts with velocity $V > 0$ in the positive direction of the x^1 axis.

Without loss of generality we can choose $(x^0, x^1) = (0,0) = (x'^0, x'^1)$. A value of time coordinate associated with any inertial observer in its own rest frame is time shown by its own clock. Our assumption means that clocks of both observers (each at the origin of its own reference frame) are *synchronized i.e.* at intersection point of x^0 and x'^0 axes they show exactly the same time.

Let us consider a line of simultaneity which corresponds with certain instant of time $x^0 = a$ in the reference frame S. This line, shown in Figure 1.20, has intersection with a world line of observer S' (the axis x'^0). The clock of observer S' marks $x'^0 = a'$ at the point of intersection (event A). The event A has following components ininertial frames of reference S and S':

- in S': $(x'^0_A, x'^1_A) = (a', 0)$ – the event loclalized at the x'^0 axis,
- in S: $(x^0_A, x^1_A) = (a, \beta a)$.

The value of spatial coordinate of event A is $x^1_A = \beta a$. It can be obtained from the ratio

$$\tan\phi = \beta = \frac{\beta a}{a} = \frac{x^1_A}{x^0_A}$$

where ϕ is the angle formed by x^0 and x'^0 axes. The invariance of the interval Δs^2_{0A} leads to the equation $a'^2 - 0^2 = a^2 - \beta^2 a^2$, which implies

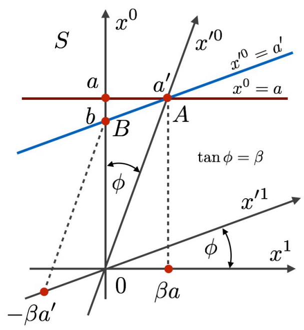


Figure 1.20: Time dilation according to the observer S.

that

$$a' = \sqrt{1-\beta^2}a = \frac{1}{\gamma}a. \tag{1.18}$$

Thus *the observer in S sees the clock S′ running slow.*

Does it mean that S' sees the clock S running fast? In order to answer this question we consider the line of simultaneity in S' that contains A. Such a line consists of events with time coordinate $x'^0 = a'$ in the reference frame S'. This line (parallel to the x'^1 axis) crosses the world line of the observer S *i.e.* the x^0 axis at $x^0 = b$ (event B). The coordinates of B read:

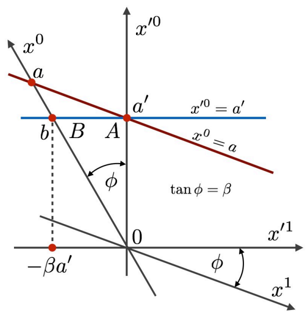


Figure 1.21: Time dilation according to the observer S'.

- in S: $(x^0_B, x^1_B) = (b, 0)$ – the event at the x^0 axis,
- w S: $(x'^0_B, x'^1_B) = (a', -\beta a')$.

The minus sign reflects the fact that S drifts with relation to S' in negative direction of x'^1 axis. Thus $x'^1_B < 0$ gives

$$\tan\phi = \beta = -\frac{-\beta a'}{a'} = -\frac{x'^1_B}{x'^0_B}.$$

Invariance of the interval Δs^2_{0B} leads to the equation $b^2 - 0^2 = a'^2 - (-\beta a')^2$ which has solution

$$b = \sqrt{1-\beta^2}a' = \frac{1}{\gamma}a'. \tag{1.19}$$

The observer in S′ claims that the clock in S is running slow. This result is *symmetric* with the previous one in the following sense

Clocks in drifting reference frame S' are running slow according to external observer S

$$\frac{b}{a'} = \sqrt{1-\beta^2} = \frac{a'}{a}$$

which means that *none of the inertial observers is distinguished*.

Proper time

A worldline of any *inertial* observer is plotted as straight line on the Minkowski diagram. The length a segment with endpoints p and q divided by the speed of light has interpretation of a time interval that separates the events p and q. In the limit $q \to p$ this segment became infinitesimal and its length is given by a differential expression (the line element) which has the form $\frac{1}{c}\sqrt{ds^2}$. In the reference frame S' this interval is equal to

$$dt' = \frac{1}{c}\sqrt{ds^2}$$

where $dx'^i \equiv 0$ because S' is a *rest frame* of the inertial observer.

The instantaneous rest frame

If the observer moves with *non-constant velocity* then at each point at the world line its own reference frame coincides with one infinitely many inertial reference frames. Such a reference frame is called *the instantaneous rest frame* (IR). The time which shows a clock of the observer that moves along the (curved) worldline is called its *proper time* and it is denoted by τ. The differential of the proper time is directly related with the line element in Minkowski spacetime

$$d\tau := \frac{1}{c}\sqrt{ds^2}.$$

The infinitesimal length of the world line of such an observer is equal to $c\,d\tau$. Note, that *there is no proper time for photons* because of $ds^2 = 0$ (there is no rest frame for photons). Let $x^i = x^i(t)$ be parametric equations of certain worldline

$$ds^2 = (dx^0)^2 - (d\boldsymbol{x})^2 = (1 - \frac{1}{c^2}\boldsymbol{V}^2)(dx^0)^2.$$

Thus $d\tau = \sqrt{1-\boldsymbol{\beta}(t)^2}dt$.The parameter τ can be obtained as follows

Proper time

$$\tau(t) = \int_0^t dt'\sqrt{1-\boldsymbol{\beta}(t')^2} + const. \tag{1.20}$$

This function can be inverted (it is strictly increasing) and so one gets $t(\tau)$. Usually we shall assume from the very beginning that the world line of an observer (e.g. point-like particle) is given by a set of four functions of τ *i.e.* $x^\mu = x^\mu(\tau)$. A world line which connects events A and B has

$$\tau_B - \tau_A = \int_{t_A}^{t_B} dt\sqrt{1-\boldsymbol{\beta}(t)^2} \le t_B - t_A.$$

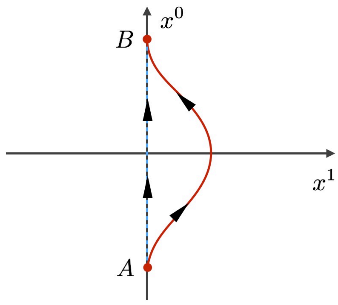


Figure 1.22: The twin paradox. World lines of two observers with common events A and B.

This result shows that a clock in IR runs *slower* than in any other inertial reference frame.

Let us consider two observers: inertial and non-inertial one. The event A in Figure 1.22 is a point at which both observers occupy the same position and their clocks are *synchronized*. One can alternatively say that they have equal age (that is why they are called twins). They follow different world lines such that they meet again at B. At the instant of time when they encounter the clock of the observer which was moving along the curved world line *is late* comparing with the other clock. This phenomenon is called a *twin paradox*. The *paradox* term associated with this thought experiment comes from the fact that one could (wrongly) expect that the situation is symmetric and the twin observers would have the same age when they meet whereas their actual ages are different. The solution of the paradox comes from the observation that *there is no symmetry*. One of the observers accelerates whereas the other one moves with constant velocity.

Twin paradox: two different curves with ends on the same pair of events has in generality different lengths. Length of the curve is proportional to proper time of observers.

Four-velocity and four-momentum

Let $x^\mu(\tau)$ be components of four-vector which describes position of the particle in Minkowski spacetime in certain inertial reference frame S. Equivalently, we say that $x^\mu = x^\mu(\tau)$ is parametric equation of their world line.

Four-velocity

Definition. The *four-velocity* is defined as a derivative with respect to proper time. It has components

$$u^\mu := \frac{dx^\mu}{d\tau}. \tag{1.21}$$

This four-vector is *tangent* to the world line of the particle. The components (1.21) are of the form

$$u^0 = c\frac{dt}{d\tau} = c\gamma(t) \qquad u^i = \frac{dx^i}{dt}\frac{dt}{d\tau} = \gamma(t)v^i(t)$$

which gives

$$u^\mu \to (\gamma c, \gamma v^i).$$

The four velocity has components $u'^\mu \to (c, 0)$ in the reference frame IR of an accelerated (non-inertial) observer. A relativistic square of the four-velocity is a Lorentz scalar which takes the value

Four velocity is a time-like four-vector

$$u^\mu u_\mu = c^2.$$

Certainly this is a *time-like* four-vector.

Let us consider a massive particle. Such particles move with subliminal velocity and so they have four-velocity. We shall define their four-momentum.

Definition. The *four-momentum* of a massive particle is a product of the mass and its four-velocity

Four-momentum of massive particles

$$p^\mu := m\,u^\mu \to (\gamma\,mc, \gamma\,mv^i)$$

where

$$p^\mu p_\mu = m^2c^2. \tag{1.22}$$

The components p^1, p^2 and p^3 of the four-momentum constitute a relativistic generalisation of the standard (three-)momentum components. The meaning of the component p^0 can be established from small velocity limit $v \ll c$. The Taylor series expansion gives

Interpretation of component p^0

$$p^0 = mc\left[1 + \frac{1}{2}\frac{\boldsymbol{v}^2}{c^2} + \dots\right] = \frac{1}{c}\left[mc^2 + \frac{1}{2}m\boldsymbol{v}^2 + \dots\right] \equiv \frac{E}{c}.$$

We recognise the term $\frac{1}{2}m\boldsymbol{v}^2$ which is non-relativistic kinetic energy of massive particle. It suggests that p_0 is the energy of the particle divided by the speed of light. The constant term mc^2 has interpretation of energy which the particle has at their rest frame. Such term is absent (irrelevant) in classical mechanics because the Lagrangian of a particle is determined up to constant term. The expression (1.22) gives

Relativistic relation between energy and linear momentum

$$E^2 = m^2c^4 + c^2\boldsymbol{p}^2.$$

It shows that square of relativistic energy is proportional to square of three-momentum. Any relativistic quantum theory that describe elementary particles must be consistent with this relation.

The four momentum can be also introduced for photons. It is light-like four-vector with components

Four momentum of photons can be introduced in spite of lack of proper time

$$p^\mu \to \left(\frac{E}{c}, \frac{E}{c}\hat{\boldsymbol{n}}\right) \qquad \text{where} \qquad \hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{n}} = 1. \tag{1.23}$$

Note that in quantum mechanics energy and linear momentum of photons are given by

$$E = \hbar\omega \qquad p^i = \hbar k^i$$

what leads to dispersion relation

$$\frac{E^2}{c^2} = \boldsymbol{p}^2 \qquad \Rightarrow \qquad \frac{\omega^2}{c^2} = \boldsymbol{k}^2.$$

This relation, which is present in quantum mechanics, reflects relativistic character of relation energy– linear momentum for photons.

Four-acceleration

Definition. The *four-acceleration* is a second derivative of $x^\mu(\tau)$

$$a^\mu := \frac{d^2x^\mu}{d\tau^2} = \frac{du^\mu}{d\tau} \tag{1.24}$$

where $u^\mu \to (\gamma c, \gamma\boldsymbol{v})$. It has components

$$\frac{du^\mu}{d\tau} \to \left(c\frac{d\gamma}{d\tau}, \frac{d\gamma}{d\tau}\boldsymbol{v} + \gamma\frac{d\boldsymbol{v}}{d\tau}\right). \tag{1.25}$$

The derivatives (1.25) have the explicit form

$$\frac{d\boldsymbol{v}}{d\tau} = \frac{dt}{d\tau}\frac{d\boldsymbol{v}}{dt} = \gamma\dot{\boldsymbol{v}},$$
$$\frac{d\gamma}{d\tau} = \frac{dt}{d\tau}\frac{d\gamma}{dt} = \gamma\left[-\frac{1}{2}\frac{-2\boldsymbol{\beta}\cdot\frac{d\boldsymbol{\beta}}{dt}}{(\sqrt{1-\beta^2})^3}\right] = \frac{1}{c}\gamma^4\boldsymbol{\beta}\cdot\frac{d\boldsymbol{\beta}}{dt} \equiv \frac{1}{c}\gamma^4\boldsymbol{\beta}\cdot\dot{\boldsymbol{v}}$$

what gives

$$a^\mu \to \left(\gamma^4(\boldsymbol{\beta}\cdot\dot{\boldsymbol{v}}), \gamma^2\dot{\boldsymbol{v}} + \gamma^4(\boldsymbol{\beta}\cdot\dot{\boldsymbol{v}})\boldsymbol{\beta}\right) \tag{1.26}$$
$$= \left(\gamma^4(\boldsymbol{\beta}\cdot\dot{\boldsymbol{v}}), \gamma^4(1-\beta^2)\dot{\boldsymbol{v}} + \gamma^4(\boldsymbol{\beta}\cdot\dot{\boldsymbol{v}})\boldsymbol{\beta}\right)$$
$$= \gamma^4\left(\boldsymbol{\beta}\cdot\dot{\boldsymbol{v}}, \dot{\boldsymbol{v}} - \boldsymbol{\beta}\times(\dot{\boldsymbol{v}}\times\boldsymbol{\beta})\right). \tag{1.27}$$

The velocity vanishes $\boldsymbol{\beta} = 0$ ($\gamma = 1$) in IR frame of the accelerated observer, however, the acceleration in this frame $\dot{\boldsymbol{v}}$ is different from zero. It gives

$$a'^{\mu} \to (0, \dot{\boldsymbol{v}}') \qquad \text{where} \qquad \dot{\boldsymbol{v}}' := \frac{d\boldsymbol{v}'}{dt'}. \tag{1.28}$$

The Lorentz scalar $a^{\mu}a_{\mu}$ reads

Space-like character of four-acceleration

$$a'^{\mu}a'_{\mu} = a^{\mu}a_{\mu} = -\left(\frac{d\boldsymbol{v}'}{dt'}\right)^2 < 0. \tag{1.29}$$

It allows us to conclude that the four-acceleration is a *space-like four-vector*. Note that the acceleration has *no absolute character* in special relativity *i.e.*

$$\frac{d\boldsymbol{v}'}{dt'} \neq \frac{d\boldsymbol{v}}{dt}.$$

1.3 Lorentz transformations

A desired transformation that relates any two inertial reference frames must have the form which guarantee the invariance of the spacetime interval. In this section we shall restrict our attention to *coordinate transformations*

$$x'^{\mu} = x'^{\mu}(x^0, x^1, x^2, x^3)$$

called also *passive transformations*. Thus two sets $\{x^{\mu}\}$ and $\{x'^{\mu}\}$ are collections of coordinates describing *the same event* in two different inertial reference frames.

In some physical problems, for example in the literature devoted to physics of elementary particles, it is commonly accepted to use *active transformations i.e.* transformations that act on four-vectors (*e.g.* four-momentum) mapping them on some other four-vectors. For instance a four-momentum of an elementary particle at rest is mapped by certain *boost transformation* on a four-momentum of an elementary particle in motion.

For instance, *translation*

Translation

$$x'^{\mu} = x^{\mu} + a^{\mu}, \tag{1.30}$$

where a^{μ} are constant Cartesian components of a four-vector, is a coordinate transformation that preserves the line element.

All other transformations which preserve the line element and that are not of the form (1.30) are called *Lorentz transformations*. The Lorentz transformations map a zero four-vector on a zero four-vector.

General condition

A general Lorentz transformation is formally a transformation between coordinates so it can be written in terms of elements of the Jacobi matrix[5]

$$L^{\mu}{}_{\nu} \equiv \frac{\partial x'^{\mu}}{\partial x^{\nu}}. \tag{1.31}$$

[5] We stress that according to adopted here convention for elements of the Jacobi matrix $J^{\mu}{}_{\nu} := \frac{\partial x^{\mu}}{\partial x'^{\nu}}$ the Lorentz transformation introduced in (1.31) correspond with elements of *the inverse Jacobi matrix*

$$(J^{-1})^{\mu}{}_{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} =: L^{\mu}{}_{\nu}.$$

The Cartesian components dx^{μ} of the infinitesimal four-vector of position in the reference frame S and their components dx'^{μ} in the reference frame S' satisfy[6]

$$dx'^{\mu} = L^{\mu}{}_{\nu} dx^{\nu}. \tag{1.32}$$

[6] Although in this section we consider dx^{μ} as components of contravariant vector $dx^{\mu}\mathbf{e}_{\mu} \in T_pM$ the transformation rule (1.32) of these components has formally the same form as transformaton of *differential forms* $dx^{\mu} \in T_p^*M$.

The transformation (1.31) must be *global* so it should contain only *constant parameters*. The linear element is of the form

$$ds^2 = \boldsymbol{g}(dx, dx) = \eta_{\mu\nu} dx^{\mu} dx^{\nu} \tag{1.33}$$

where $\eta_{\mu\nu}$ stands for Cartesian components of the metric tensor in Minkowski spacetime. Components of the inverse tensor are obtained from general condition $\eta^{\mu\alpha}\eta_{\alpha\nu} = \delta^{\mu}_{\nu}$. Note that components of both tensors coincide in Cartesian coordinates *i.e.*

$$\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1) = \eta^{\mu\nu}.$$

The condition of the invariance of a linear element, $ds'^2 = ds^2$, can be written in the form

$$\eta_{\mu\nu}(L^{\mu}{}_{\alpha} dx^{\alpha})(L^{\nu}{}_{\beta} dx^{\beta}) = \eta_{\alpha\beta} dx^{\alpha} dx^{\beta}.$$

This condition must be satisfied for any dx^{α}. It is possible if

Algebraic condition defining Lorentz transformations (ten constraints)

$$\boxed{L^{\mu}{}_{\alpha} L^{\nu}{}_{\beta} \eta_{\mu\nu} = \eta_{\alpha\beta}} \tag{1.34}$$

i.e. when transformation (1.31) is such that *it preserves the components of the metric tensor.* It means that universal value of the speed of light in any inertial reference frame is geometrically represented by condition of invariance of the interval and algebraically by condition (1.34). Thus we have

$$\underbrace{c' = c}_{\text{postulate}} \quad \Leftrightarrow \quad \underbrace{ds'^2 = ds^2}_{\text{geometric condition}} \quad \Leftrightarrow \quad \underbrace{L^{\mu}{}_{\alpha} L^{\nu}{}_{\beta} \eta_{\mu\nu} = \eta_{\alpha\beta}}_{\text{algebraic condition}}.$$

The expression (1.34) can be written in the matrix form. We define two matrices with elements given by components of the Lorentz transformation and components of the metric tensor

$$\hat{L} := [L^{\mu}{}_{\nu}], \qquad \hat{\eta} := [\eta_{\mu\nu}].$$

In order to call a given element of the matrix we shall use notation

$$(\hat{L})^{\mu}{}_{\nu}, \qquad (\hat{\eta})_{\mu\nu}, \qquad \ldots$$

The left index μ *numbers lines* of matrices and right index ν *numbers theirs columns*. The transpose matrix $\hat{L}^T$ is such that its elements are given by elements of the Lorentz transformation $(\hat{L}^T)^{\mu}{}_{\nu} := L^{\nu}{}_{\mu}$. The condition (1.34) takes the form

Condition satisfied by Lorentz matrices

$$\boxed{\hat{L}^T\hat{\eta}\hat{L} = \hat{\eta},} \tag{1.35}$$

where multiplication of matrices has explicit form[7]

$$(\hat{L}^T\hat{\eta}\hat{L})_{\alpha\beta} \equiv (\hat{L}^T)^{\alpha}{}_{\mu}(\hat{\eta})_{\mu\nu}(\hat{L})^{\nu}{}_{\beta} = L^{\mu}{}_{\alpha}\eta_{\mu\nu}L^{\nu}{}_{\beta}.$$

[7] Note that when indices represent lines and rows of matrices then they might be on the same level. What really matters is their positions as number of row and number of column.

Multiplying the condition (1.35) by $\hat{L}\hat{\eta}^{-1}$ from the left we get equation

$$(\hat{L}\hat{\eta}^{-1}\hat{L}^T)\hat{\eta}\hat{L} = \hat{L}$$

which implies that

Alternative form of condition that defines Lorentz matrices

$$\boxed{\hat{L}\hat{\eta}^{-1}\hat{L}^T = \hat{\eta}^{-1}.} \tag{1.36}$$

The condition (1.36) says that Lorentz transformations preserve the inverse metric tensor. Left hand side of this equation has the from

$$(\hat{L}\hat{\eta}^{-1}\hat{L}^T)^{\mu\nu} \equiv (\hat{L})^{\mu}{}_{\alpha}(\hat{\eta}^{-1})^{\alpha\beta}(\hat{L}^T)^{\beta}{}_{\nu} = L^{\mu}{}_{\alpha}\eta^{\alpha\beta}L^{\nu}{}_{\beta}.$$

Thus using index notation one gets

$$\boxed{L^{\mu}{}_{\alpha}L^{\nu}{}_{\beta}\eta^{\alpha\beta} = \eta^{\mu\nu}.} \tag{1.37}$$

Conditions (1.35) and (1.36) are equivalent.

Lorentz group

An important fact about Lorentz transformations is that they have mathematical structure of *group*. For further convenience we shall recall definition of abstract group.

Abstract group

Definition. *An abstract group* G is a set of elements furnished with composition law (or product) defined for every pair of elements of G and that satisfies:

(a). if g_1 and g_2 are elements of G, then the product g_1g_2 is also an element of G (*closure property*);

(b). the composition law is *associative*, that is $(g_1g_2)g_3 = g_1(g_2g_3)$ for every g_1, g_2 and $g_3 \in G$;

(c). there exists an unique element e in G called *identity element* such that $eg = ge = g$ for every $g \in G$;

(d). for every element g of G, there exists an unique *inverse element*, denoted g^{-1}, such that $g^{-1}g = gg^{-1} = e$.

In order to show that Lorentz transformations form group we shall check if they obey all necessary conditions (a-d). The symbol $\mathcal{L}$ is adopted for denoting the Lorentz group.

Composition rule

Ad. *(a).* Let us consider two successive Lorentz transformations

$$dx'^{\nu} = (L_1)^{\nu}{}_{\lambda} dx^{\lambda}, \qquad dx''^{\mu} = (L_2)^{\mu}{}_{\nu} dx'^{\nu}.$$

A resultant transformation can be cast in the form

$$dx''^{\mu} = (L_2)^{\mu}{}_{\nu} (L_1)^{\nu}{}_{\lambda} dx^{\lambda} = (L_3)^{\mu}{}_{\lambda} dx^{\lambda} \tag{1.38}$$

i.e. the result of two consecutive Lorentz transformations can be represented as a single Lorentz transformation $(L_3)^{\mu}{}_{\lambda}$. The composition law (1.38) can be represented in the matrix form

$$\hat{L}_3 = \hat{L}_2 \hat{L}_1. \tag{1.39}$$

Thus consecutive changes of inertial frames can be represented by multiplication of respective Lorentz matrices. The result of any such multiplication is another Lorentz matrix. *The elements of Lorentz group are given by Lorentz matrices and the composition law is just matrix multiplication*. The closure under multiplication of matrices means that $\hat{L}_2\hat{L}_1 \in \mathcal{L}$ for any $\hat{L}_1, \hat{L}_2 \in \mathcal{L}$.

Associativity

Ad. *(b).* The product of any three matrices is associative so certainly holds for the Lorentz matrices.

Unit element

Ad. *(c).* The unit element of the group is represented by the identity matrix

$$e = \mathbb{1}_4 \tag{1.40}$$

i.e. the Lorentz transformation which does not change coordinates associated with given inertial reference frame.

Inverse element

Ad. *(d).* The existence of the inverse Lorentz matrix follows from non-vanishing of the determinant of Lorentz matrices. Taking determinant of both sides of relation (1.35) we get $\det \hat{L}^T \det \hat{\eta} \det \hat{L} = \det \hat{\eta}$ what results in $(\det \hat{L})^2 = 1$ and so $\det \hat{L} = \pm 1$. The inverse Lorentz matrix can be obtained directly from (1.35). Multilpying this relation from the left by $\hat{\eta}^{-1}$ and from the right by $\hat{L}^{-1}$ one gets

$$\hat{L}^{-1} = \hat{\eta}^{-1} \hat{L}^T \hat{\eta} \tag{1.41}$$

which means that with any matrix $\hat{L}$ there is associated the matrix $\hat{L}^{-1}$ such that

$$\hat{L}^{-1}\hat{L} = \mathbb{1}_4 = \hat{L}\hat{L}^{-1}.$$

Note that the equality $\hat{L}^{-1}\hat{L} = \mathbb{1}_4$ is consistent with (1.35)

$$\hat{L}^{-1}\hat{L} = \hat{\eta}^{-1}(\hat{L}^T\hat{\eta}\hat{L}) = \hat{\eta}^{-1}\hat{\eta} = \mathbb{1}_4$$

whereas the equality $\hat{L}\hat{L}^{-1} = \mathbb{1}_4$ is consistent with (1.36)

$$\hat{L}\hat{L}^{-1} = (\hat{L}\hat{\eta}^{-1}\hat{L}^T)\hat{\eta} = \hat{\eta}^{-1}\hat{\eta} = \mathbb{1}_4.$$

Relation (1.41) can be written in the index notation

$$(\hat{L}^{-1})^\mu{}_\nu = (\hat{\eta}^{-1})^{\mu\alpha}(\hat{L}^T)^\alpha{}_\beta(\hat{\eta})_{\beta\nu} = \eta^{\mu\alpha}L^\beta{}_\alpha\eta_{\alpha\nu}$$

and thus

$$\boxed{(\hat{L}^{-1})^\mu{}_\nu = \eta^{\mu\alpha}L^\beta{}_\alpha\eta_{\beta\nu} =: L_\nu{}^\mu} \tag{1.42}$$

Elements of the inverse Lorentz matrix

Note that rising and lowering the indices in (1.42) is (a convenient) abuse of notation because the Lorentz matrix is not a tensor.[8]

The relation (1.36) can be also written in terms of the inverse Lorentz matrices. Multiplying it by $\hat{L}^{-1}$ from the left and by $(\hat{L}^T)^{-1}$ from the right we get expression

$$\hat{L}^{-1}\hat{\eta}^{-1}(\hat{L}^{-1})^T = \hat{\eta}^{-1}$$

which lhs is given by

$$(\hat{L}^{-1})^\mu{}_\alpha\eta^{\alpha\beta}((\hat{L}^{-1})^T)^\beta{}_\nu = (\hat{L}^{-1})^\mu{}_\alpha\eta^{\alpha\beta}(\hat{L}^{-1})^\nu{}_\beta = L_\alpha{}^\mu\eta^{\alpha\beta}L_\beta{}^\nu.$$

Thus

$$\boxed{L_\alpha{}^\mu L_\beta{}^\nu\eta^{\alpha\beta} = \eta^{\mu\nu}.} \tag{1.43}$$

Condition for Lorentz transformation in terms of elements of inverse Lorentz matrix

[8] When adopting the symbol $L_\nu{}^\mu$ to denote elements of the inverse Lorentz Matrix we must by quite careful with transposition of the Lorentz matrix. Note that $(\hat{L})_\nu{}^\mu \neq (\hat{L}^T)^\mu{}_\nu$ because

$$[L_\nu{}^\mu] := \left[\frac{\partial x^\mu}{\partial x'^\nu}\right]$$

whereas

$$[L^\mu{}_\nu]^T := \left[\frac{\partial x'^\mu}{\partial x^\nu}\right]^T = \left[\frac{\partial x'^\nu}{\partial x^\mu}\right].$$

Classification of Lorentz transformations

Lorentz transformations can be classified in dependence on the sign of the element $L^0{}_0$ and the value of determinant of the Lorentz matrix $\hat{L}$.

Since $(\det\hat{L})^2 = 1$ then determinant of the Lorentz matrix takes one of two possible values $\det\hat{L} = \pm 1$. Setting $\alpha = 0$ i $\beta = 0$ in the condition (1.34) we get

$$(L^0{}_0)^2 = 1 + \sum_{i=1}^{3}(L^i{}_0)^2$$

which gives $L^0{}_0 \geq 1$ or $L^0{}_0 \leq -1$. Transformations with $L^0{}_0 \geq 1$ are called *ortochronus* $\mathcal{L}^\uparrow$ (preserve direction of time) whereas $L^0{}_0 \leq -1$ are called *anti-ortochronous* $\mathcal{L}^\downarrow$. All *proper* transformations $\mathcal{L}_+$ are given by $\det\hat{L} = 1$ whereas *improper* ones $\mathcal{L}_-$ have $\det\hat{L} = -1$. Table below shows classification of Lorentz transformations.

Note that only proper ortochronous transformations $\mathcal{L}_+^\uparrow$ form a *subgroup* because they contain *unit element* (an identity transformation).

Table 1.1: Classification of Lorentz transformations

	$L^0{}_0 \geq 1$		$L^0{}_0 \leq -1$	
$\det \hat{L} = +1$	$\mathcal{L}_+^\uparrow$	boosts, rotations	$\mathcal{L}_+^\downarrow$	total reflections
$\det \hat{L} = -1$	$\mathcal{L}_-^\uparrow$	spatial reflections	$\mathcal{L}_-^\downarrow$	temporal reflecions

Reflections

Reflections

We give here explicit form of Lorentz matrices that describe reflections in Minkowski spacetime. Reflections are the simplest transformations that preserve components of the metric tensor. There are three kinds of reflections:

- *temporal reflections* $T : (dx^0, d\boldsymbol{x}) \to (-dx^0, d\boldsymbol{x})$,
- *spatial reflections* $P : (dx^0, d\boldsymbol{x}) \to (dx^0, -d\boldsymbol{x})$,
- *total reflections* $TP : (dx^0, d\boldsymbol{x}) \to (-dx^0, -d\boldsymbol{x})$.

Their Lorentz matrices have the form

$$\hat{L}_T = \left(\begin{array}{c|c} -1 & 0 \\ \hline 0 & \mathbb{1}_3 \end{array}\right), \qquad \hat{L}_P = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -\mathbb{1}_3 \end{array}\right), \qquad \hat{L}_{TP} = \left(\begin{array}{c|c} -1 & 0 \\ \hline 0 & -\mathbb{1}_3 \end{array}\right).$$

Rotations

Rotations as Lorentz transformations

Rotations in three dimensional Euclidean space $\mathbb{E}^3$ are given by the set of linear transformations $x'^i = R_{ij} x^j$ which are *orthogonal* $(\hat{R}^T \hat{R} = \mathbb{1}_3)$ and have determinant equal to unity $\det(\hat{R}) = 1$. When considered as transformations in Minkowski spacetime, rotations are given by the Lorentz matrix

$$\hat{L} = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \hat{R} \end{array}\right) \tag{1.44}$$

which satisfies the condition $\hat{L}^T \hat{\eta} \hat{L} = \hat{\eta}$ *i.e.*

$$\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \hat{R}^T \end{array}\right) \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -\mathbb{1} \end{array}\right) \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \hat{R} \end{array}\right) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -\mathbb{1} \end{array}\right).$$

We shall consider rotations of the coordinate reference frame *i.e.* *passive rotations*. The simplest rotations are given by transformations that leave invariant one spatial coordinate:

Particular rotations

1. $x'^1 = x^1$: rotations in the x^2x^3-plane (around the x^1-axis), given by $R_1(\phi_1)$,
2. $x'^2 = x^2$: rotations in the x^3x^1-plane (around the x^2-axis), given by $R_2(\phi_2)$,
3. $x'^3 = x^3$: rotations in the x^1x^2-plane (around the x^3-axis). given by $R_3(\phi_3)$,

where

$$\hat{R}_1(\phi_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_1 & \sin\phi_1 \\ 0 & -\sin\phi_1 & \cos\phi_1 \end{pmatrix} \tag{1.45}$$

$$\hat{R}_2(\phi_2) = \begin{pmatrix} \cos\phi_2 & 0 & -\sin\phi_2 \\ 0 & 1 & 0 \\ \sin\phi_2 & 0 & \cos\phi_2 \end{pmatrix} \tag{1.46}$$

$$\hat{R}_3(\phi_3) = \begin{pmatrix} \cos\phi_3 & \sin\phi_3 & 0 \\ -\sin\phi_3 & \cos\phi_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.47}$$

Euler angles

According to Euler's theorem any rotation can be expressed in terms of three *Euler angles.* The Euler angles gives three consecutive rotations:

1. Rotation about the x^3-axis represented by

$$\hat{R}_3(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.48}$$

The x^1 and x^2 axes are denoted by x'^1 and x'^2 after rotation, see Figure 1.23, and they point out in direction of versors $\mathbf{e}'_1$ and $\mathbf{e}'_2$:

$$\mathbf{e}'_1 = \cos\alpha\,\mathbf{e}_1 + \sin\alpha\,\mathbf{e}_2, \qquad \mathbf{e}'_2 = -\sin\alpha\,\mathbf{e}_1 + \cos\alpha\,\mathbf{e}_2, \qquad \mathbf{e}'_3 = \mathbf{e}_3.$$

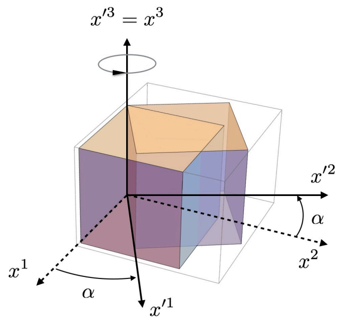


Figure 1.23: Rotation about the x^3-axis.

2. Rotation about the x'^1-axis represented by

$$\hat{R}_1(\beta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & \sin\beta \\ 0 & -\sin\beta & \cos\beta \end{pmatrix}. \tag{1.49}$$

This transformation gives $x'^2 \to x''^2$ and $x'^3 \equiv x^3 \to x''^3$, see Figure 1.24. New axes point out in direction of versors $\mathbf{e}''_k$:

$$\mathbf{e}''_1 = \mathbf{e}'_1, \qquad \mathbf{e}''_2 = \cos\beta\,\mathbf{e}'_2 + \sin\beta\mathbf{e}'_3, \qquad \mathbf{e}''_3 = -\sin\beta\,\mathbf{e}'_2 + \cos\beta\mathbf{e}'_3.$$

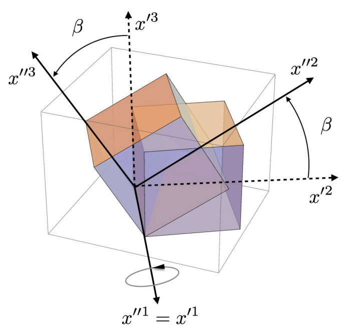


Figure 1.24: Rotation about the x'^1-axis.

Rotation about the x''^3-axis represented by

$$\hat{R}_3(\gamma) = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.50}$$

After rotation, see Figure 1.25, new axes x'''^1, x'''^2 and x'''^3 point out into directions given by versors $\mathbf{e}'''_k$:

$$\mathbf{e}'''_1 = \cos\gamma\,\mathbf{e}''_1 + \sin\gamma\,\mathbf{e}''_2, \qquad \mathbf{e}'''_2 = -\sin\gamma\,\mathbf{e}''_1 + \cos\gamma\,\mathbf{e}''_2, \qquad \mathbf{e}'''_3 = \mathbf{e}'_3.$$

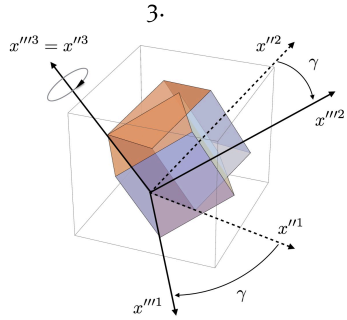


Figure 1.25: Rotation about the x''^3-axis.

Composition of all three consecutive rotations gives the following rotation matrix

$$\hat{R}(\alpha,\beta,\gamma) := \hat{R}_3(\gamma)\hat{R}_1(\beta)\hat{R}_3(\alpha) \tag{1.51}$$

where

$$\hat{R}(\alpha,\beta,\gamma) = \left(\begin{array}{c|c|c} \cos\alpha\cos\gamma - \cos\beta\sin\alpha\sin\gamma & \cos\gamma\sin\alpha + \cos\alpha\cos\beta\sin\gamma & \sin\beta\sin\gamma \\ \hline -\cos\beta\cos\gamma\sin\alpha - \cos\alpha\sin\gamma & \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & \cos\gamma\sin\beta \\ \hline \sin\alpha\sin\beta & -\sin\alpha\sin\beta & \cos\beta \end{array}\right).$$

It gives transformation from the reference frame $\{x^i\}$ to reference frame $\{x'''^i\}$. Let us denote $X^i \equiv x'''^i$ and $\hat{\boldsymbol{E}}_i \equiv \hat{\boldsymbol{e}}'''_i$. Any vector $\boldsymbol{x} \in \mathbb{E}^3$ can be written in the form

$$\boldsymbol{x} = x^i \hat{\boldsymbol{e}}_i = X^k \hat{\boldsymbol{E}}_k \tag{1.52}$$

where

$$X^k = (\hat{R})_{ki} x^i,$$
$$\hat{\boldsymbol{E}}_k = \hat{\boldsymbol{e}}_j (\hat{R}^{-1})_{jk} = \hat{\boldsymbol{e}}_j (\hat{R}^T)_{jk}.$$

General form of Lorentz transformations

In this section we shall analyze the form and physical interpretation of Lorentz transformations.[9] Let us consider transformation from the inertial frame S to another inertial frame S'. The transformation of coordinates $x'^\mu = L^\mu{}_\nu x^\nu$ have the explicit form

$$x'^0 = L^0{}_0 x^0 + L^0{}_j x^j \qquad \text{and} \qquad x'^i = L^i{}_0 x^0 + L^i{}_j x^j. \tag{1.53}$$

Since $L^0{}_0 \neq 0$ we get

$$x^0 = \frac{1}{L^0{}_0}\left(x'^0 - L^0{}_j x^j\right).$$

Plugging this expression into the second formula (1.53) we find

$$x'^i = c\frac{L^i{}_0}{L^0{}_0} t' + \left(L^i{}_j - \frac{L^i{}_0 L^0{}_j}{L^0{}_0}\right) x^j. \tag{1.54}$$

[9] See the original text: H. Arodź, *Didactic derivation of the special theory of relativity from the Klein-Gordon equation*, Eur. J. Phys. **35** (2014) 055015

If the material point remains at rest with relation to the reference frame S *i.e.* its spatial coordinates x^i are fixed, then a change of its spatial coordinates x'^i as the function of time in the reference frame S' has origin in relative motion of frames S and S'. The expression (1.54) gives the components of the velocity of the reference frame S with relation to S'

The velocity of the reference frame S with relation to S'

$$\boxed{w'^i := c\frac{L^i{}_0}{L^0{}_0}.} \tag{1.55}$$

We shall repeat these steps for the inverse Lorentz transformation $x^\mu = L_\nu{}^\mu x'^\nu$ *i.e.* transformation from S' to S. It has the form

$$x^0 = L_0{}^0 x'^0 + L_j{}^0 x'^j \qquad \text{and} \qquad x^i = L_0{}^i x'^0 + L_j{}^i x'^j.$$

Note, that expression (1.42) implies that

$$L_0{}^0 = \eta^{0\alpha} L^\beta{}_\alpha \eta_{\beta 0} = \eta^{00} L^0{}_0 \eta_{00} = L^0{}_0 \neq 0$$

what allows to divide by $L_0{}^0$

Equality $L_0{}^0 = L^0{}_0$

$$x'^0 = \frac{1}{L_0{}^0}\left(x^0 - L_j{}^0 x'^j\right).$$

The coordinates x^i read

$$x^i = c\frac{L_0{}^i}{L_0{}^0} t + \left(L_j{}^i - \frac{L_0{}^i L_j{}^0}{L_0{}^0}\right) x'^j. \tag{1.56}$$

If now the material point remains at rest in the reference frame S' then x'^j are fixed numbers. The change of its coordinates x^i in function of time t in the inertial reference frame S has its origin in relative motion of S and S'. According to (1.56) the velocity of the frame S' with relation to S has components

The velocity of the reference frame S' with relation to S

$$\boxed{v^i := c\frac{L_0{}^i}{L_0{}^0} = -c\frac{L^0{}_i}{L^0{}_0}} \tag{1.57}$$

where we have used $L_0{}^i = \eta^{i\alpha} L^\beta{}_\alpha \eta_{\beta 0} = -\delta_{ij} L^0{}_j \eta_{00} = -L^0{}_i$. Note, that the generic Lorentz matrix $(\hat{L})^\mu{}_\nu$ is **not symmetric**. It follows that in general $v^i \neq -w'^i$. The equality takes place for some special cases. This will be clear from subsequent considerations. Squares of velocities read[10]

The generic Lorentz matrix is not symmetric

[10] Here $\sum_i v^i v^i \equiv v^i v^i$ etc.

$$v^i v^i = c^2 \frac{L^0{}_i L^0{}_i}{(L^0{}_0)^2}, \qquad w'^i w'^i = c^2 \frac{L^i{}_0 L^i{}_0}{(L^0{}_0)^2}. \tag{1.58}$$

One gets from (1.37) that $L^0{}_i L^0{}_i = (L^0{}_0)^2 - 1$ and similarly from (1.34) that $L^i{}_0 L^i{}_0 = (L^0{}_0)^2 - 1$. Plugging these results to (1.58) we get

Equality of squares of velocities

$$v^i v^i = c^2\left(1 - \frac{1}{(L^0{}_0)^2}\right) = w'^i w'^i. \tag{1.59}$$

Some important conclusions can be drawn from (1.59).

1. First of all, the equality of squares of velocities $v^i v^i = w'^i w'^i$ and not their components, $v^i \neq -w'^i$, means that vectors $\boldsymbol{v}$ and $\boldsymbol{w}'$ form an angle different from π.
2. Second, the fact that $(L^0{}_0)^2 \geq 1$ implies that $|\boldsymbol{v}| < c$ and $|\boldsymbol{w}'| < c$. The existence of velocity $\boldsymbol{v}$ which has modulus smaller that the speed of light c is a consequence o formulas $x'^\mu = L^\mu{}_\nu x^\nu$ and $L^\mu{}_\alpha L^\nu{}_\beta \eta_{\mu\nu} = \eta_{\alpha\beta}$.

Subluminal character of relative velocities

In further part of the text we shall present *explicit form of elements of the Lorentz matrix*.

- The form of $L^0{}_0$ follows directly from (1.59)

Element $L^0{}_0$

$$L^0{}_0 = \pm\gamma \tag{1.60}$$

where

$$\gamma := \frac{1}{\sqrt{1-\beta^2}}, \qquad \beta \equiv |\boldsymbol{\beta}|, \qquad \boldsymbol{\beta} := \frac{\boldsymbol{v}}{c}.$$

- The element $L^0{}_j$ can be obtained from (1.57) and it reads

Element $L^0{}_j$

$$L^0{}_j = \mp\gamma\beta^j. \tag{1.61}$$

- The form of element $L^i{}_j$ follows from expression (1.34) with $\alpha = i$ and $\beta = j$

Element $L^i{}_j$

$$L^0{}_i L^0{}_j - L^k{}_i L^k{}_j = -\delta_{ij}.$$

Plugging the above results we can cast it in the form

$$L^k{}_i L^k{}_j = \delta_{ij} + \gamma^2\beta^i\beta^j. \tag{1.62}$$

There can be associated a square 3×3 symmetric matrix with expression (1.62). We define three by three dimensional real matrix $\hat{H}$ which has elements equal to expressions $L^k{}_l$ *i.e.*

Definition of $(\hat{H})_{ij}$

$$(\hat{H})_{kl} := L^k{}_l.$$

The expression (1.62) written in matrix form reads

$$\hat{H}^T\hat{H} = \mathbb{1}_3 + \gamma^2\boldsymbol{\beta}\otimes\boldsymbol{\beta}.$$

According to *polar decomposition*[11] of matrices any three by three matrix can be written as a product of an *orthogonal* matrix $\hat{R}$ *i.e.* $\hat{R}^T\hat{R} = \mathbb{1}_3$ and an *symmetric* matrix $\hat{\Delta}$ with *non-negative eigenvalues*. Thus plugging expressions

$$\hat{H} = \hat{R}\hat{\Delta}, \qquad \hat{H}^T = \hat{\Delta}\hat{R}^T$$

into (1.62) one gets

$$\Delta^2_{ij} = \delta_{ij} + \gamma^2\beta^i\beta^j. \tag{1.63}$$

[11] Polar decomposition can be prrformed for complex matrices $\hat{Z} = \hat{U}\hat{P}$ where $\hat{U}^\dagger\hat{U} = \mathbb{1}$ and $\hat{P}$ is positive-semidefinite hermitian matrix. The matrix $\hat{P}$ is unique. Denotting $z := \det\hat{Z}$, $r := \det\hat{P}$ and $e^{i\varphi} := \det\hat{U}$ we have $z = re^{i\varphi}$.

$\boldsymbol{\beta}$ is an eigenvector of the matrix $\hat{\Delta}^2$ associated with the eigenvalue γ^2:

$$\Delta^2_{ij}\beta^j = \beta^i + \gamma^2\beta^2\beta^i = \Big(1+\gamma^2(1-\tfrac{1}{\gamma^2})\Big)\beta^i = \gamma^2\delta_{ij}\beta^j.$$

We can derive explicit form of the matrix $\hat{\Delta}$ assuming the ansatz

$$\Delta_{ij} = a\delta_{ij} + b\beta^i\beta^j$$

where a and b are two free coefficients. Then taking the square of $\hat{\Delta}$ one gets

$$\Delta^2_{ij} = \Delta_{ik}\Delta_{kj} = a^2\delta_{ij} + (2ab + b^2\beta^2)\beta^i\beta^j. \tag{1.64}$$

Comparing (1.64) with (1.63) we get two equations

$$a^2 = 1 \qquad \text{and} \qquad \Big(1-\tfrac{1}{\gamma^2}\Big)b^2 + 2ab - \gamma^2 = 0.$$

They have solutions $a = \pm 1$ and $b = \frac{\gamma^2}{\gamma^2-1}(\pm\gamma - a)$. Plugging expression for b one gets[12]

$$\Delta_{ij} = a\delta_{ij} \pm \frac{\gamma^2}{\gamma^2-1}(\gamma \mp a)\beta^i\beta^j.$$

[12] We do not plug explicit value of $a = \pm 1$ in order to avoid confusion with another (not related with it) sign "$\pm\gamma$" in expression for b.

It leads to expression

$$\Delta_{ij}\beta^j = a\beta^i \pm \underbrace{\frac{\gamma^2}{\gamma^2-1}\beta^2}_{1}(\gamma \mp a)\beta^i = \pm\gamma\beta^i \tag{1.65}$$

independently on the value of a. The negative sign must be rejected because Δ_{ij} has, by definition, non-negative eigenvalues. We are left with two values of a. To get the correct solution we observe that in the limit $\beta^i \to 0$ the transformation must tend to the identity transformation. It means that only $a = +1$ is correct. Finally we get

$$\boxed{\Delta_{ij} = \delta_{ij} + \frac{\gamma^2}{\gamma+1}\beta^i\beta^j \equiv \delta_{ij} + \frac{\gamma-1}{\beta^2}\beta^i\beta^j.} \tag{1.66}$$

Thus we have the final form of $L^i{}_j = R_{ik}\Delta_{kj}$, where R_{ik} is an orthogonal matrix $\hat{R}^T\hat{R} = \mathbb{1}_3$ with $(\det\hat{R})^2 = 1$. The matrix $\hat{R}$ gives *rotations* for $\det\hat{R} = +1$ and *spatial reflections* for $\det\hat{R} = -1$.

Element $L^i{}_0$

- Expression for $L^i{}_0$ can be obtained from (1.37) for $\mu = 0$ and $\nu = i$. It gives $L^0{}_0 L^i{}_0 - L^0{}_j L^i{}_j = 0$ and so

$$L^i{}_0 = \frac{L^i{}_j L^0{}_j}{L^0{}_0} = \frac{(R_{ik}\Delta_{kj})(\mp\gamma\beta^j)}{\pm\gamma} = -\gamma R_{ik}\beta^k \tag{1.67}$$

where we have used (1.65).

Summarising, we get that the Lorentz matrix[13] has the form

[13] The "±" sign in (1.68) has origin in (1.60) and (1.61).

Decomposition of the Lorentz matrix on temporal reflections, rotations/spatial reflections and boosts

$$
\begin{aligned}
(\hat{L})^{\mu}{}_{\nu} &= \left(\begin{array}{c|c} \pm\gamma & \mp\gamma\beta^j \\ \hline -\gamma R_{ik}\beta^k & R_{ik}\Delta_{kj} \end{array}\right) \\
&= \left(\begin{array}{c|c} \pm 1 & 0 \\ \hline 0 & \delta_{il} \end{array}\right) \left(\begin{array}{c|c} \gamma & -\gamma\beta^j \\ \hline -\gamma R_{lk}\beta^k & R_{lk}\Delta_{kj} \end{array}\right) \\
&= \left(\begin{array}{c|c} \pm 1 & 0 \\ \hline 0 & \delta_{il} \end{array}\right) \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & R_{lk} \end{array}\right) \left(\begin{array}{c|c} \gamma & -\gamma\beta^j \\ \hline -\gamma\beta^k & \Delta_{kj} \end{array}\right).
\end{aligned}
\tag{1.68}
$$

The matrix product (1.68) shows that the general Lorentz transformation can be decomposed on temporal reflections, rotations/spatial reflections and boosts. We define two matrices

The orthogonal transformation

$$
\hat{\mathcal{O}} := \left(\begin{array}{c|c} \pm 1 & 0 \\ \hline 0 & \mathbb{1}_3 \end{array}\right) \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \hat{R} \end{array}\right) = \left(\begin{array}{c|c} \pm 1 & 0 \\ \hline 0 & \hat{R} \end{array}\right) \tag{1.69}
$$

and

The Lorentz boost matrix

$$
\hat{\Lambda}(\boldsymbol{v}) := \left(\begin{array}{c|c} \gamma & -\gamma\boldsymbol{\beta}^T \\ \hline -\gamma\boldsymbol{\beta} & \mathbb{1}_3 + \frac{\gamma-1}{\beta^2}\boldsymbol{\beta}\otimes\boldsymbol{\beta} \end{array}\right) \tag{1.70}
$$

where $\hat{\mathcal{O}}$ is a four by four orthogonal matrix, $\hat{\mathcal{O}}^T\hat{\mathcal{O}} = \mathbb{1}_4$, and $\hat{\Lambda}$ is symmetric. $\hat{\Lambda}(\boldsymbol{v})$ represents a transformation that relates two inertial reference frames that move with respect to each other with constant velocity (boost transformations). The inverse boost transformation is a boost with inverted velocity

$$
\hat{\Lambda}^{-1}(\boldsymbol{v}) = \hat{\Lambda}(-\boldsymbol{v}) \tag{1.71}
$$

where $\boldsymbol{\beta} = \frac{\boldsymbol{v}}{c}$. The general Lorentz matrix is the matrix product

Polar decomposition of the Lorentz matrix

$$
\boxed{\hat{L} = \hat{\mathcal{O}}\hat{\Lambda}(\boldsymbol{v}).} \tag{1.72}
$$

Expression (1.72) is called *polar decomposition of the Lorentz matrix*.

Relativistic composition of velocities

Let us consider composition of two Lorentz boosts parametrized by velocities $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. We take $\hat{\mathcal{O}}_1 = \hat{\mathcal{O}}_2 = \mathbb{1}_4$ so

$$
\hat{L}_1 = \hat{\Lambda}(\boldsymbol{v}_1) \qquad \text{and} \qquad \hat{L}_2 = \hat{\Lambda}(\boldsymbol{v}_2).
$$

The resultant Lorentz transformation is given by a matrix which has the general form $\hat{L} = \hat{L}_2\hat{L}_1$. It can be written in the form

Composition of two arbitrary boosts

$$\hat{\Lambda}(\boldsymbol{v}_2)\hat{\Lambda}(\boldsymbol{v}_1) = \hat{\mathcal{O}}\hat{\Lambda}(\boldsymbol{v}) \qquad \text{where} \qquad \boldsymbol{v} = \boldsymbol{v}(\boldsymbol{v}_2, \boldsymbol{v}_1) \tag{1.73}$$

i.e. the order of arguments in $\boldsymbol{v}(\cdot,\cdot)$ is exactly as in the product of matrices $\hat{\Lambda}_2(\cdot)\hat{\Lambda}_1(\cdot)$. The orthogonal transformation $\hat{\mathcal{O}}$ is not an identity transformation if $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are not parallel. The orthogonal matrix $\hat{\mathcal{O}}$ must be some function of velocities $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ *i.e.* $\hat{\mathcal{O}} = \hat{\mathcal{O}}(\boldsymbol{v}_2, \boldsymbol{v}_1)$.

$\hat{\mathcal{O}}$ depends on velocities $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$

The matrix which represents composition of boosts is of the form

$$\begin{aligned}\hat{\Lambda}(\boldsymbol{v}_2)\hat{\Lambda}(\boldsymbol{v}_1) &= \left(\begin{array}{c|c} \gamma_2 & -\gamma_2\boldsymbol{\beta}_2^T \\ \hline -\gamma_2\boldsymbol{\beta}_2 & \hat{\Delta}_2 \end{array}\right)\left(\begin{array}{c|c} \gamma_1 & -\gamma_1\boldsymbol{\beta}_1^T \\ \hline -\gamma_1\boldsymbol{\beta}_1 & \hat{\Delta}_1 \end{array}\right) \\ &= \left(\begin{array}{c|c} \gamma_1\gamma_2(1+\boldsymbol{\beta}_1\cdot\boldsymbol{\beta}_2) & -\gamma_1\gamma_2\boldsymbol{\beta}_1^T - \gamma_2\boldsymbol{\beta}_2^T\hat{\Delta}_1 \\ \hline -\gamma_1\gamma_2\boldsymbol{\beta}_2 - \gamma_1\hat{\Delta}_2\boldsymbol{\beta}_1 & \gamma_1\gamma_2\boldsymbol{\beta}_2\otimes\boldsymbol{\beta}_1 + \hat{\Delta}_2\hat{\Delta}_1 \end{array}\right).\end{aligned} \tag{1.74}$$

On the other hand, according to (1.68), this matrix must have the form

$$\hat{\mathcal{O}}(\boldsymbol{v}_2, \boldsymbol{v}_1)\hat{\Lambda}(\boldsymbol{v}) = \left(\begin{array}{c|c} \gamma & -\gamma\boldsymbol{\beta}^T \\ \hline -\gamma\hat{R}\boldsymbol{\beta} & \hat{R}\hat{\Delta} \end{array}\right) \tag{1.75}$$

where $\boldsymbol{\beta} = \frac{\boldsymbol{v}}{c}$, $\gamma = (1-\beta^2)^{-1/2}$. The minus sign, which corresponds with anti-orthochronous transformations, was excluded in (1.75) because we consider a composition of two orthochronous transformations $\hat{L}_1$ and $\hat{L}_2$.

In order to get the resulting velocity $\boldsymbol{v}(\boldsymbol{v}_2, \boldsymbol{v}_1)$ and the factor $\gamma = \gamma(v)$ it is enough to study the first line of matrix equality (1.73). Thus, comparing (1.74) and (1.75) we get

$$\boxed{\gamma = \gamma_1\gamma_2\left(1 + \frac{\boldsymbol{v}_1\cdot\boldsymbol{v}_2}{c^2}\right)} \tag{1.76}$$

and

$$\begin{aligned}\boldsymbol{\beta} &= \frac{[\gamma_1\gamma_2\boldsymbol{\beta}_1^T + \gamma_2\boldsymbol{\beta}_2^T\hat{\Delta}_1]^T}{\gamma} = \frac{\gamma_1\gamma_2}{\gamma}\left[\boldsymbol{\beta}_1 + \frac{1}{\gamma_1}\hat{\Delta}_1\boldsymbol{\beta}_2\right] \\ &= \frac{\gamma_1\gamma_2}{\gamma}\left[\frac{\boldsymbol{\beta}_2}{\gamma_1} + \left(1 + \frac{\gamma_1}{\gamma_1+1}\boldsymbol{\beta}_1\cdot\boldsymbol{\beta}_2\right)\boldsymbol{\beta}_1\right].\end{aligned} \tag{1.77}$$

Multiplying both sides of (1.77) by constant c we get *relativistic law for velocity composition*

Relativistic composition of velocities

$$\boxed{\boldsymbol{v}(\boldsymbol{v}_2, \boldsymbol{v}_1) = \frac{1}{1 + \frac{\boldsymbol{v}_1 \cdot \boldsymbol{v}_2}{c^2}} \left[\frac{\boldsymbol{v}_2}{\gamma_1} + \left(1 + \frac{\gamma_1}{\gamma_1 + 1} \frac{\boldsymbol{v}_1 \cdot \boldsymbol{v}_2}{c^2} \right) \boldsymbol{v}_1 \right].} \tag{1.78}$$

We shall adopt a symbol "$\vdash$" for denoting the operation of composition of velocities

$$\boxed{\boldsymbol{v}_2 \vdash \boldsymbol{v}_1 \equiv \boldsymbol{v}(\boldsymbol{v}_2, \boldsymbol{v}_1).} \tag{1.79}$$

The composition of velocities (1.78) has the following properties:

1. The resultant velocity, obtained from composition of two subluminal velocities $|\boldsymbol{v}_1| < c$ and $|\boldsymbol{v}_2| < c$, is also *subluminal*, $|\boldsymbol{v}| \leq c$.

Subluminal value of resulting velocity

2. It is *noncommutative*

Noncommutativity

$$\boldsymbol{v}_2 \vdash \boldsymbol{v}_1 \neq \boldsymbol{v}_1 \vdash \boldsymbol{v}_2$$

since $\boldsymbol{v}(\cdot,\cdot)$ is nonsymmetric in its arguments: $\boldsymbol{v}(\boldsymbol{v}_2, \boldsymbol{v}_1) \neq \boldsymbol{v}(\boldsymbol{v}_1, \boldsymbol{v}_2)$.

3. It is *nonassociative*

Nonassociativity

$$(\boldsymbol{v}_1 \vdash \boldsymbol{v}_2) \vdash \boldsymbol{v}_3 \neq \boldsymbol{v}_1 \vdash (\boldsymbol{v}_2 \vdash \boldsymbol{v}_3).$$

We shall discuss these properties below.

Ad. 1. The resulting velocity never exceeds the light speed. It can be seen after taking square of expression $\boldsymbol{\beta}$ which yields[14]

$$\boxed{\boldsymbol{\beta}^2 = 1 - \frac{1}{\gamma^2} = 1 - \frac{(1-\beta_1^2)(1-\beta_2^2)}{(1+\boldsymbol{\beta}_1 \cdot \boldsymbol{\beta}_2)^2}.} \tag{1.80}$$

For fixed $\beta_2 < 1$ and $\beta_1 \to 1$ the expression (1.80) behaves as

$$\boldsymbol{\beta}^2 \approx 1 - \underbrace{\frac{2(1-\beta_2^2)}{(1+\beta_2 \cos\alpha)^2}}_{\text{fixed}} (1-\beta_1) \tag{1.81}$$

and it has limit $\boldsymbol{\beta}^2 = 1$ for $\beta_1 \to 1$. A similar argument for β_1 fixed and $\beta_2 \to 1$ gives the same result. In Figure 1.26 we plot the absolute value of the resulting dimensionless velocity $\boldsymbol{\beta}$ for anti-parallel and parallel velocities $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$.

Ad. 2. Noncommutativity of the operation "$\vdash$" has its origin in the fact that

$$[\hat{\Lambda}(\boldsymbol{v}_1), \hat{\Lambda}(\boldsymbol{v}_2)] \neq 0$$

for $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ not being parallel. If $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are *parallel*[15] then the composition of velocities has the form

$$\boxed{\boldsymbol{v} = \frac{\boldsymbol{v}_1 + \boldsymbol{v}_2}{1 + \frac{\boldsymbol{v}_1 \cdot \boldsymbol{v}_2}{c^2}}} \tag{1.82}$$

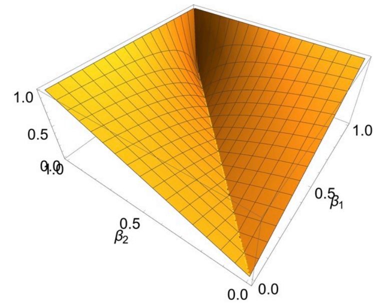
and it is *symmetric* in velocities, $\boldsymbol{v}(\boldsymbol{v}_2, \boldsymbol{v}_1) = \boldsymbol{v}(\boldsymbol{v}_1, \boldsymbol{v}_2)$.

[14] Exercise: Derive the expression (1.80) by explicit squaring of both sides of (1.77).

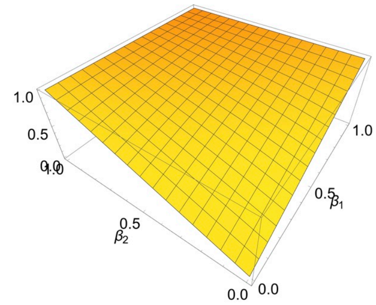
[15] Exercise: Find the form of (1.74) for parallel velocities.

Proof: We take $\boldsymbol{\beta}_2 = \lambda\boldsymbol{\beta}_1$ what gives

$$\begin{aligned}
\boldsymbol{\beta} &= \frac{1}{1+\boldsymbol{\beta}_1\cdot\boldsymbol{\beta}_2}\left[\frac{\lambda\boldsymbol{\beta}_1}{\gamma_1} + \left(1+\frac{\gamma_1}{1+\gamma_1}\lambda\beta_1^2\right)\boldsymbol{\beta}_1\right] \\
&= \frac{1}{1+\boldsymbol{\beta}_1\cdot\boldsymbol{\beta}_2}\left[\frac{\lambda\boldsymbol{\beta}_1}{\gamma_1} + \boldsymbol{\beta}_1 + \lambda\frac{\gamma_1}{1+\gamma_1}\frac{(\gamma_1-1)(\gamma_1+1)}{\gamma_1^2}\boldsymbol{\beta}_1\right] \\
&= \frac{1}{1+\boldsymbol{\beta}_1\cdot\boldsymbol{\beta}_2}\left[\frac{\lambda\boldsymbol{\beta}_1}{\gamma_1} + \boldsymbol{\beta}_1 + \lambda\boldsymbol{\beta}_1 - \frac{\lambda\boldsymbol{\beta}_1}{\gamma_1}\right] \\
&= \frac{\boldsymbol{\beta}_1+\boldsymbol{\beta}_2}{1+\boldsymbol{\beta}_1\cdot\boldsymbol{\beta}_2}. \qquad (1.83)
\end{aligned}$$



(a)



(b)

Figure 1.26: The composition of velocities. Expression $\beta = |\boldsymbol{\beta}|$ where $\boldsymbol{\beta} = \boldsymbol{\beta}(\boldsymbol{\beta}_2, \boldsymbol{\beta}_1)$ for $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ being (a) anti-parallel and (b) parallel.

Ad. 3. Let us consider a triple product matrices $\hat{\Lambda}$. Since the matrix product is *associative* then we get the identity

$$\left(\hat{\Lambda}(\boldsymbol{v}_1)\hat{\Lambda}(\boldsymbol{v}_2)\right)\hat{\Lambda}(\boldsymbol{v}_3) = \hat{\Lambda}(\boldsymbol{v}_1)\left(\hat{\Lambda}(\boldsymbol{v}_2)\hat{\Lambda}(\boldsymbol{v}_3)\right).$$

Applying (1.73) to left hand side of this expression we get

$$\begin{aligned}
LHS &= \hat{\mathcal{O}}(\boldsymbol{v}_1,\boldsymbol{v}_2)\hat{\Lambda}(\boldsymbol{v}_1\vdash\boldsymbol{v}_2)\hat{\Lambda}(\boldsymbol{v}_3) \\
&= \hat{\mathcal{O}}(\boldsymbol{v}_1,\boldsymbol{v}_2)\hat{\mathcal{O}}(\boldsymbol{v}_1\vdash\boldsymbol{v}_2,\boldsymbol{v}_3)\hat{\Lambda}\left((\boldsymbol{v}_1\vdash\boldsymbol{v}_2)\vdash\boldsymbol{v}_3\right). \qquad (1.84)
\end{aligned}$$

The right hand side of the identity has the form

$$\begin{aligned}
RHS &= \hat{\Lambda}(\boldsymbol{v}_1)\hat{\mathcal{O}}(\boldsymbol{v}_2,\boldsymbol{v}_3)\hat{\Lambda}(\boldsymbol{v}_2\vdash\boldsymbol{v}_3) \\
&= \hat{\Lambda}(\boldsymbol{v}_1)\hat{\mathcal{O}}(\boldsymbol{v}_2,\boldsymbol{v}_3)\hat{\Lambda}^{-1}(\boldsymbol{v}_1)\hat{\Lambda}(\boldsymbol{v}_1)\hat{\Lambda}(\boldsymbol{v}_2\vdash\boldsymbol{v}_3) \\
&= \hat{\Lambda}(\boldsymbol{v}_1)\hat{\mathcal{O}}(\boldsymbol{v}_2,\boldsymbol{v}_3)\hat{\Lambda}^{-1}(\boldsymbol{v}_1)\hat{\mathcal{O}}(\boldsymbol{v}_1,\boldsymbol{v}_2\vdash\boldsymbol{v}_3)\hat{\Lambda}(\boldsymbol{v}_1\vdash(\boldsymbol{v}_2\vdash\boldsymbol{v}_3)). \qquad (1.85)
\end{aligned}$$

The matrix

Orthogonal matrix

$$\hat{\mathcal{O}}(\boldsymbol{v}_1,\boldsymbol{v}_2)\hat{\mathcal{O}}(\boldsymbol{v}_1\vdash\boldsymbol{v}_2,\boldsymbol{v}_3)$$

is *orthogonal* so the expression (1.84) constitutes the polar decomposition of the Lorentz matrix obtained as triple composition of boosts $\hat{\Lambda}(\boldsymbol{v}_1)\hat{\Lambda}(\boldsymbol{v}_2)\hat{\Lambda}(\boldsymbol{v}_3)$. On the other hand (1.85) is a polar decomposition only if

Not necessarily orthogonal matrix

$$\hat{\Lambda}(\boldsymbol{v}_1)\hat{\mathcal{O}}(\boldsymbol{v}_2,\boldsymbol{v}_3)\hat{\Lambda}^{-1}(\boldsymbol{v}_1)$$

is an orthogonal matrix. It turns out that for a general directions of velocities $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ this expression *is not an orthogonal matrix*. If it was orthogonal then (1.85) would provide the polar decomposition of the Lorentz matrix. Taking into account that *the polar decomposition is unique* one would conclude that the composition of velocities would be associative (arguments of Lorentz matrices *i.e.* $(\boldsymbol{v}_1\vdash\boldsymbol{v}_2)\vdash\boldsymbol{v}_3$ in (1.84) and $\boldsymbol{v}_1\vdash(\boldsymbol{v}_2\vdash\boldsymbol{v}_3)$ in (1.85) would be the same). If the matrix $\hat{\mathcal{O}}(\boldsymbol{v}_2,\boldsymbol{v}_3)$ would be an identity matrix then $\hat{\Lambda}(\boldsymbol{v}_1)\hat{\mathcal{O}}(\boldsymbol{v}_2,\boldsymbol{v}_3)\hat{\Lambda}^{-1}(\boldsymbol{v}_1)$ would be also an identity matrix which would lead to polar decomposition and associativity of composition of velocities. We conclude that the main reason of non-associativity of such decomposition is the presence of matrix $\hat{\mathcal{O}}(v_2, v_3)$. The presence of this matrix means that, in general, *composition of two Lorentz boosts is not a boost*.

Composition of two boosts is not a boost in generality

Generators of the Lorentz group

Proper orthochronous Lorentz transformations constitute a *subgroup* (we shall call it "*the Lorentz group*") of the most general Lorentz group. It can be verified by checking all necessary conditions, namely,

Restriction to $\mathcal{L}_+^\uparrow$

1. $\hat{L}_2\hat{L}_1 \in \mathcal{L}_+^\uparrow$, for $\hat{L}_1$, $\hat{L}_2 \in \mathcal{L}_+^\uparrow$;
2. $\hat{L}_3(\hat{L}_2\hat{L}_1) = (\hat{L}_3\hat{L}_2)\hat{L}_1$;
3. there exists an unit element $e = \mathbb{1}_4$;
4. for any element $\hat{L} \in \mathcal{L}_+^\uparrow$ there exist exactly one element $\hat{L}^{-1}$ such that $\hat{L}^{-1}\hat{L} = \hat{L}\hat{L}^{-1} = \mathbb{1}_4$.

The Lorentz group is a special orthogonal group denoted by $SO(1,3)$. It is an example of Lie group (continuous group). Each element of this group can be written as the exponential

$$\hat{L} = \exp(\pm\hat{\Omega}) \equiv \sum_{n=0}^{\infty} \frac{1}{n!}(\pm\hat{\Omega})^n. \tag{1.86}$$

The $(\pm)$ sign in (1.86) allows us to distinguish active $(+)$ and passive $(-)$ transformations. In this section we shall consider only *passive transformations*. Note that any finite Lorentz transformation can be expressed as a composition of infinitely many infinitesimal transformations

Composition of infinitely many infinitesimal transformations

$$\begin{aligned}\hat{L} &= \lim_{n\to\infty}\left(1 \pm \frac{\hat{\Omega}}{n}\right)^n = \lim_{n\to\infty}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\frac{(\pm\hat{\Omega})^k}{n^k} \\ &= \lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{k!}\frac{(n-k+1)\cdots(n-1)n}{n^k}(\pm\hat{\Omega})^k \\ &= \sum_{k=0}^{\infty}\frac{1}{k!}(\pm\hat{\Omega})^k. \end{aligned}\tag{1.87}$$

In odder to study *the Lie algebra* associated with the group $SO(1,3)$ we look at condition (1.34) when transformations are infinitesimal. It can be put in the form

$$\hat{L}^T\hat{\eta} = \hat{\eta}\hat{L}^{-1}$$

where $\hat{L}^T \equiv \exp(-\hat{\Omega}^T)$ and $\hat{L}^{-1} \equiv \exp(\hat{\Omega})$. Then plugging the series expansions we have

$$\left[\mathbb{1}_4 - \hat{\Omega}^T + \frac{1}{2}(\hat{\Omega}^T)^2 + \ldots\right]\hat{\eta} = \hat{\eta}\left[\mathbb{1}_4 + \hat{\Omega} + \frac{1}{2}\hat{\Omega}^2 + \ldots\right]. \tag{1.88}$$

Equation (1.88) must be satisfied in all orders of expansion. In the *zero order* (1.88) reduces to equality $\hat{\eta} = \hat{\eta}$, so there is no condition on $\hat{\Omega}$. Such a condition appears in the *first order* of expansion

$$-(\hat{\eta}\hat{\Omega})^T = \hat{\eta}\hat{\Omega} \tag{1.89}$$

where we have used the property $\hat{\eta}^T = \hat{\eta}$. Thus (1.89) implies that

Antisymmetric matrix

$$\hat{\omega} := \hat{\eta}\hat{\Omega}$$

is an antisymmetric matrix $\hat{\omega}^T = -\omega$. This matrix has components

$$\omega_{\mu\nu} = \eta_{\mu\alpha}\Omega^{\alpha}{}_{\nu}. \tag{1.90}$$

Since the matrix $\hat{\omega}$ is antisymmetric and real-valued then it has *six independent elements*. These elements can be chosen in the following way

$\frac{4(4-1)}{2} = 6$ independent free parameters

$$\hat{\omega} = \left(\begin{array}{c|ccc} 0 & \psi_1 & \psi_2 & \psi_3 \\ \hline -\psi_1 & 0 & \phi_3 & -\phi_2 \\ -\psi_2 & -\phi_3 & 0 & \phi_1 \\ -\psi_3 & \phi_2 & -\phi_1 & 0 \end{array}\right) \tag{1.91}$$

and so the matrix $\hat{\Omega}$ reads

$$\hat{\Omega} = \hat{\eta}^{-1}\hat{\omega} = \left(\begin{array}{c|ccc} 0 & \psi_1 & \psi_2 & \psi_3 \\ \hline \psi_1 & 0 & -\phi_3 & \phi_2 \\ \psi_2 & \phi_3 & 0 & -\phi_1 \\ \psi_3 & -\phi_2 & \phi_1 & 0 \end{array}\right). \tag{1.92}$$

The matrix (1.91) can be represented in the form of linear combination containing matrices associated with *generators of boosts* $\hat{K}_k$ and *generators of rotations* $\hat{J}_k$

Expansion on generators

$$\hat{\Omega} = -i\sum_{k=1}^{3}\hat{K}_k\,\psi_k - i\sum_{k=1}^{3}\hat{J}_k\,\phi_k. \tag{1.93}$$

There are three generations of boosts that are *anti-hermitian*, $\hat{K}^\dagger = -\hat{K}$. They have the form

Generators of boosts

$$\hat{K}_1 := \left(\begin{array}{c|ccc} 0 & i & 0 & 0 \\ \hline i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \hat{K}_2 := \left(\begin{array}{c|ccc} 0 & 0 & i & 0 \\ \hline 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \hat{K}_3 := \left(\begin{array}{c|ccc} 0 & 0 & 0 & i \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{array}\right)$$

Similarly, there are three generators of rotations that are *hermitian*, $\hat{J}^\dagger = \hat{J}$. They have the form

Generators of rotations

$$\hat{J}_1 := \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{array}\right) \quad \hat{J}_2 := \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{array}\right) \quad \hat{J}_3 := \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The Lie algebra of the $SO(1,3)$ group is given by commutation relations of generators $\hat{J}_k$ and $\hat{K}_k$:

Lie algebra of $SO(1,3)$

$$[\hat{J}_i, \hat{J}_j] = i\epsilon_{ijk}\hat{J}_k, \tag{1.94}$$
$$[\hat{J}_i, \hat{K}_j] = i\epsilon_{ijk}\hat{K}_k, \tag{1.95}$$
$$[\hat{K}_i, \hat{K}_j] = -i\epsilon_{ijk}\hat{J}_k. \tag{1.96}$$

Any element $\hat{\Omega}$ of the Lie algebra of $SO(1,3)$ can be represented as contraction of the matrix elements $\omega_{\mu\nu} := (\hat{\omega})_{\mu\nu}$ where

$$\omega_{0i} = -\omega_{i0} := \psi_i \qquad \text{and} \qquad \omega_{ij} = \epsilon_{ijk}\phi_k. \tag{1.97}$$

with elements of an antisymmetric matrix $M^{\mu\nu} = -M^{\nu\mu}$ with components being generators

$$M^{0i} = -M^{i0} := \hat{K}_i \qquad\qquad M^{ij} := \epsilon_{ijk}\hat{J}_k. \tag{1.98}$$

Let us check:

$$\begin{aligned}\frac{1}{2}\omega_{\mu\nu}M^{\mu\nu} &= \frac{1}{2}(\omega_{0i}M^{0i} + \omega_{i0}M^{i0}) + \frac{1}{2}\omega_{ij}M^{ij} \\ &= \psi_i\hat{K}_i + \frac{1}{2}\underbrace{\epsilon_{ijk}\epsilon_{ijl}}_{2\delta_{kl}}\phi_k\hat{J}_l \\ &= \psi_i\hat{K}_i + \phi_k\hat{J}_k = i\hat{\Omega}.\end{aligned} \tag{1.99}$$

The *passive* Lorentz transformations $\hat{L} = \exp(-\hat{\Omega})$ can be cast in the form

$$\boxed{\hat{L} = \exp\left(\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right).} \tag{1.100}$$

Note, that elements of $M^{\mu\nu}$ are generators in the matrix realisation. In order to get components of these generators we need another pair of Lorentz indices. The generators $M^{\mu\nu}$ have components[16]

[16] Exercise: check it by explicit calculation.

$$(M^{\mu\nu})^{\alpha}{}_{\beta} = \eta^{\alpha\rho}(M^{\mu\nu})_{\rho\beta} = i\eta^{\alpha\rho}(\delta^{\mu}_{\rho}\delta^{\nu}_{\beta} - \delta^{\mu}_{\beta}\delta^{\nu}_{\rho})$$

then

Components of Lorentz group generators

$$\boxed{(M^{\mu\nu})^{\alpha}{}_{\beta} = i(\eta^{\alpha\mu}\delta^{\nu}_{\beta} - \eta^{\alpha\nu}\delta^{\mu}_{\beta}).} \tag{1.101}$$

In particular, the generators $\hat{K}_i$ and $\hat{J}_i$ have the form

$$(\hat{K}_i)^{\alpha}{}_{\beta} = (M^{0i})^{\alpha}{}_{\beta} = i(\eta^{\alpha 0}\delta^{i}_{\beta} - \eta^{\alpha i}\delta^{0}_{\beta}),$$
$$(\hat{J}_i)^{\alpha}{}_{\beta} = (M^{ij})^{\alpha}{}_{\beta} = i(\eta^{\alpha i}\delta^{j}_{\beta} - \eta^{\alpha j}\delta^{i}_{\beta}).$$

Plugging (1.101) into (1.100) one gets

$$\begin{aligned}\exp\left(\frac{i}{2}\omega_{\mu\nu}(M^{\mu\nu})^{\alpha}{}_{\beta}\right) &= \exp\left(-\frac{1}{2}\omega_{\mu\nu}\eta^{\alpha\rho}(\delta^{\mu}_{\rho}\delta^{\nu}_{\beta}-\delta^{\mu}_{\beta}\delta^{\nu}_{\rho})\right)\\ &= \exp\left(-\frac{1}{2}\eta^{\alpha\rho}(\omega_{\rho\beta}-\omega_{\beta\rho})\right)\\ &= \exp\left(-\eta^{\alpha\rho}\omega_{\rho\beta}\right) = \exp\left(-\Omega^{\alpha}{}_{\beta}\right).\end{aligned}$$

The commutation relations (1.94)- (1.96) can be represented[17] by a single commutator containing generators $M^{\mu\nu}$

[17] Exercise.

Lorentz algebra

$$\boxed{[M^{\mu\nu},M^{\rho\sigma}] = i\left(\eta^{\nu\rho}M^{\mu\sigma}-\eta^{\mu\rho}M^{\nu\sigma}+\eta^{\mu\sigma}M^{\nu\rho}-\eta^{\nu\sigma}M^{\mu\rho}\right).} \quad (1.102)$$

In order to proof this expression we express the generators $M^{\mu\nu}$ in terms of their elements.
Proof.

$$\begin{aligned}([M^{\mu\nu},M^{\rho\sigma}])^{\alpha}{}_{\beta} &= (M^{\mu\nu}M^{\rho\sigma})^{\alpha}{}_{\beta}-(M^{\rho\sigma}M^{\mu\nu})^{\alpha}{}_{\beta}\\ &= (M^{\mu\nu})^{\alpha}{}_{\lambda}(M^{\rho\sigma})^{\lambda}{}_{\beta}-(M^{\rho\sigma})^{\alpha}{}_{\lambda}(M^{\mu\nu})^{\lambda}{}_{\beta}\\ &= i^2(\eta^{\alpha\mu}\delta^{\nu}_{\lambda}-\eta^{\alpha\nu}\delta^{\mu}_{\lambda})(\eta^{\lambda\rho}\delta^{\sigma}_{\beta}-\eta^{\lambda\sigma}\delta^{\rho}_{\beta})\\ &\quad - i^2(\eta^{\alpha\rho}\delta^{\sigma}_{\lambda}-\eta^{\alpha\sigma}\delta^{\rho}_{\lambda})(\eta^{\lambda\mu}\delta^{\nu}_{\beta}-\eta^{\lambda\nu}\delta^{\mu}_{\beta})\\ &= i^2(\eta^{\alpha\mu}\eta^{\nu\rho}\delta^{\sigma}_{\beta}-\eta^{\alpha\mu}\eta^{\nu\sigma}\delta^{\rho}_{\beta}-\eta^{\alpha\nu}\eta^{\mu\rho}\delta^{\sigma}_{\beta}+\eta^{\alpha\nu}\eta^{\mu\sigma}\delta^{\rho}_{\beta}\\ &\quad -\eta^{\alpha\rho}\eta^{\sigma\mu}\delta^{\nu}_{\beta}+\eta^{\alpha\rho}\eta^{\sigma\nu}\delta^{\mu}_{\beta}+\eta^{\alpha\sigma}\eta^{\rho\mu}\delta^{\nu}_{\beta}-\eta^{\alpha\sigma}\eta^{\rho\nu}\delta^{\mu}_{\beta})\\ &= i\eta^{\nu\rho}\overbrace{[i(\eta^{\alpha\mu}\delta^{\sigma}_{\beta}-\eta^{\alpha\sigma}\delta^{\mu}_{\beta})]}^{(M^{\mu\sigma})^{\alpha}{}_{\beta}} - i\eta^{\mu\rho}\overbrace{[i(\eta^{\alpha\nu}\delta^{\sigma}_{\beta}-\eta^{\alpha\sigma}\delta^{\nu}_{\beta})]}^{(M^{\nu\sigma})^{\alpha}{}_{\beta}}\\ &\quad + i\eta^{\mu\sigma}\underbrace{[i(\eta^{\alpha\nu}\delta^{\rho}_{\beta}-\eta^{\alpha\rho}\delta^{\nu}_{\beta})]}_{(M^{\nu\rho})^{\alpha}{}_{\beta}} - i\eta^{\nu\rho}\underbrace{[i(\eta^{\alpha\mu}\delta^{\rho}_{\beta}-\eta^{\alpha\rho}\delta^{\mu}_{\beta})]}_{(M^{\mu\rho})^{\alpha}{}_{\beta}}.\end{aligned}$$

Boost in direction of the x^1-axis

We consider the Lorentz boost with $\psi_1 \equiv \psi$ and $\psi_2 = \psi_3 = 0$. In such a case the boost transformation $\hat{L}(\psi)$ is generated by $\hat{K}_1$ and it reads

$$\hat{L}(\psi) = \exp(i\psi\hat{K}_1) = \mathbb{1}_4 + \sum_{n=1}^{\infty}\frac{(-\psi)^n}{n!}\left(\begin{array}{cc|cc} 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ \hline 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right)^n$$

$$= \mathbb{1}_4 + \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ \hline 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right)\underbrace{\sum_{k=1}^{\infty}\frac{\psi^{2k}}{(2k)!}}_{\cosh(\psi)-1} + \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ \hline 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right)\underbrace{\sum_{k=1}^{\infty}\frac{(-\psi)^{2k-1}}{(2k-1)!}}_{-\sinh(\psi)}.$$

This expression is equal to the Lorentz matrix of the boost transformation in direction x^1

$$\hat{L}(\psi) = \left(\begin{array}{cc|cc} \cosh(\psi) & -\sinh(\psi) & 0 & 0 \\ -\sinh(\psi) & \cosh(\psi) & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right). \tag{1.103}$$

The parameter ψ is called *rapidity* and has following relation with the velocity of the inertial frame S' in S

Rapidity parameter

$$\beta = \tanh(\psi). \tag{1.104}$$

One gets in this parametrisation

$$\cosh(\psi) = \frac{\cosh(\psi)}{\sqrt{\cosh^2(\psi) - \sinh^2(\psi)}} = \frac{1}{\sqrt{1-\beta^2}} = \gamma \tag{1.105}$$

and

$$\sinh(\psi) = \tanh(\psi)\cosh(\psi) = \beta\gamma. \tag{1.106}$$

The presence of hyperbolic functions allows to interpret the Lorentz boost as *hyperbolic rotation* in Minkowski spacetime.

The composition of two parallel velocities can be represented by their rapidities. We choose axes x^1 and x'^1 in direction of velocities . The composition of two successive boosts with rapidities ψ_1 and ψ_2 gives

Composition of parallel velocities

$$dx''^\mu = L^\mu{}_\alpha(\psi_2)dx'^\alpha = L^\mu{}_\alpha(\psi_2)L^\alpha{}_\nu(\psi_1)dx^\nu = L^\mu{}_\nu(\psi)dx^\nu. \tag{1.107}$$

which can be written as a product of two Lorentz matrices

$$\hat{L}(\psi_2)\hat{L}(\psi_1) = \hat{L}(\psi). \tag{1.108}$$

Equation (1.108) has explicit form

$$\begin{pmatrix} A & -B & 0 & 0 \\ -B & A & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh(\psi) & -\sinh(\psi) & 0 & 0 \\ -\sinh(\psi) & \cosh(\psi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where

$$\begin{aligned} A &= = \cosh\psi_1\cosh\psi_2 + \sinh\psi_1\sinh\psi_2 = \cosh(\psi_1+\psi_2) \\ B &= \sinh\psi_1\cosh\psi_2 + \cosh\psi_1\sinh\psi_2 = \sinh(\psi_1+\psi_2). \end{aligned}$$

It allows to obtain the function $\psi = \psi(\psi_2, \psi_1)$. We conclude that composition of two boosts in a given direction results is an usual sum of their rapidities

The *addition* of rapidities

$$\boxed{\psi = \psi_1 + \psi_2.} \tag{1.109}$$

It allows to obtain the resulting velocity $\beta = \tanh(\psi)$ in dependence on velocities $\beta_1 \tanh(\psi_1)$ and $\beta_2 = \tanh(\psi_2)$

The *composition* of velocities

$$\begin{aligned}\beta = \tanh(\psi_1 + \psi_2) &= \frac{\sinh\psi_1 \cosh\psi_2 + \cosh\psi_1 \sinh\psi_2}{\cosh\psi_1 \cosh\psi_2 + \sinh\psi_1 \sinh\psi_2} \\ &= \frac{\tanh\psi_1 + \tanh\psi_2}{1 + \tanh\psi_1 \tanh\psi_2} = \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2}. \qquad (1.110)\end{aligned}$$

Rotation in the x^2x^3–plane

Let us consider a rotation generated by $\hat{J}_1$. The parameter of this rotation is denoted by $\phi_1 \equiv \phi$. The other parameters are set zero, $\phi_2 = \phi_2 = 0$. The exponential form can be transformed in following way

$$\hat{L}(\phi) = \exp(i\phi\hat{J}_1) = \mathbb{1}_4 + \sum_{n=1}^{\infty} \frac{(-\phi)^n}{n!} \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right)^n . \qquad (1.111)$$

Since the matrix

$$\hat{C} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad (1.112)$$

has properties $\hat{C}^{2k} = (-1)^k \, \mathbb{1}_2$ and $\hat{C}^{2k-1} = (-1)^{k-1}\hat{C}$
one gets

$$\begin{aligned}\hat{L}(\phi) = \mathbb{1}_4 &+ \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \underbrace{\sum_{k=1}^{\infty} (-1)^k \frac{\psi^{2k}}{(2k)!}}_{\cos(\phi)-1} \\ &+ \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right) \overbrace{\underbrace{\sum_{k=1}^{\infty} (-1)^{k-1} \frac{(-\phi)^{2k-1}}{(2k-1)!}}_{-\sin(\phi)}}^{\sum_{l=0}^{\infty} (-1)^l \frac{(-\phi)^{2l+1}}{(2l+1)!}} .\end{aligned}$$

The the Lorentz matrix $\hat{L}(\phi)$ reads

$$\hat{L}(\phi) = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & \cos\phi & \sin\phi \\ 0 & 0 & -\sin\phi & \cos\phi \end{array}\right) . \qquad (1.113)$$

The parameter ϕ has interpretation of *angle of rotation* in the x^2x^3–plane.

Note that the composition of two rotations in the x^2x^3-plane given by matrices $\hat{L}(\phi_1)$ and $\hat{L}(\phi_2)$ is a rotation with matrix $\hat{L}(\phi)$ such that

$$\hat{L}(\phi_2)\hat{L}(\phi_1) = \hat{L}(\phi) \qquad (1.114)$$

where $\phi = \phi_1 + \phi_2$. The addition of angles of rotations follows from

$$\cos(\phi_1)\cos(\phi_2) - \sin(\phi_1)\sin(\phi_2) = \cos(\phi_1 + \phi_2)$$
$$\sin(\phi_1)\cos(\phi_2) + \cos(\phi_1)\sin(\phi_2) = \sin(\phi_1 + \phi_2).$$

1.4 Poincaré group

Basic notions on the Poincaré group

The Lorentz group $\mathcal{L}$ is parametrised by six parameters. It does not contain translations. The extension of the Lorentz group which includes translations is called *the Pioncaré group* $\mathcal{P}$. The contravariant components a four-vector transform under the Poincaré transformation in the following way

Poincaré transformation

$$x'^\mu = L^\mu{}_\alpha x^\alpha + a^\mu \tag{1.115}$$

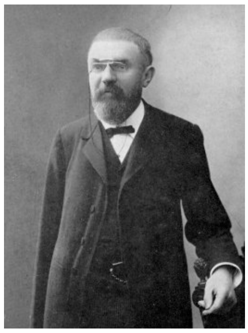


Figure 1.27: Henri Poincaré (1854-1912).

where $L^\mu{}_\alpha$ are components of the Lorentz matrix and a^μ are constant components of a four-vector a. A given four vector x has components x^μ in S and x'^μ in S'. When considering (1.115) as *passive transformation* (our case) the reference frame S' experiences translation by the vector "$-a$" in S.

Each element of the Poincaré group $\hat{P} \in \mathcal{P}$ can be represented by a pair

$$\boxed{\hat{P} := (\hat{L}, a)} \tag{1.116}$$

where $\hat{L} \in \mathcal{L}_+^\uparrow$ is an element of the proper orthochronous group and $a \in T_4$ is a constant four-vector (element of the translation group in four dimensions). The composition of two Poincaré transformations can be deduced from transformations of coordinates of reference frames S, S' and S''. Let x^μ, x'^μ and x''^μ be components of certain four–vector x in the inertial reference frames S, S' and S''. Components of x''^μ can be obtained either from components of x'^μ or x^μ. One gets

$$x''^\mu = (\hat{P}_2 x')^\mu = (\hat{P}_2(\hat{P}_1 x))^\mu = ((\hat{P}_2\hat{P}_1)x)^\mu \tag{1.117}$$

where we have used notation[18]

$$(\hat{P}_n x)^\mu \equiv (\hat{L}_n)^\mu{}_\alpha x^\alpha + a_n^\mu$$

[18] When the Poincaré transformations act in active way then $\hat{P}_n x$ is a new four-vector. In our case (passive transformations) the symbol $\hat{P}_n x$ must be used carefully, otherwise it may be confusing.

i.e. $\hat{P}_n x$ represents transformation of components of a four-vector. One gets from (1.117) that

Composition of Poincaré transformations

$$\begin{aligned} x''^\mu &= (\hat{L}_2)^\mu{}_\alpha x'^\alpha + a_2^\mu = (\hat{L}_2)^\mu{}_\alpha [(\hat{L}_1)^\alpha{}_\nu x^\nu + a_1^\alpha] + a_2^\mu \\ &= (\hat{L}_2\hat{L}_1)^\mu{}_\nu x^\nu + (\hat{L}_2)^\mu{}_\alpha a_1^\alpha + a_2^\mu. \end{aligned} \tag{1.118}$$

Expression (1.118) gives the following rule of composition of the Poincaré transformations

$$\boxed{(\hat{L}_2, a_2)(\hat{L}_1, a_1) = (\hat{L}_2\hat{L}_1, \hat{L}_2 a_1 + a_2).} \tag{1.119}$$

In order to show that the Poincaré transformations form a group we have to check if they satisfy all necessary requirements.

1. The composition rule (1.119) assures that $\hat{P}_2\hat{P}_1 \in \mathcal{P}$ for any two elements $\hat{P}_1$, $\hat{P}_2 \in \mathcal{P}$.

Composition rule

2. The composition of any three Poincaré transformations is *associative* *i.e.*

$$\hat{P}_3(\hat{P}_2\hat{P}_1) = (\hat{P}_3\hat{P}_2)\hat{P}_1. \tag{1.120}$$

Associativity

Indeed, the l.h.s. of (1.120) reads

$$\begin{aligned}\hat{P}_3(\hat{P}_2\hat{P}_1) &= (\hat{L}_3, a_3)(\hat{L}_2\hat{L}_1, \hat{L}_2 a_1 + a_2)\\ &= (\hat{L}_3\hat{L}_2\hat{L}_1, \hat{L}_3\hat{L}_2 a_1 + \hat{L}_3 a_2 + a_3)\end{aligned}$$

whereas the r.h.s. of (1.120) is of the form

$$\begin{aligned}(\hat{P}_3\hat{P}_2)\hat{P}_1 &= (\hat{L}_3\hat{L}_2, \hat{L}_3 a_2 + a_3)(\hat{L}_1, a_1)\\ &= (\hat{L}_3\hat{L}_2\hat{L}_1, \hat{L}_3\hat{L}_2 a_1 + \hat{L}_3 a_2 + a_3).\end{aligned}$$

3. *The identity element* is given by a pair

Identity element

$$\boxed{e := (\mathbb{1}_4, 0)} \tag{1.121}$$

and it satisfies

$$\begin{aligned}e\hat{P} &= (\mathbb{1}_4, 0)(\hat{L}, a) = (\mathbb{1}_4\,\hat{L}, \mathbb{1}_4 a + 0) = \hat{P},\\ \hat{P}e &= (\hat{L}, a)(\mathbb{1}_4, 0) = (\hat{L}\,\mathbb{1}_4, \hat{L}\,0 + a) = \hat{P}.\end{aligned}$$

4. *The inverse element* has the form

Inverse element

$$\boxed{\hat{P}^{-1} := (\hat{L}^{-1}, -\,\hat{L}^{-1}a).} \tag{1.122}$$

It obeys

$$\begin{aligned}\hat{P}^{-1}\hat{P} &= (\hat{L}^{-1}, -\hat{L}^{-1}a)(\hat{L}, a) = (\hat{L}^{-1}\hat{L}, \hat{L}^{-1}a - \hat{L}^{-1}a) = (\mathbb{1}_4, 0) = e,\\ \hat{P}\hat{P}^{-1} &= (\hat{L}, a)(\hat{L}^{-1}, -\hat{L}^{-1}a) = (\hat{L}\hat{L}^{-1}, -\hat{L}\hat{L}^{-1}a + a) = (\mathbb{1}_4, 0) = e.\end{aligned}$$

Each element of the Poincaré group possesses *unambiguous* decomposition into product of elements such that one of them belongs to the four-dimensional group of translations $(\mathbb{I}_4, a) \in T_4$ and the other one is an element of the proper orthochronous Lorentz group $(\hat{L}, 0) \in \mathcal{L}_+^\uparrow$. It can be written in the form

Decomposition on translations and boosts

$$\boxed{(\hat{L}, a) = (\mathbb{1}_4, a)(\hat{L}, 0).} \tag{1.123}$$

Note that *he order of elements does matter* since

The Poincaré group as a semidirect product of T_4 and $\mathcal{L}_+^\uparrow$

$$(\hat{L}, 0)(\mathbb{1}_4, a) = (\hat{L}, \hat{L}a) \neq (\hat{L}, a).$$

The form of the decomposition (1.123) indicates that the Poincaré group is a *semidirect product* of T_4 group and $\mathcal{L}_+^\uparrow$ group. [19]

The Poincaré group constitutes a very important concept in theoretical physics. It is a *symmetry group of each relativistic field theory*.

[19] For a *direct product* of groups it would be $(\hat{L}_2, a_2)(\hat{L}_1, a_1) = (\hat{L}_2\hat{L}_1, a_1 + a_2)$ what has no reflection in the composition rule (1.119).

The Poincaré algebra

The Poincaré algebra is given by *commutation relations* involving generators of translations P^μ and generators of the Lorentz transformations $M^{\mu\nu}$. The commutator $[M^{\mu\nu}, M^{\rho\sigma}]$ is given by (1.102). We still need to get commutators $[P^\mu, P^\nu]$ and $[P^\lambda, M^{\mu\nu}]$. Note that although $M^{\mu\nu}$ can be realised as a matrix (for each fixed μ and ν we have one such matrix) we do not have a matrix form form for generators P^μ. For this reason commutators of generators are just Lie brackets – they are antisymmetric compositions (but not matrix products) of these generators.

The infinitesimal Poincaré transformation for $a = 0$ (the Lorentz transformation) has the form

$$(\hat{L}, 0) = \exp\left(\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right) = \mathbb{1} + \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu} + \mathcal{O}(\omega^2). \tag{1.124}$$

Pairs $(\hat{L}, 0)$ can be identified with exponentials $\exp\left(\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right)$ because they are two equivalent realisations of the abstract group $SO(1,3)$. Similarly, the infinitesimal Poincaré transformation for $\hat{L} = \mathbb{1}$ (pure translation) is given by

$$(\mathbb{1}, a) = \exp\left(ia_\mu P^\mu\right) = \mathbb{1} + ia_\mu P^\mu + \mathcal{O}(a^2). \tag{1.125}$$

Let us consider the following equality

Composition of translations

$$(\mathbb{1}, b)(\mathbb{1}, a)(\mathbb{1}, b)^{-1} = (\mathbb{1}, a). \tag{1.126}$$

For a and b being infinitesimal this equality takes the form

$$(\mathbb{1} + ib_\mu P^\mu)(\mathbb{1} + ia_\lambda P^\lambda)(\mathbb{1} - ib_\nu P^\nu) = \mathbb{1} + ia_\lambda P^\lambda.$$

The l.h.s. of the last equality reads

$$\begin{aligned} LHS &= \mathbb{1} + \cancel{ib_\mu P^\mu} + ia_\lambda P^\lambda - \cancel{ib_\nu P^\nu} + i^2 b_\mu a_\lambda P^\mu P^\lambda - i^2 b_\nu a_\lambda P^\lambda P^\nu + \mathcal{O}(b^2) \\ &= \mathbb{1} + i^2 b_\mu a_\nu [P^\mu, P^\nu] + \mathcal{O}(b^2). \end{aligned}$$

The l.h.s. contains term proportional to $b_\mu a_\nu$ whereas the r.h.s. does not. Thus the generators of translations commute $[P^\mu, P^\nu] = 0$ which is necessary condition for satisfying the equality (1.126).

In order to compute the commutator $[P^\lambda, M^{\mu\nu}]$ we consider an element of the Poincaré group $(\hat{L}, \hat{L}a)$ which can be written in two equivalent ways

Composition of translations and boosts

$$(\hat{L}, 0)(\mathbb{1}, a) = (\mathbb{1}, \hat{L}a)(\hat{L}, 0).$$

Multiplying by $(\hat{L}, 0)^{-1}$ we get

$$(\hat{L}, 0)(\mathbb{1}, a)(\hat{L}, 0)^{-1} = (\mathbb{1}, \hat{L}a) \tag{1.127}$$

Expanding l.h.s. of (1.127) up to linear terms in $\omega_{\mu\nu}$ and a_λ we get

$$\begin{aligned} LHS &= (\mathbb{1} + \tfrac{i}{2}\omega_{\mu\nu}M^{\mu\nu})(\mathbb{1} + ia_\lambda P^\lambda)(\mathbb{1} - \tfrac{i}{2}\omega_{\alpha\beta}M^{\alpha\beta}) \\ &= \mathbb{1} + ia_\lambda P^\lambda + \cancel{\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}} - \cancel{\frac{i}{2}\omega_{\alpha\beta}M^{\alpha\beta}} + \frac{i^2}{2}\omega_{\mu\nu}a_\rho[M^{\mu\nu}, P^\rho] + \mathcal{O}(\omega^2). \end{aligned}$$

The r.h.s. of the equality (1.127) reads

$$\begin{aligned} RHS &= \mathbb{1} + i(\hat{L}a)^\lambda P_\lambda = \mathbb{1} + i(\delta^\lambda_\rho + \tfrac{i}{2}\omega_{\mu\nu}(M^{\mu\nu})^\lambda{}_\rho)a^\rho P_\lambda \\ &= \mathbb{1} + ia_\lambda P^\lambda + \frac{i^3}{2}a^\rho\omega_{\mu\nu}(\eta^{\lambda\mu}\delta^\nu_\rho - \eta^{\lambda\nu}\delta^\mu_\rho)P_\lambda \\ &= \mathbb{1} + ia_\lambda P^\lambda + \frac{i^2}{2}a_\rho\omega_{\mu\nu}[-i(\eta^{\mu\rho}P^\nu - \eta^{\nu\rho}P^\mu)]. \end{aligned}$$

Comparing terms proportional to $a_\rho\omega_{\mu\nu}$ we get commutation relation

$$[P^\rho, M^{\mu\nu}] = i(\eta^{\mu\rho}P^\nu - \eta^{\nu\rho}P^\mu).$$

Summarising, we have seen that the Poincaré algebra consists on the following commutation relations

Poincaré algebra of generators of the Poincaré group

$$[P^\mu, P^\nu] = 0, \tag{1.128}$$
$$[P^\rho, M^{\mu\nu}] = i(\eta^{\mu\rho}P^\nu - \eta^{\nu\rho}P^\mu), \tag{1.129}$$
$$[M^{\mu\nu}, M^{\rho\sigma}] = i\left(\eta^{\nu\rho}M^{\mu\sigma} - \eta^{\mu\rho}M^{\nu\sigma} + \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\nu\sigma}M^{\mu\rho}\right). \tag{1.130}$$

1.5 *Meaning of Special Theory of Relativity*

The Special Theory of Relativity is an extremely important achievement in theoretical physics. First, this theory has changed our way of thinking about space and time. Time is no longer considered as a parameter, instead, it has been promoted to the status of coordinate. In this sense, time is not absolute. In consequence, simultaneity became an observer dependent concept. Second, the theory of relativity unified the principles of conservation of energy and momentum into one law - conservation of four-momentum. Third, the formalism of the Special Theory of Relativity allowed us for full unification of electricity and magnetism. Electric and magnetic phenomena are manifestations of the dynamics of a single structure – the electromagnetic field. Finally, this theory gives the speed of light new status – universal constant.

1. New concept of space and time

2. Unification of energy and momentum in one structure

3. Unification of electricity and magnethism

Exercices

1. *Derive the Lorentz matrix for transformation describing uniform motion of two inertial frames with parallel axes. The reference frame S' has velocity $\boldsymbol{V} = V\boldsymbol{e}_1$ with respect to S.*

   The coordinates x^0 and x^1 transform so we expect that Lorentz matrix is of the form

$$L^{\mu}{}_{\nu} = \begin{pmatrix} L^0{}_0 & L^0{}_1 & 0 & 0 \\ L^1{}_0 & L^1{}_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.131}$$

   The metric tensor is preserved by the Lorentz transformation. It gives three conditions

$$(L^0{}_0)^2 - (L^1{}_0)^2 = 1, \tag{1.132}$$
$$(L^1{}_1)^2 - (L^0{}_1)^2 = 1, \tag{1.133}$$
$$L^0{}_0 L^0{}_1 - L^1{}_0 L^1{}_1 = 0. \tag{1.134}$$

   Two first conditions can be solved immediately using appropriate parametrisation

$$L^0{}_0 = \cosh\psi, \qquad L^1{}_0 = -\sinh\psi,$$
$$L^1{}_1 = \cosh\psi', \qquad L^0{}_1 = -\sinh\psi'.$$

   The last condition leads to $\tanh\psi' = \tanh\psi$, which implies that $\psi' = \psi$. The meaning of parameter ψ can be obtained from world line of the inertial observer S'. In its own reference frame the world line is described by condition $dx'^1 = 0$. This condition can be cast in the form

$$L^1{}_0 dx^0 + L^1{}_1 dx^1 = 0.$$

   The world line of the observer S' in the reference frame of the observer S is given by $dx^1 = \beta dx^0$. Plugging this expression to the last equation we get

$$\frac{L^1{}_0}{L^1{}_1} + \beta = 0 \qquad \Rightarrow \qquad \tanh\psi = \beta.$$

   Note that

$$L^0{}_0 = L^1{}_1 = \frac{\cosh\psi}{\sqrt{\cosh^2\psi - \sinh^2\psi}} = \frac{1}{\sqrt{1-\tanh^2\psi}} = \frac{1}{\sqrt{1-\beta^2}} \equiv \gamma.$$

   The remaining parameters can be obtained as follows

$$L^1{}_0 = L^0{}_1 = -\sinh\psi = -\cosh\psi\tanh\psi = -\beta\gamma.$$

Finally we get the Lorentz matrix:

$$(\hat{L})^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.135}$$

2. *Find the general form of the Lorentz boost for velocity $\boldsymbol{\beta} = \beta^i \boldsymbol{e}_i$, where $i = \{1, 2, 3\}$, generalising a boost transformation in direction of the x^1–axis.*

First of all we note that the form of particular transformation suggests that spatial components of the four-vector perpendicular to the velocity vector does not change under the transformation *i.e.*

$$dx'^0 = \gamma(dx^0 - \beta dx_{\parallel}) \tag{1.136}$$

$$d\boldsymbol{x}'_{\parallel} = \gamma(d\boldsymbol{x}_{\parallel} - \boldsymbol{\beta} dx^0) \tag{1.137}$$

$$d\boldsymbol{x}'_{\perp} = d\boldsymbol{x}_{\perp} \tag{1.138}$$

where $\beta \equiv |\boldsymbol{\beta}|$. Any vector $d\boldsymbol{x}$ can be decomposed on components parallel and perpendicular to the vector $\boldsymbol{\beta}$:

$$d\boldsymbol{x} = \underbrace{\frac{\boldsymbol{\beta}(d\boldsymbol{x} \cdot \boldsymbol{\beta})}{\beta^2}}_{d\boldsymbol{x}_{\parallel}} + \underbrace{\frac{\boldsymbol{\beta} \times (d\boldsymbol{x} \times \boldsymbol{\beta})}{\beta^2}}_{d\boldsymbol{x}_{\perp}}. \tag{1.139}$$

Since $\beta dx_{\parallel} = \boldsymbol{\beta} \cdot d\boldsymbol{x}$ then

$$dx'^0 = \gamma(dx^0 - \boldsymbol{\beta} \cdot d\boldsymbol{x}). \tag{1.140}$$

The spatial part $d\boldsymbol{x}'$ of the four-vector dx'^{μ} reads

$$\begin{aligned} d\boldsymbol{x}' &= d\boldsymbol{x}'_{\parallel} + d\boldsymbol{x}'_{\perp} \\ &= d\boldsymbol{x}'_{\parallel} + d\boldsymbol{x}_{\perp} \\ &= \gamma(d\boldsymbol{x}_{\parallel} - \boldsymbol{\beta} dx^0) + d\boldsymbol{x}_{\perp} \\ &= \gamma(d\boldsymbol{x}_{\parallel} - \boldsymbol{\beta} dx^0) + d\boldsymbol{x} - d\boldsymbol{x}_{\parallel} \\ &= d\boldsymbol{x} - \gamma \boldsymbol{\beta} dx^0 + \frac{\gamma - 1}{\beta^2}(\boldsymbol{\beta} \cdot d\boldsymbol{x})\boldsymbol{\beta} \end{aligned} \tag{1.141}$$

The expression $dx'^{\mu} = L^{\mu}{}_{\nu} dx^{\nu}$ implies that

$$dx'^0 = L^0{}_0 dx^0 + L^0{}_j dx^j, \tag{1.142}$$

$$dx'^i = L^i{}_0 dx^0 + L^i{}_j dx^j. \tag{1.143}$$

Expressions (1.140) and (1.141) can be written in the form

$$\begin{aligned} dx'^0 &= \gamma dx^0 - \gamma \beta^j dx^j, \\ dx'^i &= -\gamma \beta^i dx^0 + \left[\delta_{ij} + \frac{\gamma - 1}{\beta^2} \beta^i \beta^j\right] dx^j. \end{aligned}$$

Comparing this with (1.142) and (1.143) we find that

$$L^0{}_0 = \gamma \qquad L^0{}_i = L^i{}_0 = -\gamma\beta^i \qquad L^i{}_j = \delta_{ij} + \frac{\gamma-1}{\beta^2}\beta^i\beta^j \quad (1.144)$$

$$L^\mu{}_\nu = \left(\begin{array}{c|c} L^0{}_0 & L^0{}_j \\ \hline L^i{}_0 & L^i{}_j \end{array}\right).$$

3. *Find the rule of composition of parallel velocities*

It is enough to restrict our considerations to a pair of coordinates (x^0, x^1). The inertial reference frameS' moves along axis x^1 with constant velocity V. Let $v \equiv \frac{dx^1}{dt}$ be the velocity of certain massive particle in S and $v' \equiv \frac{dx'^1}{dt'}$ its velocity in S'. The Inverse transformation is given by $(L^{-1})^\mu{}_\nu$ and it is of the form

$$dx^0 = \gamma dx'^0 + \gamma\beta dx'^1 \qquad dx^1 = \gamma\beta dx'^0 + \gamma dx'^1.$$

It gives

$$dx^0 = \gamma\left(1 + \beta\frac{dx'^1}{dx'^0}\right)dx'^0 = \gamma\left(1 + \frac{Vv'}{c^2}\right)dx'^0$$
$$dx^1 = \gamma\left(\beta + \frac{dx'^1}{dx'^0}\right)dx'^0 = \frac{\gamma}{c}\left(V + v'\right)dx'^0$$

The velocity of the particle measured in S reads

$$v = c\frac{dx^1}{dx^0} = \frac{V + v'}{1 + \frac{Vv'}{c^2}}. \qquad (1.145)$$

This form formula is meaningful also for photons $v' = c$. The photon velocity in S reads

$$v = \frac{V + c}{1 + \frac{V}{c}} = c.$$

It reflects the fact that the speed of light has universal character.

Chapter 2
Maxwell's Equations

The form of transformation which preserve Maxwell's equations is not straightforward from their standard (non-covariant) from. The transformation of field equations must be consistent with transformation rule of electric and magnetic field. The existence of such transformation rule is expected. For instance, electrically charged particles at rest in the laboratory reference frame S are identified with sources of the electric field (electrostatic in S). On the other hand, these particles remain in motion in S' and thus they form electric current which is the source of magnetic field in S'.

2.1 *Electromagnetic potentials*

The set of Maxwell's equations read

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \tag{2.1}$$

$$\nabla \times \mathbf{B} - \frac{1}{c}\partial_t \mathbf{E} = \frac{4\pi}{c}\mathbf{J}, \tag{2.2}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{2.3}$$

$$\nabla \times \mathbf{E} + \frac{1}{c}\partial_t \mathbf{B} = 0. \tag{2.4}$$

It is known fact that for $\mathbf{E}$ and $\mathbf{B}$ given in terms of *electromagnetic potentials* the sourceless Maxwell's equations (2.3) and (2.4) have the form of *identities*. The vector potential $\mathbf{A}$ is a vector field such that magnetic field is obtained as $\mathbf{B} := \nabla \times \mathbf{A}$. The equation (2.3) is an identity,[1] $\nabla \cdot (\nabla \times \mathbf{A}) \equiv 0$, because it contains antisymmetric combinations of symmetric second-order partial derivatives. The Faraday's law (2.4) can be cast in the form

$$\nabla \times \left(\mathbf{E} + \frac{1}{c}\partial_t \mathbf{A} \right) = 0.$$

This equation reduces to identity when expression $\mathbf{E} + \frac{1}{c}\partial_t \mathbf{A}$ is proportional to the gradient of certain function $\varphi(t, \boldsymbol{x})$. In order to establish the correspondence with electrostatic potential we choose the minus sign in

[1] Independently on particular form of $\mathbf{A}$.

$\mathbf{E} + \frac{1}{c}\partial_t \mathbf{A} := -\nabla\varphi$. The scalar potential φ depends on variables t and x^i. In some special cases it can be a function of only x^i (electrostatic potential). In terms of electromagnetic potentials one gets the electric field strength and the magnetic field in the form

$$\mathbf{E} = -\frac{1}{c}\partial_t \mathbf{A} - \nabla\varphi, \tag{2.5}$$

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{2.6}$$

The electromagnetic potentials possess the gauge freedom *i.e.* they can be substituted by new potentials

$$\varphi' = \varphi - \frac{1}{c}\partial_t \chi, \qquad \mathbf{A}' = \mathbf{A} + \nabla\chi \tag{2.7}$$

leaving the fields unchanged *i.e.* $\mathbf{E}' = \mathbf{E}$ oraz $\mathbf{B}' = \mathbf{B}$. The gauge transformation (2.7) is the electromagnetic field *internal symmetry*.

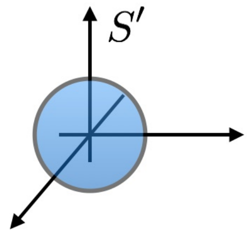


Figure 2.1: Spherical electric charge density $\rho' = qn_0$ at its rest frame.

Four-current

The density of electric charge ρ and three components of the electric current density J^i combined together

$$J^\mu \to (c\rho, J^i). \tag{2.8}$$

transform as components of contravariant four-vector. This statement follows from the fact that *electric charge is invariant under the Lorentz transformations*[2].

Let us consider static configuration of electric charges in certain inertial reference frame[3] S'. The electric charge density of this configuration is given by $\rho' = qn_0$ where n_0 is the concentration of electrically charged particles in their rest frame. For simplicity, we shall assume $n_0 = const$. According to the Lorentz-FitzGerald contraction the electric charge density is higher by the factor γ *i.e.* $\rho = qn_0\gamma$ in the reference frame S in which the configuration moves with a velocity $\mathbf{v}$.

Moreover, the moving free charges contribute to the electric current density $\mathbf{J} = qn_0\gamma\mathbf{v}$ in this reference frame. It follows that J^μ is proportional to a four-velocity $u^\mu \to (\gamma c, \gamma\mathbf{v})$. It allows to conclude that J^μ transform exactly as contravariant components of some four-vector. If the electric charge configuration has some velocity in S then its four-current has the form (2.8).

Note, that the four-current transformation law $J'^\mu = L^\mu{}_\nu J^\nu$ can be also deduced from the continuity equation $\partial_0(c\rho) + \partial_i J^i = 0$. This equation describes the conservation of electric charge and so it must have the same form in all inertial reference frames. The l.h.s. of the equation of continuity can be written in two inertial reference frames $\partial'_\mu J'^\mu = \partial_\nu J^\nu$. Since partial derivatives transform as components of *covariant* four-vector, then components of four-current density must transform as components of *contravariant* four-vector.

[2] According to experiments hydrogen atoms and deuteruim atoms are electrically neutral. Protons and neutrons in deuterium atoms interact via strong interaction what significantly increases theirs kinetic energy. If the motion of the proton would have any affect on its electric charge then it would not be possible the existence of neutral deuterium atoms. Neutrality of the deuterium means that motion of electrically charged particle has no influence on their electric charge

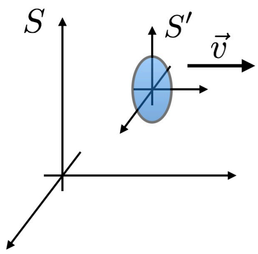


Figure 2.2: Electric charge density $\rho = qn_0\gamma$ at the laboratory frame. The charge configuration has velocity v in this reference frame.

[3] For instance, it can be represented by a charged spherical conductor

2.2 *Electromagnetic field tensor*

We shall consider components with respect to Cartesian basis $\{\mathbf{e}_i\}_{i=1,2,3}$ of electric and magnetic field and components of the vector potential

$$\mathbf{E} \to E^i, \qquad \mathbf{B} \to B^i, \qquad \mathbf{A} \to A^i.$$

Partial derivatives with respect to the coordinates x^i, $i = 1, 2, 3$, and with respect to time t are denoted by ∂_i and ∂_t. The gradient operator has Cartesian components

$$\nabla \to \partial_i.$$

For further convenience we define the symbol

$$\partial_0 := \frac{1}{c}\frac{\partial}{\partial t}$$

The set of derivatives of any function $f(x^0, \boldsymbol{x})$ *i.e.* $(\partial_0 f, \partial_1 f, \partial_2 f, \partial_3 f)$ transform as covariant components of a four-vector *i.e.*

$$\partial'_\mu f = L_\mu{}^\nu \partial_\nu f.$$

Taking into account that function f is arbitrary we can write the last formula in the form $\partial'_\mu = L_\mu{}^\nu \partial_\nu$.

We shall *assume* that the sequence of four elements φ and A^1, A^2, A^3 transform as contravariant components of a four-vector. Thus, we denote

$$A^\mu \to (A^0, A^i) \qquad \text{where} \qquad A^0 \equiv \varphi.$$

In order to justify this assumption we note that Maxwell's equations imply that the electromagnetic potentials must obey equations

$$(\partial_0^2 - \nabla^2)\varphi = \frac{4\pi}{c}(c\rho), \qquad (\partial_0^2 - \nabla^2)A^i = \frac{4\pi}{c}J^i \qquad (2.9)$$

where the *Lorenz gauge condition*[4] $\partial_t \varphi + \partial_i A^i = 0$ has been imposed.[5]

Since the d'Alembert operator $\partial_0^2 - \nabla^2$ is Lorentz invariant and the components $(c\rho, J^i) \to J^\mu$ transform as contravariant components of the four-current then the sequence (φ, A^i) must also transform as contravariant components of a four vector. Note, that the Lorenz condition is invariant under Lorentz transformations when $A'^\mu = L^\mu{}_\nu A^\nu$.

The *covariant* components $A_\mu \to (A^0, -A^i)$ can be obtained from contravariant ones by contraction with components of the metric tensor $A_\mu = \eta_{\mu\nu} A^\nu$. Note, that expression A^i appears in **two different contexts**.

1. Spatial components $\mu = 1, 2, 3$ of the four-potential that differ from covariant components $A_i = \eta_{i\nu} A^\nu = -\delta_{ij} A^j = -A^i$. In this case the involved metric tensor is $\eta_{\mu\nu}$ because the four vector lives in the Minkowski spacetime.

[4] Ludvig Valentin Lorenz 1829 - 1891. In the literature frequently confused with Hendrik Antoon Lorentz.

[5] If the original potentials φ' and A'^i give $\partial_t \varphi' + \partial_i A'^i = f(t, x^i)$ then the function $\chi(t, x^i)$ which is a solution of equation $(\partial_0^2 - \nabla^2)\chi(t, x^i) = f(t, x^i)$ allows to obtain new set of potentials φ and A^i that already satisfy the Lorenz condition. Note that Lorenz condition do not fixes the potentials completely. New potentials has still gauge freedom given by functions χ which are solutions of the wave equation $(\partial_0^2 - \nabla^2)\chi(t, x^i) = 0$.

2. On the other hand, they appear as Cartesian *contravariant* components A^i of the vector potential $\boldsymbol{A}$. Such components are mapped on the *covariant* components by the metric tensor of a Euclidean space which has the form $g_{ij} = \delta_{ij}$ in the Cartesian coordinates. It follows that [6]

$$A_i = \delta_{ij} A^j = A^i.$$

Components of electric and magnetic field can be cast in the form

$$E^i = -\partial_0 A^i - \partial_i A^0 = \partial_0 A_i - \partial_i A_0 \equiv F_{0i}, \tag{2.10}$$

$$B^i = \epsilon_{ijk} \partial_j A^k = -\frac{1}{2} \epsilon_{ijk} (\partial_j A_k - \partial_k A_j) \equiv -\frac{1}{2} \epsilon_{ijk} F_{jk} \tag{2.11}$$

where, by construction, F_{0i} and F_{jk} are antisymmetric expressions. They constitute components of the *electromagnetic field tensor* (called also *electromagnetic strength tensor*, *Faraday tensor*[7])

$$\mathbf{F} = F_{\mu\nu}\, \mathbf{e}^{\mu} \otimes \mathbf{e}^{\nu}$$

where

$$\boxed{F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu.} \tag{2.12}$$

Components $F_{\mu\nu}$ transform as covariant components of a second-rank tensor because A_μ and ∂_ν transform as covariant components of a four-vector. In terms of electric and magnetic field they read

$$F_{0i} = E^i, \qquad F_{ij} = -\epsilon_{ijk} B^k. \tag{2.13}$$

In order get the second relation (2.13) we contract both sides of (2.11) with the Levi-Civita symbol. It gives

$$-\epsilon_{abi} B^i = \frac{1}{2} \epsilon_{abi} \epsilon_{ijk} F_{jk} = \frac{1}{2} (\delta_{aj}\delta_{bk} - \delta_{ak}\delta_{bj}) F_{jk} = \frac{1}{2} (F_{ab} - F_{ba}) = F_{ab}.$$

Covariant components of the electromagnetic field tensor can be identified with the following matrix

$$F_{\mu\nu} \to \left(\begin{array}{c|ccc} 0 & E^1 & E^2 & E^3 \\ \hline -E^1 & 0 & -B^3 & B^2 \\ -E^2 & B^3 & 0 & -B^1 \\ -E^3 & -B^2 & B^1 & 0 \end{array} \right), \tag{2.14}$$

whereas the matrix

$$F^{\mu\nu} \to \left(\begin{array}{c|ccc} 0 & -E^1 & -E^2 & -E^3 \\ \hline E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{array} \right). \tag{2.15}$$

contains its *contravariant* components $F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta} F_{\alpha\beta}$.

[6] In order to avoid confusion one could define the four-potential as $\mathcal{A}^\mu$ and preserve the letter A^i exclusively for three components of the vector potential. In such a case $\mathcal{A}_i = -\mathcal{A}^i$ where $\mathcal{A}^i \equiv A^i = A_i$.

[7] See *Gravitation* Charles W. Misner, Kip S. Thorne, and John Archibald Wheeler.

2.3 Covariant form of Maxwell's equations

With help of the electromagnetic field tensor we can put Maxwell's equations in their covariant form.

- *Gauss's law* for the electric field can be cast in the following form

$$\partial_i E^i = 4\pi\rho \qquad \rightarrow \qquad \partial_i F^{i0} = \frac{4\pi}{c} J^0 \qquad \rightarrow \qquad \partial_\mu F^{\mu 0} = \frac{4\pi}{c} J^0.$$

- *Ampere-Maxwell law* can be also written in terms of the electromagnetic field tensor

$$\underbrace{\varepsilon_{ijk}\partial_j B^k}_{\partial_j(\varepsilon_{ijk}B^k)} - \partial_0 E^i = \frac{4\pi}{c} J^i \qquad \rightarrow \qquad -\partial_j F^{ij} - \partial_0 F^{i0} = \frac{4\pi}{c} J^i,$$

which gives

$$\partial_\mu F^{\mu i} = \frac{4\pi}{c} J^i.$$

Thanks to tensor approach Gauss's law and Ampere-Maxwell's can be wrapped up together in a single, frame-independent law

First pair of Maxwell's equations

$$\boxed{\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu.} \tag{2.16}$$

- *Gauss's law for magnetic field* reads

$$\partial_i B^i = 0 \qquad \rightarrow \qquad \partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} = 0.$$

- *Faraday's law* is of the form

$$\varepsilon_{ijk}\partial_j E^k + \partial_0 B^i = 0$$

and it is equivalent to the following set of equations

$$\begin{cases} \partial_2 E^3 - \partial_3 E^2 + \partial_0 B^1 = 0 \\ \partial_3 E^1 - \partial_1 E^3 + \partial_0 B^2 = 0 \\ \partial_1 E^2 - \partial_2 E^1 + \partial_0 B^3 = 0 \end{cases} \rightarrow \quad \begin{cases} \partial_2 F_{03} + \partial_3 F_{20} + \partial_0 F_{32} = 0 \\ \partial_3 F_{01} + \partial_1 F_{30} + \partial_0 F_{13} = 0 \\ \partial_1 F_{02} + \partial_2 F_{10} + \partial_0 F_{21} = 0 \end{cases}. \tag{2.17}$$

Gauss' law for magnetic field and Faraday's law form the *second pair of Maxwell's equations*. They can be wrapped up together in a single law

Second pair of Maxwell's equations

$$\boxed{\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0.} \tag{2.18}$$

or equivalently as

$$\partial_{[\alpha} F_{\beta\gamma]} = 0$$

where $[\ldots]$ stands for anti-symmetrization of any group of indices. In the case of three indices it reads

$$\partial_{[\alpha} F_{\beta\gamma]} := \frac{1}{3!}\left(\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} - \partial_\beta F_{\alpha\gamma} - \partial_\alpha F_{\gamma\beta} - \partial_\gamma F_{\beta\alpha}\right).$$

The expression (2.18) is known as *Bianchi identities* in electromagnetism. Indeed, for $F_{\mu\nu}$ given in terms of four-potential A_μ the expression (2.18) vanishes identically.

The Levi-Civita symbol

The second pair of Maxwell's equations can be put in similar form as the first pair. It can be done in terms of *dual electromagnetic field tensor*[8].

[8] This object naturally appears in the approach based on differential forms

First, we define the antisymmetric *symbol* in four dimensions

Permutation symbol

$$\epsilon_{\mu\nu\alpha\beta} \equiv \epsilon^{\mu\nu\alpha\beta} := \begin{cases} +1 & \text{for} \quad \text{even permutation of } 0123, \\ -1 & \text{for} \quad \text{odd permutation of } 0123, \\ 0 & \text{for} \quad \text{repetition of indices} \end{cases} \tag{2.19}$$

known also as *the Levi-Civita permutation symbol*.

In order to establish its transformation law we consider some general transformation of coordinates

Diffeomorphism

$$\{x^\mu\} \to \{x'^\mu\} \tag{2.20}$$

which, in contradistinction to Lorentz transformations, is not necessarily globally constant. We assume that transformation (2.20) is a diffeomorphism (invertible and has at least first partial derivatives). The coordinates without prime are called "old" whereas those with prime are called "new". *The Jacobian matrix* $\hat{J}$ of the transformation contains first partial derivatives and it has the form[9]

[9] We use convention in which the Jacobian matrix contains derivatives of old coordinates with respect to the new ones. In such approach the Jacobian determinant appears in the volume element. For instance, when changing of coordinates from Cartesian $\{x^i\}$ to polar $\{x'^i\}$ ones then

$$J \equiv \det\hat{J} \equiv \frac{\partial(x,y)}{\partial(r,\varphi)} = r$$

and so $dxdy = rdrd\varphi$.

$$\hat{J} := \left[\frac{\partial x^\mu}{\partial x'^\nu}\right], \qquad \hat{J}^{-1} = \left[\frac{\partial x'^\mu}{\partial x^\nu}\right],$$

where the existence of the inverse transformation is assured by our assumption about the form of the transformation. The upper index numbers the lines of the matrix and the lower one numbers its columns. The Jacobian determinant must not be zero

$$J := \det(\hat{J}) = \epsilon_{\mu\nu\alpha\beta}\frac{\partial x^\mu}{\partial x'^0}\frac{\partial x^\nu}{\partial x'^1}\frac{\partial x^\alpha}{\partial x'^2}\frac{\partial x^\beta}{\partial x'^3} \neq 0 \tag{2.21}$$

and

$$J^{-1} := \det(\hat{J}^{-1}) = \frac{1}{J}.$$

The Levi-Civita symbol is not a tensor because it has not tensor law of transformation. Its transformation law can be obtained from an alternative expression for the Jacobian determinant

$$J = \epsilon'^{\mu\nu\alpha\beta}\frac{\partial x^0}{\partial x'^\mu}\frac{\partial x^1}{\partial x'^\nu}\frac{\partial x^2}{\partial x'^\alpha}\frac{\partial x^3}{\partial x'^\beta} \tag{2.22}$$

where there appears a symbol $\epsilon'^{\mu\nu\alpha\beta}$ with indices that label new coordinates on the rhs of (2.22). Any permutation of two rows changes the sign of the determinant (2.22). Thus we can write

$$J\epsilon^{\rho\sigma\gamma\delta} = \epsilon'^{\mu\nu\alpha\beta}\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\frac{\partial x^\gamma}{\partial x'^\alpha}\frac{\partial x^\delta}{\partial x'^\beta}. \tag{2.23}$$

Contracting the equation (2.23) with $\frac{\partial x'^\kappa}{\partial x^\rho}\frac{\partial x'^\lambda}{\partial x^\sigma}\frac{\partial x'^\omega}{\partial x^\gamma}\frac{\partial x'^\chi}{\partial x^\delta}$ one gets

Transformation formula of the Levi-Civita symbol $\epsilon^{\mu\nu\alpha\beta}$

$$J\epsilon^{\rho\sigma\gamma\delta}\frac{\partial x'^\kappa}{\partial x^\rho}\frac{\partial x'^\lambda}{\partial x^\sigma}\frac{\partial x'^\omega}{\partial x^\gamma}\frac{\partial x'^\chi}{\partial x^\delta} = \epsilon'^{\kappa\lambda\omega\chi}. \tag{2.24}$$

The presence of Jacobian determinant spoils the tensor law of transformation.[10] Similarly, taking arbitrary permutation of columns in (2.21) and dividing by J we get

[10] Such objects are called *relative tensors* of weight $W = +1$.

Transformation formula of the Levi-Civita symbol $\epsilon_{\mu\nu\alpha\beta}$

$$\epsilon'_{\rho\sigma\gamma\delta} = \frac{1}{J}\epsilon_{\mu\nu\alpha\beta}\frac{\partial x^\mu}{\partial x'^\rho}\frac{\partial x^\nu}{\partial x'^\sigma}\frac{\partial x^\alpha}{\partial x'^\gamma}\frac{\partial x^\beta}{\partial x'^\delta}. \tag{2.25}$$

In order to define the Levi-Civita tensor we observe that Jacobian determinant of any rank-two tensor cannot be a scalar. In particular we are interested here in the metric tensor. Its covariant components transform under (2.20) according to

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}g_{\alpha\beta}(x) \tag{2.26}$$

where x' on the lhs is a function of x *i.e.* $x' = x'(x)$. The formula (2.26) can be cast in the matrix form

$$\hat{g}'(x') = \hat{J}^T\hat{g}(x)\hat{J}. \tag{2.27}$$

Taking determinant of both sides of (2.27) we get

$$g'(x') = J^2 g(x). \tag{2.28}$$

where

$$g(x) := \det(\hat{g}(x)), \qquad g'(x') := \det(\hat{g}'(x')).$$

The signature[11] of the quadratic form cannot be changed by transformation of coordinates hence the sign of the determinant must be consistent with the sign of determinant of metrics in any frame. In particular, we know that in Cartesian basis $\det(\hat{\eta}) = -1$ so $g(x) < 0$ and $g'(x') < 0$. Multiplying (2.28) by -1 and taking the square root of its both sides we get

[11] Numbers of positive and negative elements when the form has its canonical form.

$$\sqrt{-g'(x')} = \mathrm{sgn}(J)\, J\sqrt{-g(x)}. \tag{2.29}$$

Next, dividing the lhs (rhs) of equation (2.24) by the lhs (rhs) of (2.29) one gets expression

$$\frac{1}{\sqrt{-g'}}\epsilon'^{\kappa\lambda\omega\chi} = \mathrm{sign}(J)\frac{1}{\sqrt{-g}}\epsilon^{\rho\sigma\gamma\delta}\frac{\partial x'^\kappa}{\partial x^\rho}\frac{\partial x'^\lambda}{\partial x^\sigma}\frac{\partial x'^\omega}{\partial x^\gamma}\frac{\partial x'^\chi}{\partial x^\delta}$$

which is the pseudo-tensor transformation law. When restrict considerations to transformations with $\mathrm{sign}(J) = 1$ one gets that expression $\frac{1}{\sqrt{-g(x)}}\epsilon^{\mu\nu\alpha\beta}$ transforms as a tensor. The *Levi-Civita pseudotensor* is defined as[12]

[12] The minus sign is optional. Our choice agrees with Landau Lifshitz convention adopted in *The Classical Theory of Fields*.

$$\boxed{\varepsilon^{\mu\nu\alpha\beta}(x) := -\frac{1}{\sqrt{-g(x)}}\epsilon^{\mu\nu\alpha\beta}.} \tag{2.30}$$

Since (2.30) is a tensor we can define another tensor contracting (2.30) with components of the metric tensor (lowering its indices)

$$\begin{aligned}\varepsilon_{\rho\sigma\gamma\delta}(x) :=& g_{\rho\mu}g_{\sigma\nu}g_{\gamma\alpha}g_{\delta\beta}\varepsilon^{\mu\nu\alpha\beta}(x)\\ =& -g_{\rho\mu}g_{\sigma\nu}g_{\gamma\alpha}g_{\delta\beta}\epsilon^{\mu\nu\alpha\beta}\frac{1}{\sqrt{-g(x)}}\\ =& -\det(\hat{g}(x))\frac{1}{\sqrt{-g(x)}}\epsilon_{\rho\sigma\gamma\delta}.\end{aligned}$$

Thus

$$\boxed{\varepsilon_{\rho\sigma\gamma\delta}(x) := \sqrt{-g(x)}\epsilon_{\rho\sigma\gamma\delta}.} \tag{2.31}$$

Using (2.25) and (2.29) we get that (2.31) transforms as

$$\begin{aligned}\varepsilon'_{\rho\sigma\gamma\delta}(x') &= \sqrt{-g'(x')}\epsilon'_{\rho\sigma\gamma\delta}\\ &= \mathrm{sgn}(J)J\sqrt{-g(x)}\,\frac{1}{J}\epsilon_{\mu\nu\alpha\beta}\frac{\partial x^\mu}{\partial x'^\rho}\frac{\partial x^\nu}{\partial x'^\sigma}\frac{\partial x^\alpha}{\partial x'^\gamma}\frac{\partial x^\beta}{\partial x'^\delta}\\ &= \mathrm{sgn}(J)\varepsilon_{\mu\nu\alpha\beta}(x)\frac{\partial x^\mu}{\partial x'^\rho}\frac{\partial x^\nu}{\partial x'^\sigma}\frac{\partial x^\alpha}{\partial x'^\gamma}\frac{\partial x^\beta}{\partial x'^\delta}.\end{aligned} \tag{2.32}$$

It shows that (2.31) is a pseudotensor.

Note that contraction of pseudotensors (2.30) and (2.31) is a scalar

$$\varepsilon^{\mu\nu\alpha\beta}(x)\varepsilon_{\mu\nu\alpha\beta}(x) = -4!$$

Dual electromagnetic field tensor

Now we are ready to express the second pair of Maxwell's equations in different but equivalent form. We are going back to Cartesian coordinates where $\sqrt{-g} = 1$ and thus $\varepsilon^{\mu\nu\alpha\beta} = -\epsilon^{\mu\nu\alpha\beta}$ and $\varepsilon_{\mu\nu\alpha\beta} = +\epsilon_{\mu\nu\alpha\beta}$. The electromagnetic dual tensor is defined in the following way

Dual electromagnetic tensor

$$\boxed{{}^*F^{\mu\nu} := \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}} \tag{2.33}$$

and it has the form ${}^*F^{\mu\nu} = -\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$ in Cartesian coordinates.

It gives

$$\begin{aligned}{}^*F^{01} &= -\frac{1}{2}\epsilon^{0123}F_{23} - \frac{1}{2}\epsilon^{0132}F_{32} = -\epsilon^{0123}F_{23} = -F_{23} = B^1,\\ {}^*F^{02} &= -\frac{1}{2}\epsilon^{0231}F_{31} - \frac{1}{2}\epsilon^{0213}F_{13} = -\epsilon^{0231}F_{31} = -F_{31} = B^2,\\ {}^*F^{03} &= -\frac{1}{2}\epsilon^{0312}F_{12} - \frac{1}{2}\epsilon^{0321}F_{21} = -\epsilon^{0312}F_{12} = -F_{12} = B^3,\end{aligned}$$

and similarly

$$
\begin{aligned}
{}^*F^{12} &= -\frac{1}{2}\epsilon^{1230}F_{30} - \frac{1}{2}\epsilon^{1203}F_{03} = -\epsilon^{0123}F_{03} = -F_{03} = -E^3, \\
{}^*F^{23} &= -\frac{1}{2}\epsilon^{2310}F_{10} - \frac{1}{2}\epsilon^{2301}F_{01} = -\epsilon^{0123}F_{01} = -F_{01} = -E^1, \\
{}^*F^{31} &= -\frac{1}{2}\epsilon^{3120}F_{20} - \frac{1}{2}\epsilon^{3102}F_{02} = -\epsilon^{0123}F_{02} = -F_{02} = -E^2.
\end{aligned}
$$

The contravariant components of the dual tensor can be arranged in the form of matrix

$$
{}^*F^{\mu\nu} \to \left(\begin{array}{c|ccc} 0 & B^1 & B^2 & B^3 \\ \hline -B^1 & 0 & -E^3 & E^2 \\ -B^2 & E^3 & 0 & -E^1 \\ -B^3 & -E^2 & E^1 & 0 \end{array}\right). \tag{2.34}
$$

Comparing (2.34) with (2.15) we conclude that dual transformation maps electric field into negative of magnetic field and magnetic field into electric field

$$
{}^*\boldsymbol{E} = -\boldsymbol{B} \qquad \text{and} \qquad {}^*\boldsymbol{B} = \boldsymbol{E}. \tag{2.35}
$$

Double dual transformation changes the sign of the field.

Contraction of Bianchi identities (2.18) with constant expression

$$
-\frac{1}{3!}\epsilon^{\alpha\nu\beta\gamma}
$$

gives

$$
\partial_\alpha\left[-\frac{1}{6}\epsilon^{\alpha\nu\beta\gamma}F_{\beta\gamma}\right] + \partial_\beta\left[-\frac{1}{6}\epsilon^{\alpha\nu\beta\gamma}F_{\gamma\alpha}\right] + \partial_\gamma\left[-\frac{1}{6}\epsilon^{\alpha\nu\beta\gamma}F_{\alpha\beta}\right] = 0.
$$

Changing labels of indices in the above equation one gets

$$
\partial_\mu \underbrace{\left[-\frac{1}{6}\epsilon^{\mu\nu\beta\gamma}F_{\beta\gamma}\right]}_{\frac{1}{3}{}^*F^{\mu\nu}} + \partial_\mu \underbrace{\left[-\frac{1}{6}\epsilon^{\alpha\nu\mu\gamma}F_{\gamma\alpha}\right]}_{\frac{1}{3}{}^*F^{\mu\nu}} + \partial_\mu \underbrace{\left[-\frac{1}{6}\epsilon^{\alpha\nu\beta\mu}F_{\alpha\beta}\right]}_{\frac{1}{3}{}^*F^{\mu\nu}} = 0.
$$

Finally, the second pair of Maxwell's equation reads

Alternative form of second pair of Maxwell's equations

$$
\boxed{\partial_\mu {}^*F^{\mu\nu} = 0} \tag{2.36}
$$

where $\nu = 0$ corresponds with magnetic Gauss's law and $\nu = 1, 2, 3$ gives three components of Faraday's law.

Covariance of Maxwell's equations

The expression $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ represent components the rank-two $\binom{2}{0}$ tensor so it transforms according to

$$
F'^{\mu\nu} = L^\mu{}_\alpha L^\nu{}_\beta F^{\alpha\beta}.
$$

where A^μ and ∂^ν transform as components of rank-one $\binom{1}{0}$ tensor (four-vector).

First pair of Maxwell's equation in the inertial reference frame S' which moves with the velocity $\boldsymbol{V}$ with respect to the reference frame S have the form $\partial'_\mu F'^{\mu\nu} - \frac{4\pi}{c} J'^\nu = 0$. This equations are equivalent to the following ones

$$\underbrace{(L^{-1})^\rho{}_\mu L^\mu{}_\alpha}_{\delta^\rho_\alpha} L^\nu{}_\beta \partial_\rho F^{\alpha\beta} - \frac{4\pi}{c} L^\nu{}_\beta J^\beta = 0$$

which can be written in the form

$$L^\nu{}_\beta \left[\partial_\alpha F^{\alpha\beta} - \frac{4\pi}{c} J^\beta \right] = 0. \tag{2.37}$$

Each equation in (2.37) in S' (with given value of ν) is linear combination of Maxwell's equations in S where coefficients of the combination are elements of the Lorentz matrix. First pair of Maxwell's equations (2.37) takes the form

Transformation of first pair of Maxwell's equations

$$\boxed{L^\nu{}_0 \underbrace{\left[\partial_\alpha F^{\alpha 0} - \frac{4\pi}{c} J^0 \right]}_{\text{Gauss's law in S}} + L^\nu{}_i \underbrace{\left[\partial_\alpha F^{\alpha i} - \frac{4\pi}{c} J^i \right]}_{\text{Ampere's law in S}} = 0.} \tag{2.38}$$

Similarly, components of the dual electromagnetic tensor in S' are given by components of this tensor in S[13]

$${}^*F'^{\mu\nu} = \pm L^\mu{}_\alpha L^\nu{}_\beta {}^*F^{\alpha\beta}.$$

[13] The sign $\pm 1 = \mathrm{sgn} J$, where $J^{-1} = \det(\hat{L})$ is a Jacobian of a transformation, appears due to transformation rule of the Levi-Civita pseudotensor $\varepsilon^{\mu\nu\alpha\beta}(x)$ given by (2.32). In further part we shall choose $+1$ because we are interested in boost transformations which are proper Lorentz transformations.

Hence the second pair of Maxwell's equations reads

$$\underbrace{(L^{-1})^\rho{}_\mu L^\mu{}_\alpha}_{\delta^\rho_\alpha} L^\nu{}_\beta \partial_\rho {}^*F^{\alpha\beta} = 0$$

and it can be cast in the form

Transformation of second pair of Maxwell's equations

$$\boxed{L^\nu{}_0 \underbrace{\left[\partial_\alpha {}^*F^{\alpha 0} \right]}_{\text{Gauss' law in S}} + L^\nu{}_i \underbrace{\left[\partial_\alpha {}^*F^{\alpha i} \right]}_{\text{Faraday's law in S}} = 0.} \tag{2.39}$$

The important fact about transformation of Maxwell's equation is that left hand sides of these equations in a given reference frame are linear combinations of left hand sides of the equations in another frame. The coefficients of linear combinations are elements of the Lorentz matrix. The Lorentz transformations do not mix equations belonging to different pairs.

2.4 *Maxwell's equations and external differential forms*

The formalism of external differential forms provides natural framework for representation of Maxwell's equations. Among others, especial useful are applications of Stokes theorem. Using this theorem one gets integral form of Maxwell's equations which is *explicitly Lorentz invariant*.

External differential forms

Differential forms can be defined in spaces with any finite number of dimensions. The case $N = 4$ is of special importance in application to electromagnetism.

Let $(\mathcal{A}, \mathcal{V}^{(4)})$ be 4−dimensional affine space where $\mathcal{A}$ is an infinite collection of points (more precisely – manifold) and $\mathcal{V}^{(4)}$ is a vector space.

One-form $\overset{1}{\omega}$ is a *linear function* which maps vectors $\mathbf{v} \in \mathbf{T}(p)$ on real numbers

One-form definition

$$\overset{1}{\omega} : \mathbf{v} \mapsto \mathbb{R}$$

such that

Linearity in argument of the form

$$\overset{1}{\omega}(a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2) = a_1 \overset{1}{\omega}(\mathbf{v}_1) + a_2 \overset{1}{\omega}(\mathbf{v}_2). \tag{2.40}$$

A set of all such linear forms (covectors) at any point p form a vector space provided that

Linear space of covectors

$$(\overset{1}{\omega}_1 + \overset{1}{\omega}_2)(\mathbf{v}) \quad := \quad \overset{1}{\omega}_1(\mathbf{v}) + \overset{1}{\omega}_2(\mathbf{v}), \tag{2.41}$$

$$(a \overset{1}{\omega})(\mathbf{v}) \quad := \quad a \overset{1}{\omega}(\mathbf{v}). \tag{2.42}$$

This space, denoted by $\mathbf{T}^*(p)$, is dual to $\mathbf{T}(p)$ and it is called *tangent space of covectors* or *cotangent space*. Components of one-forms are denoted by ω_α and they are defined as sequences of numbers representing values of one-form on basis vectors $\{\mathbf{e}_\alpha\}_{\alpha=0,\dots,3}$ at the point p *i.e.*,

Cotangent space

Components of one-form

$$\omega_\alpha := \overset{1}{\omega}(\mathbf{e}_\alpha).$$

In the space of one-forms one can define *the dual basis* which consists on one-forms $\{e^\beta\}_{\beta=0,\dots,3}$ that act on the basis vectors $\mathbf{e}_\alpha \in \mathbf{T}(p)$ giving

$$e^\beta(\mathbf{e}_\alpha) := \delta^\beta_\alpha. \tag{2.43}$$

Elements of dual basis $\{e^\alpha\}_{\alpha=0,\dots 3}$ and components of one-forms $\overset{1}{\omega}$ transform in the following way

$$e'^\alpha = \frac{\partial x'^\alpha}{\partial x^\beta}(p) e^\beta, \qquad \omega'_\alpha = \frac{\partial x^\beta}{\partial x'^\alpha}(p)\omega_\beta. \tag{2.44}$$

Geometric interpretation of covectors

Let $\mathcal{O}$ be certain region of spacetime and f a differentiable function $f : \mathcal{O} \to \mathbb{R}$. The differential of this function df is a liner operation which associates a number

$$df = \frac{\partial f}{\partial x^\alpha}(p) dx^\alpha$$

with the function f at any *fixed point* $p \in \mathcal{O}$. This number depends on some (arbitrary) sequence of numbers $\{dx^0, \cdots, dx^3\}$. It can be interpreted as linear form with $N = 4$ variables. Derivatives $\frac{\partial f}{\partial x^\alpha}(p)$ form a sequence of real numbers which depends on f.

Let $\mathbf{v} \in \boldsymbol{T}(p)$ be a vector which has components $\{v^0, \cdots, v^3\}$ in a given basis $\{\mathbf{e}_\alpha\}_{\alpha=0,\cdots,3}$. The mapping between sequences $\{dx^0, \cdots, dx^3\}$ and $\{v^0, \cdots, v^3\}$ allows to associate a linear form with the differential df at p (tangent covector). It acts on vectors $\mathbf{v} \in \mathbf{T}(p)$ according to

Differential of a function at the fixed point as a model of one-form

$$df(\mathbf{v}) := \frac{\partial f}{\partial x^\alpha}(p) v^\alpha. \tag{2.45}$$

The formula (2.45) allows us to associate with the differential df at p the *linear differential form*

$$\overset{1}{\omega} := \frac{\partial f}{\partial x^\alpha}(p) e^\alpha \in \mathbf{T}^*(p). \tag{2.46}$$

Such association does not depend on the reference frame. Moreover, it is invertible – with any sequence of numbers it can be associated a sequence of derivatives $\partial_\alpha f|_p$. Since p is fixed the sequence of derivatives is a numerical sequence. The simplest function which allows to associate a sequence of numbers with a sequence of partial derivatives at p has the form

Invertibility

$$f(x^0, \cdots, x^3) = \omega_\alpha x^\alpha$$

where ω_α is a sequence of numbers. It gives

$$\partial_\alpha f|_p = \omega_\beta \delta^\beta_\alpha = \omega_\alpha.$$

It means that, there is one-to-one correspondence (isomorphism) between the space $\mathbf{T}^*(p)$ and the differential of function f at p. This correspondence preserves the structure of both spaces.

Basis

Any covector $\overset{1}{\omega}$ can be decomposed in basis of covectors $\{e^\alpha\}_{\alpha=0,\ldots,3}$ dual to vector basis *i.e.* $e^\alpha(\mathbf{e}_\beta) := \delta^\alpha_\beta$. In particular, covectors e^α can be cast in the form (2.46) what gives

$$e^\alpha = \frac{\partial f^\alpha}{\partial x^\beta}(p) e^\beta \tag{2.47}$$

where $\{f^\alpha\}_{\alpha=0,\ldots,3}$ is a sequence of functions. Each function f^α defines one covector e^α. Components β of this covector in the basis that the covector belongs to read

$$(e^\alpha)_\beta = \frac{\partial f^\alpha}{\partial x^\beta}(p) = \delta^\alpha_\beta.$$

The simplest set of functions satisfying this requirement is given by

$$f^\alpha(x^0, \cdots, x^3) := x^\alpha.$$

Differentials of such functions, associated with covectors, read

$$e^\alpha = dx^\alpha.$$

We stress that dx^α in this expression have the meaning of *linear forms* that act on elements of the space $\mathbf{T}(p)$ according to

Differentials dx^α as linear forms

$$dx^\alpha(\mathbf{v}) = dx^\alpha(v^\beta \mathbf{e}_\beta) = v^\beta dx^\alpha(\mathbf{e}_\beta) = v^\beta \delta^\alpha_\beta = v^\alpha.$$

Hence

$$\overset{1}{\omega}(\mathbf{v}) = \omega_\alpha dx^\alpha(\mathbf{v}) = \omega_\alpha v^\alpha.$$

Linear differential forms

In previous section we managed to establish one-to-one correspondence between differentials of functions at fixed point and linear forms with constant components ω_α in the basis dx^α. In this section we shall extend the idea of differential forms on expressions with non constant components ω_α.

Linear one-form in a region of spacetime

We consider certain region $\mathcal{O}$ of Minkowski spacetime[14] and linear forms at each point $p \in \mathcal{O}$. We say that *field of covectors* (or *field of linear differential forms*) is given in $\mathcal{O}$.

[14] Here we concentrate on definition of linear forms in Minkowski spacetime, however, it must be stressed that such considerations do not depend of this particular fact.

Linear one-form $\overset{1}{\omega}(p) = \omega_\alpha(p)e^\alpha$ is defined only by its coordinates $\omega_\alpha(x^0, \cdots, x^3)$ after setting the coordinate reference frame whole region $\mathcal{O}$. In such fixed coordinate frame an arbitrary *linear differential one-form* is given by expression

Linear differential one-form

$$\boxed{\overset{1}{\omega} = \omega_\alpha(x^0, \ldots, x^3)dx^\alpha.} \tag{2.48}$$

Note, that ω_α in (2.48) form a *sequence of functions* and not a sequence of numbers. This fact has some immediate consequences. Namely, for a given sequence of *numbers* we can always find such a function $f(x^0, \ldots, x^3)$ that the sequence of its partial derivatives at p corresponds with ω_α. It is not possible for generic sequence of functions $\omega_\alpha(x^0, \ldots, x^3)$. It follows that *each differential of function is a differential form whereas the inverse is not necessarily true*. The differential form $\overset{1}{\omega}$ acts on vector field $\mathbf{v} \in \mathbf{T}(p)$. It maps sequence of differentials $\{dx^\alpha\}$ onto sequence of *functions*

Each differential of a function is differential form but not each differential form is a differential of a function

$$\{dx^0, \ldots, dx^3\} \qquad \mapsto \qquad \{v^0, \ldots, v^3\} \tag{2.49}$$

where $\{v^\alpha\}$ are components of certain vector $\mathbf{v}$ in vector basis $\{\mathbf{e}_\alpha\}_{\alpha=0,\dots,3}$. This mapping defines the function $\omega\mathbf{v}(p) : \mathbf{T}(p) \mapsto \mathbb{R}$ which is given by

Function $p \to \omega\mathbf{v}(p) \in \mathbb{R}$

$$\omega\mathbf{v}(p) := \overset{1}{\omega}(\mathbf{v}) = \omega_\alpha v^\alpha. \tag{2.50}$$

Linear two-form is a function which associates a real number with *a pair of vectors* from $\mathbf{T}(p)$

Linear two-form

$$\overset{2}{\omega} : \qquad (\mathbf{v}_1, \mathbf{v}_2) \quad \to \quad \mathbb{R}.$$

This function has the following properties

- it is linear in its first argument

Linearity

$$\overset{2}{\omega}(a_1\,\mathbf{v}_1 + a_2\,\mathbf{v}_2, \mathbf{v}_3) = a_1\,\overset{2}{\omega}(\mathbf{v}_1, \mathbf{v}_3) + a_2\,\overset{2}{\omega}(\mathbf{v}_2, \mathbf{v}_3), \tag{2.51}$$

  where a_1 and a_2 are some real numbers,

- it is anti-symmetric

Skew-symmetry

$$\overset{2}{\omega}(\mathbf{v}_1, \mathbf{v}_2) = -\overset{2}{\omega}(\mathbf{v}_2, \mathbf{v}_1). \tag{2.52}$$

The set of all such forms on $\mathbf{T}(p)$ is a $\binom{4}{2} = 6$ dimensional real vector space provided that

Linear space of two-forms

$$(\overset{2}{\omega}_1 + \overset{2}{\omega}_2)(\mathbf{v}_1, \mathbf{v}_2) := \overset{2}{\omega}_1(\mathbf{v}_1, \mathbf{v}_2) + \overset{2}{\omega}_2(\mathbf{v}_1, \mathbf{v}_2) \tag{2.53}$$

$$(a\,\overset{2}{\omega})(\mathbf{v}_1, \mathbf{v}_2) := a\,\overset{2}{\omega}(\mathbf{v}_1, \mathbf{v}_2). \tag{2.54}$$

With each pair of one-forms $\overset{1}{\omega}_1$, $\overset{1}{\omega}_2$ there can be associated a two-form $\overset{1}{\omega}_1 \wedge \overset{1}{\omega}_2$ which acts on vectors $\mathbf{v}_1, \mathbf{v}_2$ according to

External product of one-forms as a prototype of two-form

$$\begin{aligned}(\overset{1}{\omega}_1 \wedge \overset{1}{\omega}_2)(\mathbf{v}_1, \mathbf{v}_2) &:= \epsilon_{ij}\overset{1}{\omega}_1(\mathbf{v}_i)\overset{1}{\omega}_2(\mathbf{v}_j) \\ &\equiv \det\begin{bmatrix} \overset{1}{\omega}_1(\mathbf{v}_1) & \overset{1}{\omega}_2(\mathbf{v}_1) \\ \overset{1}{\omega}_1(\mathbf{v}_2) & \overset{1}{\omega}_2(\mathbf{v}_2) \end{bmatrix}.\end{aligned} \tag{2.55}$$

The form $\overset{1}{\omega}_1 \wedge \overset{1}{\omega}_2$ is obtained as *external product of covectors* $\overset{1}{\omega}_1$ and $\overset{1}{\omega}_2$. Each one-form maps vectors from $\mathbf{T}(\mathbf{p})$ on real numbers and these numbers are arranged in anti-symmetric expression. It follows from determinant properties that

$$\overset{1}{\omega}_1 \wedge \overset{1}{\omega}_2 = -\overset{1}{\omega}_2 \wedge \overset{1}{\omega}_1 \tag{2.56}$$

$$(\overset{1}{\omega}_1 + \overset{1}{\omega}_2) \wedge \overset{1}{\omega}_3 = \overset{1}{\omega}_1 \wedge \overset{1}{\omega}_3 + \overset{1}{\omega}_2 \wedge \overset{1}{\omega}_3. \tag{2.57}$$

The external product can be used to define differential two-forms and more generally k–forms.

Linear differential two-form is given by formula

Linear differential two-form

$$\boxed{\overset{2}{\omega} = \frac{1}{2}\omega_{\alpha\beta}\, dx^\alpha \wedge dx^\beta.} \tag{2.58}$$

It associates a real number with each pair of tangent vectors $(\mathbf{v}_1, \mathbf{v}_2)$, according to

$$\begin{aligned}\overset{2}{\omega}(\mathbf{v}_1, \mathbf{v}_2) &= \frac{1}{2}\omega_{\alpha\beta}\,(dx^\alpha \wedge dx^\beta)(\mathbf{v}_1, \mathbf{v}_2)\\ &= \frac{1}{2}\omega_{\alpha\beta}\epsilon_{ij} dx^\alpha(\mathbf{v}_i) dx^\beta(\mathbf{v}_j)\\ &\equiv \frac{1}{2}\omega_{\alpha\beta}\det\begin{bmatrix} dx^\alpha(\mathbf{v}_1) & dx^\beta(\mathbf{v}_1)\\ dx^\alpha(\mathbf{v}_2) & dx^\beta(\mathbf{v}_2)\end{bmatrix}\\ &= \frac{1}{2}\omega_{\alpha\beta}\det\begin{bmatrix} v_1^\alpha & v_1^\beta\\ v_2^\alpha & v_2^\beta\end{bmatrix}\end{aligned} \tag{2.59}$$

where $dx^\alpha(\mathbf{v}_i) = dx^\alpha(v_i^\mu \mathbf{e}_\mu) = v_i^\mu \delta_\mu^\alpha = v_i^\alpha$. The coefficients of this form are values which it takes on each pair of vectors from the set $\{\mathbf{e}_\alpha\}_{\alpha=0,\dots,3}$,

$$\omega_{\mu\nu} := \overset{2}{\omega}(\mathbf{e}_\mu, \mathbf{e}_\nu) = \frac{1}{2}\omega_{\alpha\beta}\det\begin{bmatrix} \delta_\mu^\alpha & \delta_\mu^\beta\\ \delta_\nu^\alpha & \delta_\nu^\beta\end{bmatrix}. \tag{2.60}$$

Linear differential k-form $\overset{k}{\omega}$, where $0 \le k \le N \equiv 4$, is a covariant tensor of rank $\binom{0}{k}$ given by

Linear differential k-form

$$\boxed{\overset{k}{\omega} = \frac{1}{k!}\omega_{\alpha_1\dots\alpha_k} dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_k}} \tag{2.61}$$

where $\alpha_a = 0, 1, 2, 3$. The antisymmetric expressions

$$dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_k}$$

are elements of basis in the space of differential k–forms. The form (2.61) associates a real number with each sequence of tangent vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ according to the formula

$$\overset{k}{\omega}(\mathbf{v}_1, \dots, \mathbf{v}_k) = \frac{1}{k!}\omega_{\alpha_1\dots\alpha_k}\det\begin{bmatrix} v_1^{\alpha_1} & \dots & v_1^{\alpha_k}\\ \vdots & \dots & \vdots\\ v_k^{\alpha_1} & \dots & v_k^{\alpha_k}\end{bmatrix}. \tag{2.62}$$

Components of $\overset{k}{\omega}$ are given by values which this form takes on each k-th element sequence of vectors $\{\mathbf{e}_\alpha\}$ *i.e.*

$$\omega_{\alpha_1\dots\alpha_k} = \overset{k}{\omega}(\mathbf{e}_{\alpha_1}, \dots, \mathbf{e}_{\alpha_k}). \tag{2.63}$$

Dual forms (adjoint forms)

With each given k–form $\overset{k}{\omega}$ in N dimensions there can be associated the $(N-k)$–form[15] $\overset{N-k}{\sigma}$. This operation is defined in terms of *Hodge operator* "$*$", which depends on the scalar product (and so it requires the metric tensor) and space (spacetime) orientation.

[15] Here we are mostly interested in the case N=4. However, for generality we shall keep explicitly N for denoting number of dimensions.

The Hodge operator requires Levi-Civita pseudotensor which is proportional to Levi-Civita permutation symbol $\epsilon_{\alpha_1\ldots\alpha_N}$. The permutation symbol in N dimensions is introduced exactly in the same way as in the case $N=4$ discussed before. Let

$$\pi(0,1,\ldots,N-1) = \alpha_1,\alpha_2,\ldots,\alpha_{N-1}$$

be a permutation of numbers $0,1,\ldots,N-1$ and $n(\pi)$ stands for number of permutations i.e. number of mappings of the sequence $0,1,\ldots,N-1$ onto $\alpha_1,\alpha_2,\ldots,\alpha_{N-1}$. It is defined as

Levi -Civita permutation symbol

$$\epsilon_{\alpha_1\ldots\alpha_N} \equiv \epsilon^{\alpha_1\ldots\alpha_N} := \begin{cases} (-1)^{n(\pi)}, & \\ 0 & \text{for repetition of indices} \end{cases}.$$

Levi-Civita pseudotensor is an expression whose components has pseudotensorial transformation law. The covariant components of Levi-Civita pseudotensor are defined as follows

Covariant components of Levi - Civita pseudotensor

$$\varepsilon_{\alpha_1\ldots\alpha_N} := \sqrt{|g|}\,\epsilon_{\alpha_1\ldots\alpha_N}. \tag{2.64}$$

Its contravariant components $\varepsilon^{\alpha_1\ldots\alpha_N}$ are given by expression

Contravariant components of the Levi -Civita pseudotensor

$$\begin{aligned} \varepsilon^{\alpha_1\ldots\alpha_N} &:= g^{\alpha_1\beta_1}\ldots g^{\alpha_N\beta_N}\sqrt{|g|}\,\epsilon_{\beta_1\ldots\beta_N} \\ &= \frac{1}{g}\sqrt{|g|}\epsilon^{\alpha_1\ldots\alpha_N} \\ &= \frac{\mathrm{sgn}(g)}{\sqrt{|g|}}\epsilon^{\alpha_1\ldots\alpha_N} \end{aligned}$$

Hodge star operator is a mapping between linear forms which associates $(N-k)$–form with the form (2.61). This k-form is given by

Hodge operator

$$*\overset{k}{\omega} = \frac{1}{(N-k)!}\left[\frac{1}{k!}\sqrt{|g|}\,\omega^{\alpha_1\ldots\alpha_k}\,\epsilon_{\alpha_1\ldots\alpha_k\beta_1\ldots\beta_{N-k}}\right] dx^{\beta_1}\wedge\ldots\wedge dx^{\beta_{N-k}}$$

where

$$\boxed{{}^*\omega_{\beta_1\ldots\beta_{N-k}} := \frac{1}{k!}\sqrt{|g|}\,\omega^{\alpha_1\ldots\alpha_k}\,\epsilon_{\alpha_1\ldots\alpha_k\beta_1\ldots\beta_{N-k}}} \tag{2.65}$$

are *components of dual form (adjoint form)*.

Physical examples

The four-current one-form is a differential form with components J^μ *i.e.*

Electric current one-form

$$\mathbf{J} := J_\mu dx^\mu. \tag{2.66}$$

Its dual is a 3−form $^*\mathbf{J}$ given by expression

Dual there-form of the electric current density

$$^*\mathbf{J} = \frac{1}{3!}{}^*J_{\alpha\beta\gamma}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma \tag{2.67}$$

where $^*J_{\alpha\beta\gamma} = J^\mu \varepsilon_{\mu\alpha\beta\gamma}$. We shall restrict our considerations to Cartesian coordinates, so $\sqrt{|g|} = \sqrt{-g} = 1$. The electric current dual form has components

Cartesian coordinates

$$\begin{aligned}
{}^*J_{123} &= J^0 \varepsilon_{0123} = J^0, \\
{}^*J_{023} &= J^1 \varepsilon_{1023} = -J^1, \\
{}^*J_{031} &= J^2 \varepsilon_{2031} = -J^2, \\
{}^*J_{012} &= J^3 \varepsilon_{3012} = -J^3
\end{aligned}$$

and so

$$\begin{aligned}
{}^*\mathbf{J} = J^0 dx^1 \wedge dx^2 \wedge dx^3 - J^1 dx^0 \wedge dx^2 \wedge dx^3 \\
- J^2 dx^0 \wedge dx^3 \wedge dx^1 - J^3 dx^0 \wedge dx^1 \wedge dx^2.
\end{aligned} \tag{2.68}$$

Another important example are differential forms which involve components of electromagnetic tensor. For instance, **Faraday's two-form** is defined as follows

Two-form of the electromagnetic field

$$\boxed{\mathbf{F} := \frac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu.} \tag{2.69}$$

The explicit form of the Faraday's two-form in terms of components of electric and magnetic fields reads

$$\begin{aligned}
\mathbf{F} = E^1 dx^0 \wedge dx^1 + E^2 dx^0 \wedge dx^2 + E^3 dx^0 \wedge dx^3 \\
- B^1 dx^2 \wedge dx^3 - B^2 dx^3 \wedge dx^1 - B^3 dx^1 \wedge dx^2.
\end{aligned} \tag{2.70}$$

Similarly, **Maxwell's form** is a two-form dual to Faraday's form. It reads

Dual two-form of the electromagnetic field

$$\boxed{{}^*\mathbf{F} := \frac{1}{2}{}^*F_{\alpha\beta}\, dx^\alpha \wedge dx^\beta,} \tag{2.71}$$

where

$$^*F_{\alpha\beta} = \frac{1}{2} F^{\mu\nu} \varepsilon_{\mu\nu\alpha\beta}. \tag{2.72}$$

They have their explicit form

$$^*F_{\alpha\beta} \to \left(\begin{array}{c|ccc}
0 & -B^1 & -B^2 & -B^3 \\
\hline
B^1 & 0 & -E^3 & E^2 \\
B^2 & E^3 & 0 & -E^1 \\
B^3 & -E^2 & E^1 & 0
\end{array}\right). \tag{2.73}$$

Thus, Maxwell's two-form is given by expression

$$\begin{aligned}{}^*\mathbf{F} = &- B^1 dx^0 \wedge dx^1 - B^2 dx^0 \wedge dx^2 - B^3 dx^0 \wedge dx^3 \\ &- E^1 dx^2 \wedge dx^3 - E^2 dx^3 \wedge dx^1 - E^3 dx^1 \wedge dx^2. \end{aligned} \tag{2.74}$$

The adjoint operation interchanges electric and magnetic field in such a way that ${}^*\mathbf{E} = -\mathbf{B}$ and ${}^*\mathbf{B} = \mathbf{E}$. Note, that in the alternative convention where $\varepsilon_{0123} = -1$ the duality relations read ${}^*\mathbf{E} = \mathbf{B}$ and ${}^*\mathbf{B} = -\mathbf{E}$.

Exterior derivative

We have seen that with a differential of certain function $f : \mathcal{O} \to \mathbb{R}$ there can be associated *differential one-form* in the region $\mathcal{O}$ of Minkowski spacetime. Since the function f can be interpreted as *zero-form* then differential of function associates one-form with each zero-form.

Exterior differential

The exterior derivative (exterior differential) is a generalization of this operation for an arbitrary $k-$form. This operation associates $(k+1)$–form of class $\mathcal{C}^{r-1}$ with k–form of class $\mathcal{C}^r$:

$$\begin{aligned} d\overset{k}{\omega} &= d\left(\frac{1}{k!}\omega_{\alpha_1\ldots\alpha_k} dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_k}\right) \\ &= \frac{1}{k!} d\omega_{\alpha_1\ldots\alpha_k} \wedge dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_k} \\ &= \frac{1}{k!} \partial_{\alpha_0}\omega_{\alpha_1\ldots\alpha_k} dx^{\alpha_0} \wedge dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_k}. \end{aligned} \tag{2.75}$$

Properties of exterior differentiation

The exterior derivative has the following properties

$$d(\overset{k}{\omega}_1 + \overset{k}{\omega}_2) = d\overset{k}{\omega}_1 + d\overset{k}{\omega}_2, \tag{2.76}$$

$$d(\overset{k}{\omega} \wedge \overset{l}{\omega}) = d\overset{k}{\omega} \wedge \overset{l}{\omega} + (-1)^k \overset{k}{\omega} \wedge d\overset{l}{\omega}, \tag{2.77}$$

$$dd\overset{k}{\omega} = 0 \qquad \text{for any form,} \tag{2.78}$$

$$d\overset{0}{\omega} = \partial_\alpha \omega\, dx^\alpha. \tag{2.79}$$

Closed form

A differential form $\overset{k}{\omega}$ is called **closed** if

$$\boxed{d\overset{k}{\omega} = 0.} \tag{2.80}$$

Exact form

A differential form $\overset{k}{\omega}$ is called **exact** if

$$\boxed{\overset{k}{\omega} = d\overset{k-1}{\sigma}.} \tag{2.81}$$

Each exact form is closed because $dd\overset{k-1}{\sigma} \equiv 0$. The inverse statement holds only in regions *contractible to a point*.

Poincaré Lemma

Poincaré Lemma. *Let $\mathcal{O} \subset \mathcal{A}$ be a region contractible to a point and $\overset{k}{\omega}$, where $k = 1, 2, \cdots$, be an arbitrary closed differential $k-$form defined in $\mathcal{O}$ i.e. $d\overset{k}{\omega} = 0$. There exists a form $\overset{k-1}{\sigma}$ such that $\overset{k}{\omega} = d\overset{k-1}{\sigma}$.*

Examples of exterior differentiation

Let us consider three examples of differential forms of degree $0, 1, 2$ which depend only on spatial coordinates x^1, x^2 and x^3:

$$\overset{0}{\omega} = \omega, \tag{2.82}$$

$$\overset{1}{\omega} = \omega_1\, dx^1 + \omega_2\, dx^2 + \omega_3\, dx^3, \tag{2.83}$$

$$\overset{2}{\omega} = \omega_{12}\, dx^1 \wedge dx^2 + \omega_{23}\, dx^2 \wedge dx^3 + \omega_{31}\, dx^3 \wedge dx^1. \tag{2.84}$$

The exterior derivatives of these forms read

Exterior derivatives of time-independent zero-, one- and two-form

$$d\overset{0}{\omega} = \partial_1 \omega\, dx^1 + \partial_2 \omega\, dx^2 + \partial_3 \omega\, dx^3, \tag{2.85}$$

$$\begin{aligned} d\overset{1}{\omega} &= (\partial_1 \omega_2 - \partial_2 \omega_1)\, dx^1 \wedge dx^2 + (\partial_2 \omega_3 - \partial_3 \omega_2)\, dx^2 \wedge dx^3 \\ &\quad + (\partial_3 \omega_1 - \partial_1 \omega_3)\, dx^3 \wedge dx^1, \end{aligned} \tag{2.86}$$

$$d\overset{2}{\omega} = (\partial_1 \omega_{23} + \partial_2 \omega_{31} + \partial_3 \omega_{12})\, dx^1 \wedge dx^2 \wedge dx^3. \tag{2.87}$$

One can associate components of forms $\overset{k}{\omega}$, $k = 0, 1, 2$ with components of some vector fields. In such a case the differential forms obtained by exterior derivatives of given forms have components which can be identified with components of well-known operations: *gradient*, *curl* and *divergence*.

Relation with gradient, curl and divergence

Physical examples of application of the Poincaré Lemma

The Poincaré Lemma applied to closed forms has some implications on associated vector fields – with components equal to components of differential forms.

- If the form $\overset{1}{\omega}$ is closed *i.e.* $d\overset{1}{\omega} = 0$ (their components correspond with a components of a certain *curl-free vector field* or *irrotational vector field*), then there exist an exact form $\overset{0}{\sigma}$ (*scalar potential*) in the region contractible to a point, such that $\overset{1}{\omega} = d\overset{0}{\sigma}$. The vector field associated with $\overset{1}{\omega}$ is the gradient of this scalar potential $\varphi \equiv \overset{0}{\sigma}$.

Existence of scalar potential

- If the form $\overset{2}{\omega}$ is closed *i.e.* $d\overset{2}{\omega} = 0$ (their components correspond with components of *solenoidal vector field*), then there exist an exact form $\overset{1}{\sigma}$ (*vector potential*) in the region contractible to a point, such that $\overset{2}{\omega} = d\overset{1}{\sigma}$ (vector field associated with $\overset{2}{\omega}$ is the **curl** of the vector field).

Existence of vector potential

Let us consider the field

Electrostatic field of linear charge density of infinitely long line

$$\boldsymbol{E} = 2\lambda_0 \frac{\hat{r}}{r} = 2\lambda_0 \left[\frac{x}{x^2 + y^2} \hat{x} + \frac{y}{x^2 + y^2} \hat{y} \right]$$

This field correspond with the electric field of infinitely long uniformly charged lines where λ_0 stands for linear density of electric charge. The second field has the form

Magnetic field of infinitely long conducting wire

$$H = \frac{2I_0}{c}\frac{\hat{\phi}}{r} = \frac{2I_0}{c}\left[-\frac{y}{x^2+y^2}\hat{x} + \frac{x}{x^2+y^2}\hat{y}\right]$$

and it corresponds with magnetic field of infinitely long straight conductor with current intensity $I_0 = const$.

We define two differential one-forms with components equal to components of the above vector fields

$$\overset{1}{e} = 2\lambda_0\left[\frac{x}{x^2+y^2}dx + \frac{y}{x^2+y^2}dy\right], \tag{2.88}$$

$$\overset{1}{h} = \frac{2I_0}{c}\left[-\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy\right]. \tag{2.89}$$

It follows that both forms $\overset{1}{e}$ and $\overset{1}{h}$ are closed

Both $\overset{1}{e}$ and $\overset{1}{h}$ are closed

$$d\overset{1}{e} = 0 \qquad \Leftrightarrow \qquad \nabla \times E = 0, \tag{2.90}$$

$$d\overset{1}{h} = 0 \qquad \Leftrightarrow \qquad \nabla \times H = 0 \quad \text{in} \quad \mathbb{E}^2\backslash\{0,0\}. \tag{2.91}$$

A closed form is not necessarily exact. This can be seen as follows. We shall integrate each form along the circle belonging to the $z = const$ plane. Their center is located at the z–axis. We parametrize the circle in the following way $C(t) \to (\cos t, \sin t)$ where $t \in [0, 2\pi]$. The integrals read

$$\oint_{C(t)} \overset{1}{e} = 2\lambda_0 \int_0^{2\pi} 0\, dt = 0, \tag{2.92}$$

$$\oint_{C(t)} \overset{1}{h} = \frac{2I_0}{c}\int_0^{2\pi} 1\, dt = \frac{4\pi}{c}I_0. \tag{2.93}$$

The second integral does not vanish. It means that the integral between two points depends on the integration path that connects these points. Hence, the differential form $\overset{1}{h}$ is not exact (it cannot be represented by the exterior derivative of some zero-form). The form $\overset{1}{h}$ is not definite at $r = 0$ (vector $\hat{\phi}$ is not definite at the origin) what requires exclusion of this point from domain of the form. Consequently, the form is definite in the region which is not contractible to a point *i.e.* there is no satisfied the assumption of Poincaré Lemma. On the other hand, the form $\overset{1}{e}$ is exact and it reads

$$\overset{1}{e} = d\left[2\lambda_0 \ln\sqrt{x^2+y^2} + const\right].$$

Volume form

The form

$$\overset{4}{\omega} = \omega_{0123}\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$$

is a 4-form in Minkowski spacetime. It is closed, $d\overset{4}{\omega} = 0$, because its degree is equal to number of spacetime dimmensions. The example of 4-form is *volume form* $\overset{4}{\omega}$ which is given by

Volume form

$$\mathbf{vol} = *1 = \frac{1}{4!}\sqrt{-g}\,\epsilon_{\mu\nu\alpha\beta}\, dx^\mu \wedge dx^\nu \wedge dx^\alpha \wedge dx^\beta. \tag{2.94}$$

Maxwell's equations in formalism of differential forms

Exterior derivative of Faraday's form $\mathbf{F}$ reads

Exterior derivative of Faraday's form

$$\begin{aligned} d\mathbf{F} =& \frac{1}{2} dF_{\mu\nu} \wedge dx^\mu \wedge dx^\nu = \frac{1}{2}\partial_\lambda F_{\mu\nu}\, dx^\lambda \wedge dx^\mu \wedge dx^\nu \\ =& \frac{1}{3!}\Big[\partial_\lambda F_{\mu\nu}\, dx^\lambda \wedge dx^\mu \wedge dx^\nu + \partial_\mu F_{\nu\lambda}\, dx^\mu \wedge dx^\nu \wedge dx^\lambda \\ &+ \partial_\nu F_{\lambda\mu}\, dx^\nu \wedge dx^\lambda \wedge dx^\mu\Big] \\ =& \frac{1}{3!}\underbrace{\left[\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu}\right]}_{0} dx^\lambda \wedge dx^\mu \wedge dx^\nu. \end{aligned} \tag{2.95}$$

The coefficients of the resulting differential form are equal to electromagnetic Bianchi identities. It allows to represent *second pair of Maxwell's equations* as exterior derivative of Faraday's form, namely

Second pair of Maxwell's equations

$$\boxed{d\mathbf{F} = 0.} \tag{2.96}$$

On the other hand, first pair of Maxwell's equations can be expressed using Maxwell's form $^*\mathbf{F}$. Its exterior derivative has the form

Exterior derivative of Maxwell's form

$$\begin{aligned} d^*\mathbf{F} &= \frac{1}{2} d^*F_{\alpha\beta} \wedge dx^\alpha \wedge dx^\beta \\ &= \frac{1}{3!}\left[\partial_\gamma{}^*F_{\alpha\beta} + \partial_\alpha{}^*F_{\beta\gamma} + \partial_\beta{}^*F_{\gamma\alpha}\right] dx^\gamma \wedge dx^\alpha \wedge dx^\beta. \end{aligned} \tag{2.97}$$

In spite of formal similarity between expressions (2.95) and (2.97) their physical content is quite different. The expression (2.97) reads

$$\begin{aligned} d^*\mathbf{F} = & \left[\partial_1{}^*F_{23} + \partial_2{}^*F_{31} + \partial_3{}^*F_{12}\right] dx^1 \wedge dx^2 \wedge dx^3 \\ &+ \left[\partial_0{}^*F_{23} + \partial_2{}^*F_{30} + \partial_3{}^*F_{02}\right] dx^0 \wedge dx^2 \wedge dx^3 \\ &+ \left[\partial_0{}^*F_{31} + \partial_3{}^*F_{10} + \partial_1{}^*F_{03}\right] dx^0 \wedge dx^3 \wedge dx^1 \\ &+ \left[\partial_0{}^*F_{12} + \partial_1{}^*F_{20} + \partial_2{}^*F_{01}\right] dx^0 \wedge dx^1 \wedge dx^2, \end{aligned}$$

where dual components ${}^*F_{\mu\nu}$ are given in terms of components $F_{\alpha\beta}$. It leads to expression containing left hand sides of Maxwell's equations

$$\begin{aligned} d^*\mathbf{F} = &\underbrace{[-\partial_1 E^1 - \partial_2 E^2 - \partial_3 E^3]}_{-\nabla\cdot\mathbf{E}} dx^1 \wedge dx^2 \wedge dx^3 \\ &+ \underbrace{[-\partial_0 E^1 + \partial_2 B^3 - \partial_3 B^2]}_{-\partial_0 E^1 + (\nabla\times\mathbf{B})^1} dx^0 \wedge dx^2 \wedge dx^3 \\ &+ \underbrace{[-\partial_0 E^2 + \partial_3 B^1 - \partial_1 B^3]}_{-\partial_0 E^2 + (\nabla\times\mathbf{B})^2} dx^0 \wedge dx^3 \wedge dx^1 \\ &+ \underbrace{[-\partial_0 E^3 + \partial_1 B^2 - \partial_2 B^1]}_{-\partial_0 E^3 + (\nabla\times\mathbf{B})^3} dx^0 \wedge dx^1 \wedge dx^2. \end{aligned} \tag{2.98}$$

Making use of Maxwell's equations we substitute right hand sides of (2.98) by sources what gives

$$\begin{aligned} d^*\mathbf{F} =& \frac{4\pi}{c}[-J^0\, dx^1 \wedge dx^2 \wedge dx^3 + J^1\, dx^0 \wedge dx^2 \wedge dx^3 \\ &+ J^2\, dx^0 \wedge dx^3 \wedge dx^1 + J^3\, dx^0 \wedge dx^1 \wedge dx^2] \\ =& -\frac{4\pi}{c}[{}^*J_{123}\, dx^1 \wedge dx^2 \wedge dx^3 + {}^*J_{023}\, dx^0 \wedge dx^2 \wedge dx^3 \\ &+ {}^*J_{031}\, dx^0 \wedge dx^3 \wedge dx^1 + {}^*J_{012}\, dx^0 \wedge dx^1 \wedge dx^2] \\ =& -\frac{4\pi}{c}\frac{1}{3!}{}^*J_{\gamma\alpha\beta}\, dx^\gamma \wedge dx^\alpha \wedge dx^\beta = -\frac{4\pi}{c}{}^*\mathbf{J}. \end{aligned}$$

The first pair of Maxwell's equations takes the form

First pair of Maxwell's equations

$$\boxed{d^*\mathbf{F} = -\frac{4\pi}{c}{}^*\mathbf{J}.} \tag{2.99}$$

Note, that the property $dd = 0$ implies *the continuity equation*

Continuity equation

$$\boxed{d^*\mathbf{J} = 0,} \tag{2.100}$$

where

$$\begin{aligned} d^*\mathbf{J} &= dJ^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 - dJ^1 \wedge dx^0 \wedge dx^2 \wedge dx^3 \\ &\quad - dJ^2 \wedge dx^0 \wedge dx^3 \wedge dx^1 - dJ^3 \wedge dx^0 \wedge dx^1 \wedge dx^2 \\ &= (\partial_\mu J^\mu)\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3. \end{aligned} \tag{2.101}$$

2.5 *Integral form of Maxwell's equations*

Integral form of Maxwell's equations which commonly appears in the literature is, in fact, not a fully integral form because it contains temporal derivatives. In order to get fully integral form one needs to integrate Ampere-Maxwell's law and Faraday's law over temporal coordinate. It leads to surface integral calculated on two-dimensional surfaces parametrizes by temporal coordinate. Differential forms allow us to obtain explicitely *invariant* formulation of fully integral Maxwell's equations.

Integration of differential forms

The integral of differential form is given by integration of a function representing values that differential form takes on vectors $\mathbf{v}_i \in \mathbf{T}(p)$. Let us consider $k-$dimensional regular *surface sector* $\mathcal{P}$ embedded in some region of Minkowski spacetime $\mathcal{O} \subset \mathcal{A}$ $(k \leq N)$

Parametrization of the surface sector $\mathcal{P}$

$$\mathbb{R}^k \supset \mathcal{D} \ni (t^1, \ldots, t^k) \overset{\Phi}{\mapsto} \boldsymbol{x}(t^1, \ldots, t^k) \in \mathbb{R}^N \tag{2.102}$$

such that for all values of parameters $(t^1, \ldots, t^k) \in \mathcal{D}$ the matrix

$$\hat{\theta} := \left[\frac{\partial x^\alpha}{\partial t^j}\right], \qquad \alpha = 0, 1, 2, 3 \qquad j = 1, \ldots, k \tag{2.103}$$

has rank k.

Rank k transformation matrix

A regular surface sector is called *oriented* if a certain parametrization of the type (2.102) has been chosen, and admitted changes of parametrization

Fixing orientation of the surface sector

$$\tau^i = \tau^i(t^1, \ldots, t^k), \qquad i = 1, \ldots, k \tag{2.104}$$

are such that their Jacobian is positive

$$\det\left[\frac{\partial \tau^i}{\partial t^j}\right] > 0.$$

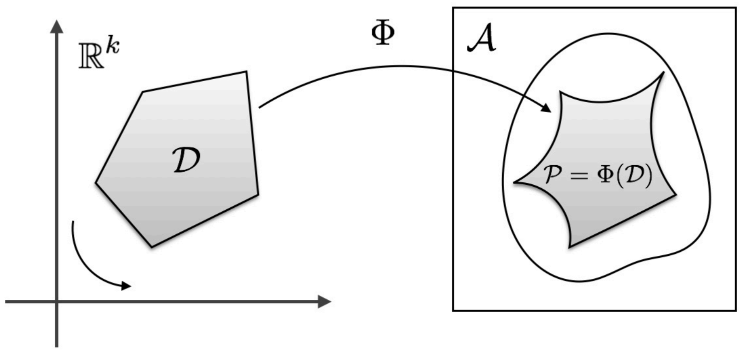


Figure 2.3: Oriented surface sector embedded in $\mathcal{O}$.

For the differential form of degree k

$$\overset{k}{\omega} = \frac{1}{k!}\omega_{\alpha_1 \ldots \alpha_k}(x^0, \ldots, x^3)\, dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_k}, \tag{2.105}$$

given in some region $\mathcal{O}$, the integral of this form over some regular surface sector $\mathcal{P}$ is defined as follows

Definition of the integral of $k-$form

$$\boxed{\int_{\mathcal{P}} \overset{k}{\omega} := \int_{\mathcal{D}} \overset{k}{\omega}(\mathbf{v_1}, \ldots, \mathbf{v}_k) dt^1 \ldots dt^k,} \tag{2.106}$$

where $\overset{k}{\omega}(\mathbf{v_1}, \ldots, \mathbf{v}_k)$ is the value of the form on vectors

$$\mathbf{v}_j := \frac{\partial x^\alpha}{\partial t^j} \mathbf{e}_\alpha(p), \qquad \alpha = 0, 1, 2, 3, \qquad j = 1, \ldots, k. \tag{2.107}$$

Since

$$dx^{\alpha_i}(\mathbf{v}_j) = dx^{\alpha_i}(v_j^\beta \mathbf{e}_\beta) = v_j^\beta dx^{\alpha_i}(\mathbf{e}_\beta) = v_j^\beta \delta_\beta^{\alpha_i} = v_j^{\alpha_i} = \frac{\partial x^{\alpha_i}}{\partial t^j} \tag{2.108}$$

are partial derivatives, then applying (2.62) one gets in (2.106) the determinant of partial derivatives. This determinant is just the Jacobian of transformation $x^\alpha = x^\alpha(t^1, \ldots, t^k)$ *i.e.*

$$\det \left[\frac{\partial x^{\alpha_i}}{\partial t^j} \right] \equiv \frac{\partial (x^{\alpha_1} \ldots x^{\alpha_k})}{\partial (t^1 \ldots t^k)} \qquad i, j = 1, 2, \ldots, k. \tag{2.109}$$

Thus the integral (2.106) reads

Integral of $k-$form

$$\boxed{\int_{\mathcal{P}} \overset{k}{\omega} := \int_{\mathcal{D}} \frac{1}{k!} \omega_{\alpha_1 \ldots \alpha_k} \frac{\partial (x^{\alpha_1} \ldots x^{\alpha_k})}{\partial (t^1 \ldots t^k)} dt^1 \ldots dt^k.} \tag{2.110}$$

When the integral of the form $\overset{k}{\omega}$ splits into a finite or countable number of integrals over orientable surfaces $\mathcal{P}_i$ which have no common points, then the integral of the form $\overset{k}{\omega}$ over hypersurface $\mathcal{P}$ is a sum of integrals over $\mathcal{P}_i$

Splitting the integral into integrals over distinct orientable surfaces

$$\int_{\mathcal{P}} \overset{k}{\omega} := \sum_i \int_{\mathcal{P}_i} \overset{k}{\omega}. \tag{2.111}$$

Let us consider an oriented hypersurface $\partial\mathcal{P}$ that consists on regular sectors $\partial\mathcal{P}_1, \ldots, \partial\mathcal{P}_M$. The hypersurface $\partial\mathcal{P} = \Phi(\partial\mathcal{D})$ is the image of faces S_i, $i = 1, \ldots, M$ of the polyhedron $\mathcal{D}$ in the space of parameters $\{t_1, \ldots t_k\}$ *i.e.*

Induced orientation at the border

$$\partial\mathcal{D} = S_1 \cup S_2 \cup \ldots \cup S_M.$$

The orientation introduced on $\mathcal{D}$ allows to introduce invariantly the orientation on $\partial\mathcal{P}$. This is so-called *induced orientation*. One can choose outward unit vector $\mathbf{n}$ of the polyhedron $\mathcal{D}$. It is perpendicular to S_i at any point p localised in its interior. The orientation on S_i is determined by the basis $\mathbf{f}_1, \ldots, \mathbf{f}_{k-1}$, and chosen in such a way that $\mathbf{n}, \mathbf{f}_1, \ldots, \mathbf{f}_{k-1}$ has orientation compatible with the orientation of $\mathbb{R}^k$.

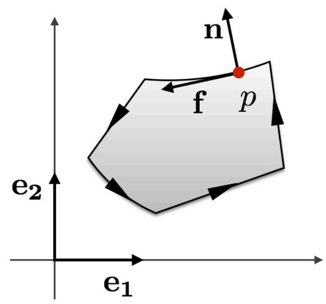


Figure 2.4: Induced orientation.

Integral of volume form

We shall consider the integral of volume form

$$\mathbf{vol} = *1 = \frac{1}{4!} \sqrt{-g}\, \epsilon_{\mu\nu\alpha\beta}\, dx^\mu \wedge dx^\nu \wedge dx^\alpha \wedge dx^\beta.$$

over some four-dimensional region of Minkowski spacetime $\mathcal{P} \subset \mathcal{A}$. This integral is given by expression

$$\int_{\mathcal{P}} \mathbf{vol} = \int_{\mathcal{D}} \frac{1}{4!}\,\varepsilon_{\mu\nu\alpha\beta}\frac{\partial(x^\mu x^\nu x^\alpha x^\beta)}{\partial(t^1t^2t^3t^4)}dt^1dt^2dt^3dt^4 = \int_{\mathcal{D}} d^4\Omega \qquad (2.112)$$

where

$$\boxed{d^4\Omega := \frac{1}{4!}\varepsilon_{\mu\nu\alpha\beta}\frac{\partial(x^\mu x^\nu x^\alpha x^\beta)}{\partial(t^1t^2t^3t^4)}dt^1dt^2dt^3dt^4} \qquad (2.113)$$

is the four-dimensional volume element in Minkowski spacetime.

Stokes Theorem

Given $(k-1)$–form $\overset{k-1}{\omega}$ of class $\mathcal{C}^1$ on $\mathcal{P} \cup \partial\mathcal{P}$ the integral of this form over $\partial\mathcal{P}$ is equal to the integral of its exterior derivative over $\mathcal{P}$

Stokes theorem for differential forms

$$\boxed{\int_{\partial\mathcal{P}} \overset{k-1}{\omega} = \int_{\mathcal{P}} d\overset{k-1}{\omega}.} \qquad (2.114)$$

This theorem is of extremal importance in physics.

As example we shall consider application of the Stokes theorem for one-form and two-form that are defined in Euclidean space $\mathbb{E}^3$.

The *one-form* $\overset{1}{\omega}$ and its exterior derivative read

Stokes theorem for one-form and its relation with Stokes theorem for vector fields

$$\begin{aligned} \overset{1}{\omega} &= \omega_1 dx^1 + \omega_2 dx^2 + \omega_3 dx^3, \\ d\overset{1}{\omega} &= (\partial_2\omega_3 - \partial_3\omega_2)dx^2 \wedge dx^3 + (\partial_3\omega_1 - \partial_1\omega_3)dx^3 \wedge dx^1 \\ &\quad + (\partial_1\omega_2 - \partial_2\omega_1)dx^1 \wedge dx^2. \end{aligned} \qquad (2.115)$$

If one associates Cartesian components of certain vector field with components of the form $\overset{1}{\omega}$

$$\boldsymbol{H} \rightarrow (\omega_1, \omega_2, \omega_3)$$

(*i.e.* $H_i \equiv \omega_i$), then Stokes theorem for differential forms takes the form of familiar Stokes theorem for vector fields

$$\int_{\partial S} \overset{1}{\omega} = \int_S d\overset{1}{\omega} \qquad \Leftrightarrow \qquad \oint_C \boldsymbol{H} \cdot d\boldsymbol{x} = \int_S \nabla \times \boldsymbol{H} \cdot d\boldsymbol{S},$$

where components of the curl operator $\nabla \times \boldsymbol{H}$ are given directly by components of the exterior derivative $d\overset{1}{\omega}$. The physical example of such a vector field is *magnetic field strength*.

Similarly, in the case of *two-form* one gets

Stokes theorem for two-form and its relation with Gauss-Green theorem for vector fields

$$\begin{aligned} \overset{2}{\omega} &= \omega_{23}dx^2 \wedge dx^3 + \omega_{31}dx^3 \wedge dx^1 + \omega_{12}dx^1 \wedge dx^2, \\ d\overset{2}{\omega} &= (\partial_1\omega_{23} + \partial_2\omega_{31} + \partial_3\omega_{12})dx^1 \wedge dx^2 \wedge dx^3. \end{aligned}$$

If one associates components of a certain vector field $\boldsymbol{D}$ with components of this two-form,

$$\boldsymbol{D} \to (\omega_{23}, \omega_{31}, \omega_{12}),$$

then the exterior derivative of this form corresponds with divergence of this vector field $\nabla \cdot \boldsymbol{D}$. In such a case the Stokes theorem for differential forms results in the Gauss theorem for vector fields

$$\int_{\partial V} \overset{2}{\omega} = \int_V d\overset{2}{\omega} \qquad \Leftrightarrow \qquad \oint_S \boldsymbol{D} \cdot d\boldsymbol{S} = \int_V \nabla \cdot \boldsymbol{D}\, dV.$$

The electric dislocation vector is an example of such a vector field.

Components of strength vectors correspond with component of one-forms

$$(\mathbf{E}, \mathbf{H}) \leftrightarrow \overset{1}{\omega}$$

whereas components of inductions correspond with components of two-forms

$$(\mathbf{D}, \mathbf{B}) \leftrightarrow \overset{2}{\omega}$$

Integral form of Maxwell's equations

Let $\mathcal{P}$ be a three-dimensional orientable surface sector $\mathcal{P}$ in Minkowski spacetime. A border of this surface sector is some piecewise regular two-dimensional closed surface $\partial\mathcal{P}$. Integrating the first pair of Maxwell's equations over the region $\mathcal{P} \cup \partial\mathcal{P}$ and applying Stokes theorem one gets

First pair of Maxwell's equations

$$\boxed{\int_{\partial\mathcal{P}} {}^*\mathbf{F} = -\frac{4\pi}{c} \int_{\mathcal{P}} {}^*\mathbf{J}.} \tag{2.116}$$

The second pair of Maxwell's equations can be integrated in a similar way and it takes the form

Second pair of Maxwell's equations

$$\boxed{\int_{\partial\mathcal{P}} \mathbf{F} = 0.} \tag{2.117}$$

Applying formula (2.110) to rhs of (2.116) one gets

$$\frac{1}{c}\int_{\mathcal{P}} {}^*\mathbf{J} = \frac{1}{c}\int_{\mathcal{D}} \frac{1}{3!} {}^*J_{\alpha\beta\gamma} \frac{\partial(x^\alpha x^\beta x^\gamma)}{\partial(t^1 t^2 t^3)} dt^1 dt^2 dt^3 \tag{2.118}$$

$$= \frac{1}{c}\int_{\mathcal{D}} J^\mu \left[\frac{1}{3!}\varepsilon_{\mu\alpha_\beta\gamma} \frac{\partial(x^\alpha x^\beta x^\gamma)}{\partial(t^1 t^2 t^3)} dt^1 dt^2 dt^3\right]$$

$$= \frac{1}{c}\int_{\mathcal{P}} J^\mu d^3\Sigma_\mu \tag{2.119}$$

where three-volume element in Minkowski spacetime reads

Three-volume element in the Minkowski spacetime

$$\boxed{d^3\Sigma_\mu := \frac{1}{3!}\varepsilon_{\mu\alpha_\beta\gamma} \frac{\partial(x^\alpha x^\beta x^\gamma)}{\partial(t^1 t^2 t^3)} dt^1 dt^2 dt^3.} \tag{2.120}$$

If the four-vector J^μ has a single non-vanishing component J^0 in the laboratory reference frame, than the integral over three-dimensional region will contain only $d^3\Sigma_0$ – three–volume element on hypersurface $x^0 = const$. Hence the expression $\frac{1}{c}\int_{\mathcal{D}} J^0 d^3\Sigma_0$ has interpretation of total electric charge contained in the region $\mathcal{D}$. It allows us to conclude that the expression

Invariant definition of electric charge

$$\boxed{Q := \frac{1}{c}\int_{\mathcal{D}} J^{\mu} d^3\Sigma_{\mu}} \tag{2.121}$$

represents total electric charge in the spacetime region $\mathcal{P}$. This expression is Lorentz invariant and thus it must remain unchanged under the change of inertial reference frame (*i.e.* under the Lorentz transformations). For generic situation all components J^{μ} are different from zero.

Applying formula (2.110) to lhs of (2.116) one gets

$$\begin{aligned}
\int_{\partial\mathcal{P}} {}^*\mathbf{F} &= \frac{1}{2}\int_{\partial\mathcal{D}} {}^*F_{\alpha\beta}\frac{\partial(x^{\alpha}x^{\beta})}{\partial(u^1u^2)}du^1du^2 \\
&= \frac{1}{2}\int_{\partial\mathcal{D}} F^{\mu\nu}\left[\frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}\frac{\partial(x^{\alpha}x^{\beta})}{\partial(u^1u^2)}du^1du^2\right] \\
&= \frac{1}{2}\int_{\partial\mathcal{P}} F^{\mu\nu}d^2S_{\mu\nu}
\end{aligned} \tag{2.122}$$

where

Two-area element at the surface in the Minkowski spacetime

$$\boxed{d^2S_{\mu\nu} := \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}\frac{\partial(x^{\alpha}x^{\beta})}{\partial(u^1u^2)}du^1du^2} \tag{2.123}$$

is two-dimensional surface element and parameters u^1, u^2 are chosen in such a way, that orientation of faces S_i (border of three dimensional region) induces the orientation which is consistent with orientation of $\mathbb{R}^3$ given by t^1, t^2, t^3. Thus, lhs of the equation (2.117) takes the form

$$\begin{aligned}
\int_{\partial\mathcal{P}} \mathbf{F} &= \frac{1}{2}\int_{\partial\mathcal{D}} F_{\alpha\beta}\frac{\partial(x^{\alpha}x^{\beta})}{\partial(u^1u^2)}du^1du^2 \\
&= \frac{1}{2}\int_{\partial\mathcal{D}} \frac{1}{2}F_{\mu\nu}(\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} - \delta^{\mu}_{\beta}\delta^{\nu}_{\alpha})\frac{\partial(x^{\alpha}x^{\beta})}{\partial(u^1u^2)}du^1du^2 \\
&= -\frac{1}{2}\int_{\partial\mathcal{D}} \left[\frac{1}{2}F_{\mu\nu}\varepsilon^{\mu\nu\lambda\sigma}\right]\left[\frac{1}{2}\varepsilon_{\alpha\beta\lambda\sigma}\frac{\partial(x^{\alpha}x^{\beta})}{\partial(u^1u^2)}du^1du^2\right] \\
&= -\frac{1}{2}\int_{\partial\mathcal{P}} {}^*F^{\lambda\sigma}d^2S_{\lambda\sigma}
\end{aligned} \tag{2.124}$$

We define two *generalized fluxes*

Generalized fluxes

$$\boxed{\Phi(\partial\mathcal{P}) := -\frac{1}{2}\oint_{\partial\mathcal{P}} F^{\mu\nu}d^2S_{\mu\nu}}$$

and

$$\boxed{\widetilde{\Phi}(\partial\mathcal{P}) := +\frac{1}{2}\oint_{\partial\mathcal{P}} {}^*F^{\mu\nu}d^2S_{\mu\nu}.}$$

They are *invariant* under Lorentz transformations. Maxwell's equations in its integral form are given as generalized fluxes eveluated on surfaces $\partial\mathcal{P}$

Maxwell's equations as generalized fluxes

$$\boxed{\Phi(\partial\mathcal{P}) = 4\pi Q, \qquad \widetilde{\Phi}(\partial\mathcal{P}) = 0.} \tag{2.125}$$

This form of Maxwell's equation is physically very sound. It shows that the content of Maxwell's equations is the *relation between generalized fluxes and the electric charges*. Moreover, the explicit invariance of integral Maxwell's equations follows from the fact that they are built of Lorentz invariants. For a particular choice of the surface $\partial\mathcal{P}$ (border of purely spatial surface) these equations reduce to electric and magneticGauss's law in its integral form. On the other hand, when the surface is the cylinder with height measured by temporal coordinate x^0, then the equations give integral version of Ampere-Maxwell law and Faraday's law.

Example

Let us consider purely spatial surface $\mathcal{P}$ in the form of cube with side a and the center located at the origin of reference frame. We choose the following parametrization

$$x^0 = const, \qquad x^1 = t^1, \qquad x^2 = t^2, \qquad x^3 = t^3.$$

The vector normal to the upper face $x^3 = \frac{a}{2}$ is given by $\mathbf{n} = \mathbf{e}_3$. The infinitesimal area element at this surface reads

Electric flux

$$dS_{03} = \varepsilon_{0312}\frac{\partial(x^1x^2)}{\partial(t^1t^2)}dt^1dt^2 = dt^1dt^2.$$

Let $\mathbf{E} \to (0,0,E)$ be the electric field $F^{03} = -E$ (uniform in space). It follows that the flux of the electric field through the face $x^3 = \frac{a}{2}$ reads

Magnetic flux

$$\Phi(x^3 = \tfrac{a}{2}) = -\frac{1}{2}\int F^{\mu\nu}d^2S_{\mu\nu} = \int_{-a/2}^{a/2}dt^1\int_{-a/2}^{a/2}dt^2E = a^2E.$$

If we substitute the electric field by some uniform magnetic field with components $\mathbf{B} \to (0,0,B)$, where ${}^*F^{03} = B$, then

$$\widetilde{\Phi}(x^3 = \tfrac{a}{2}) = +\frac{1}{2}\int {}^*F^{\mu\nu}d^2S_{\mu\nu} = \int_{-a/2}^{a/2}dt^1\int_{-a/2}^{a/2}dt^2B = a^2B.$$

Chapter 3
Electromagnetic fields of uniformly moving charges

3.1 Electromagnetic invariants

The formulas which give transformation rules for the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ can be obtained directly from the expression

$$F'^{\mu\nu}(x') = L^{\mu}{}_{\alpha} L^{\nu}{}_{\beta} F^{\alpha\beta}(x) \tag{3.1}$$

which gives transformation law of the electromagnetic field tensor. There are many *invariant* expressions that can be constructed from four-vector A_μ and its derivatives but only some of them are physically suitable. Since we are interested in *gauge invariant expressions* then we look at tensors $F^{\mu\nu}$ and $F^{\mu\nu *}$. There are only two independent and Lorentz-invariant combinations that can be constructed from these tensors, namely

Lorentz invariants quadratic in fields

$$I_1 := F^{\mu\nu} F_{\mu\nu}, \qquad I_2 := F^{\mu\nu *} F_{\mu\nu}. \tag{3.2}$$

First of these invariants I_1 is very important because it is proportional to Lagrangian density for electromagnetic field. In terms of components of electric and magnetic fields the invariants (3.2) read

$$\begin{aligned} I_1 &= 2F_{0i}F^{0i} + F_{ij}F^{ij} = -2F_{0i}F_{0i} + F_{ij}F_{ij} \\ &= -2E^i E^i + (-\epsilon_{ijk}B^k)(-\epsilon_{ijl}B^l) = 2(\boldsymbol{B}^2 - \boldsymbol{E}^2) \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} I_2 &= 2F_{0i}{}^{*}F^{0i} + F_{ij}{}^{*}F^{ij} = -2F_{0i}{}^{*}F_{0i} + F_{ij}{}^{*}F_{ij} \\ &= -2E^i(-B^i) + (-\epsilon_{ijk}B^k)(-\epsilon_{ijl}E^l) = 4\boldsymbol{E}\cdot\boldsymbol{B}. \end{aligned} \tag{3.4}$$

where ${}^{*}\boldsymbol{E} = -\boldsymbol{B}$ and ${}^{*}\boldsymbol{B} = \boldsymbol{E}$. Note that in the convention $\varepsilon_{0123} = -1$ the second invariant has opposite sign $I_2 \to -I_2$.

3.2 Transformation of fields under Lorentz boost

We consider the most general Lorentz boost from inertial frame of reference S to S'. The transformation is parameterized by three-component velocity vector $\beta^i = \frac{V^i}{c}$,

General boost

$$(\hat{L})^\mu{}_\nu = (\hat{\Lambda})^\mu{}_\nu = \left(\begin{array}{c|c} \gamma & -\gamma\beta^j \\ \hline -\gamma\beta^i & \delta_{ij} + \frac{\gamma-1}{\beta^2}\beta^i\beta^j \end{array}\right).$$

where $\mathbf{V}$ is the velocity of S' with respect to S. The electric and magnetic components are denoted by E^i and B^i and E'^i and B'^i and they give components of tensors $F^{\mu\nu}$ and $F'^{\mu\nu}$. Transformations of components E^i and B^i can be deduced form the transformation law for $F^{\mu\nu}$. Thus, the electric field in S' reads

$$\begin{aligned}
E'^i &= F'^{i0} = L^i{}_\alpha L^0{}_\beta F^{\alpha\beta} = L^i{}_0 L^0{}_j F^{0j} + L^i{}_j L^0{}_0 F^{j0} + L^i{}_j L^0{}_k F^{jk} \\
&= \left[L^0{}_0 L^i{}_j - L^i{}_0 L^0{}_j\right] F^{j0} + L^i{}_j L^0{}_k F^{jk} \\
&= \left[\gamma\left(\delta_{ij} + \frac{\gamma-1}{\beta^2}\beta^i\beta^j\right) - (-\gamma\beta^i)(-\gamma\beta^j)\right] E^j \\
&+ \left(\delta_{ij} + \frac{\gamma-1}{\beta^2}\beta^i\beta^j\right)(-\gamma\beta^k)(-\epsilon_{jkl}B^l) \\
&= \gamma E^i + \frac{\gamma(\gamma-1)}{\beta^2}(\boldsymbol{\beta}\cdot\boldsymbol{E})\beta^i - \gamma^2(\boldsymbol{\beta}\cdot\boldsymbol{B})\beta^i + \gamma\epsilon_{ikl}\beta^k B^l \\
&= \gamma\left[E^i + (\boldsymbol{\beta}\times\boldsymbol{B})^i\right] + \gamma\left(\frac{\gamma-1}{\beta^2} - \gamma\right)(\boldsymbol{\beta}\cdot\boldsymbol{E})\beta^i
\end{aligned} \tag{3.5}$$

where

$$\frac{\gamma-1}{\beta^2} - \gamma = \frac{\gamma^2}{\gamma^2-1}(\gamma-1) - \gamma = \frac{\gamma^2}{\gamma+1} - \frac{\gamma^2+\gamma}{\gamma+1} = -\frac{\gamma}{\gamma+1}.$$

Finally, we get

Electric field in S'

$$\boxed{\boldsymbol{E}' = \gamma\left[\boldsymbol{E} + \boldsymbol{\beta}\times\boldsymbol{B}\right] - \frac{\gamma^2}{\gamma+1}(\boldsymbol{\beta}\cdot\boldsymbol{E})\boldsymbol{\beta}.} \tag{3.6}$$

This formula, together with the duality transformation $^*\boldsymbol{E} = -\boldsymbol{B}$, $^*\boldsymbol{B} = \boldsymbol{E}$, leads to [1]

[1] Exercise: Derive formula (3.7) without using duality transformation.

$$\boxed{\boldsymbol{B}' = \gamma\left[\boldsymbol{B} - \boldsymbol{\beta}\times\boldsymbol{E}\right] - \frac{\gamma^2}{\gamma+1}(\boldsymbol{\beta}\cdot\boldsymbol{B})\boldsymbol{\beta}.} \tag{3.7}$$

Magnetic field in S'

In order to make clearer the meaning of obtained formulas we decompose both fields in two components: parallel and perpendicular to the vector of velocity $\boldsymbol{\beta}$. Thus, we define

Decomposition of fields on parallel and perpendicular part

$$\boldsymbol{n} := \frac{\boldsymbol{\beta}}{\beta}, \qquad \boldsymbol{E}_\parallel := (\boldsymbol{E}\cdot\boldsymbol{n})\boldsymbol{n}, \qquad \boldsymbol{E}_\perp := \boldsymbol{E} - \boldsymbol{E}_\parallel \tag{3.8}$$

and similarly for magnetic components. Formulas (3.6), (3.7) take the form

$$\boldsymbol{E}' = \gamma\left[\boldsymbol{E} + \boldsymbol{\beta}\times\boldsymbol{B}\right] - (\gamma-1)\boldsymbol{E}_\parallel, \tag{3.9}$$

$$\boldsymbol{B}' = \gamma\left[\boldsymbol{B} - \boldsymbol{\beta}\times\boldsymbol{E}\right] - (\gamma-1)\boldsymbol{B}_\parallel, \tag{3.10}$$

where we have made use of relation $\frac{\gamma^2}{\gamma+1} = \frac{\gamma-1}{\beta^2}$. Taking a scalar product with vector $\boldsymbol{n}$ we get

Parallel components do not transform

$$\boldsymbol{E}'_{\parallel} = \boldsymbol{E}_{\parallel}, \qquad \boldsymbol{B}'_{\parallel} = \boldsymbol{B}_{\parallel}. \tag{3.11}$$

Next, plugging $\boldsymbol{E} = \boldsymbol{E}_{\parallel} + \boldsymbol{E}_{\perp}$ and $\boldsymbol{B} = \boldsymbol{B}_{\parallel} + \boldsymbol{B}_{\perp}$into (3.9) and (3.10) and making use of (3.11) we obtain

Only perpendicular components participate in transformation

$$\begin{aligned} \boldsymbol{E}'_{\perp} &= \gamma\left[\boldsymbol{E}_{\perp} + \boldsymbol{\beta}\times\boldsymbol{B}_{\perp}\right], \\ \boldsymbol{B}'_{\perp} &= \gamma\left[\boldsymbol{B}_{\perp} - \boldsymbol{\beta}\times\boldsymbol{E}_{\perp}\right]. \end{aligned} \tag{3.12}$$

Expressions (3.12) show that only perpendicular components of the electric and magnetic field transform. Moreover, the Lorentz transformation mixes electric and magnetic components. It means that components E^i and B^i *do not transform as components of vectors* under Lorentz boosts. In order to justify this statement we compare the obrained transformation law with transformation of four-vectors which connects events in Minkowski spacetime. Components of any such *vector* (four-vector) transform exactly as components of dx^μ *i.e.*

Electric and magnetic field are not vectors under Lorentz boosts

Transformation of a typical Lorentz vector

$$dx'^0 = \gamma(dx^0 - \boldsymbol{\beta}\cdot d\boldsymbol{x}) = \gamma(dx^0 - \beta dx_{\parallel}) \tag{3.13}$$

$$d\boldsymbol{x}' = d\boldsymbol{x} - \gamma\boldsymbol{\beta}dx^0 + \underbrace{\frac{\gamma-1}{\beta^2}(\boldsymbol{\beta}\cdot d\boldsymbol{x})\boldsymbol{\beta}}_{(\gamma-1)d\boldsymbol{x}_{\parallel}}. \tag{3.14}$$

Multiplying (3.14) by $(\ \cdot\boldsymbol{n})\boldsymbol{n}$ we get

$$d\boldsymbol{x}'_{\parallel} = \gamma(-\boldsymbol{\beta}dx^0 + d\boldsymbol{x}_{\parallel}).$$

It follows that the perpendicular part $d\boldsymbol{x}_{\perp}$ does not transform at all! It shows that electric and magnetic components do not transform as components of vectors when a transformation is the Lorentz boost.

The fact that electric and magnetic field are often considered as *vector fields* is justified by the choice of transformations. Such transformations leave the zero-component of four-vectors unchanged. In next section we look in more details at this question.

3.3 *Transformation under spatial rotations*

We consider rotation in the x^2x^3–plane. The Lorentz matrix and the electromagnetic field tensor read

Rotation around axis x^1

$$L^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\phi & \sin\phi \\ 0 & 0 & -\sin\phi & \cos\phi \end{pmatrix}, \qquad F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}.$$

Components of the electric field in the frame S' have the form

$$E'^1 = F'^{10} = L^1{}_\alpha L^0{}_\beta F^{\alpha\beta} = L^1{}_0 L^0{}_0 F^{10} = E^1$$
$$E'^2 = F'^{20} = L^2{}_\alpha L^0{}_\beta F^{\alpha\beta} = L^2{}_2 L^0{}_0 F^{20} + L^2{}_3 L^0{}_0 F^{30} = \cos\phi\, E^2 + \sin\phi\, E^3$$
$$E'^2 = F'^{30} = L^3{}_\alpha L^0{}_\beta F^{\alpha\beta} = L^3{}_2 L^0{}_0 F^{20} + L^3{}_3 L^0{}_0 F^{30} = \sin\phi\, E^2 - \cos\phi\, E^3.$$

It follows from the above expressions that rotation mixes only electric components. It can be written in terms of three-dimensional rotation matrix

Components of electric field have vector law of transformation under rotation

$$\begin{pmatrix} E'^1 \\ E'^2 \\ E'^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} E^1 \\ E^2 \\ E^3 \end{pmatrix}. \tag{3.15}$$

Similarly, transformation of magnetic components gives

$$B'^1 = F'^{32} = L^3{}_\alpha L^2{}_\beta F^{\alpha\beta} = L^3{}_2 L^2{}_3 F^{23} + L^3{}_3 L^2{}_2 F^{32} = (\sin^2\phi + \cos^2\phi) B^1$$
$$B'^2 = F'^{13} = L^1{}_\alpha L^3{}_\beta F^{\alpha\beta} = L^1{}_1 L^3{}_2 F^{12} + L^1{}_1 L^3{}_3 F^{13} = \cos\phi\, B^2 + \sin\phi\, B^3$$
$$B'^3 = F'^{21} = L^2{}_\alpha L^1{}_\beta F^{\alpha\beta} = L^2{}_2 L^1{}_1 F^{21} + L^2{}_3 L^1{}_1 F^{31} = -\sin\phi\, B^2 + \cos\phi\, B^3.$$

Transformation of magnetic components can be cast in the form

Magnetic field transform as vector under rotations

$$\begin{pmatrix} B'^1 \\ B'^2 \\ B'^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} B^1 \\ B^2 \\ B^3 \end{pmatrix}. \tag{3.16}$$

We conclude that components of the electromagnetic field tensor

$$\boldsymbol{E} \to (F^{10}, F^{20}, F^{30}), \qquad \boldsymbol{B} \to (F^{32}, F^{13}, F^{21})$$

transform as vector components *under rotations*.

3.4 *Transformation under spatial reflections*

The Lorentz matrix for spatial reflections P is of the form

$$L^\mu{}_\nu = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -\mathbb{1} \end{array}\right).$$

The electric field components in the reference frame S' read

$$E'^i = L^i{}_\alpha L^0{}_\beta F^{\alpha\beta} = L^i{}_j L^0{}_0 F^{j0} = -\delta_{ij} F^{j0} = -F^{i0} = -E^i.$$

The magnetic components transform as

$$\begin{aligned} B'^i &= -\epsilon_{ijk} F'^{jk} = -\epsilon_{ijk} L^j{}_\alpha L^k{}_\beta F^{\alpha\beta} = -\epsilon_{ijk} L^j{}_l L^k{}_m F^{lm} \\ &= -\epsilon_{ijk} (-\delta_{jl})(-\delta_{km}) F^{lm} = -\epsilon_{ijk} F^{jk} = B^i. \end{aligned}$$

It shows that there is fundamental difference in transformation of electric and the magnetic components under spatial reflections

Magnetic field components do not change the sign under reflections

$$\boxed{\boldsymbol{E}' = -\boldsymbol{E}, \qquad \boldsymbol{B}' = \boldsymbol{B}.} \tag{3.17}$$

The magnetic field is called *pseudovector* because it dos not change the sign under reflections.

Comment. Equalities (3.17) should be understood in the following sense: for *passive transformations* (the frame transformations) components of the fields transform, whereas the fields itself remains unchanged *i.e.* $E'^i\mathbf{e}'_i = E^i\mathbf{e}_i$ where $\mathbf{e}'_i = -\mathbf{e}_i$.

3.5 *Electromagnetic field of a point-like electric charge in an uniform motion*

We consider a point-like particle with charge q. The particle moves with velocity $\mathbf{V} = c\boldsymbol{\beta}$ in the laboratory reference frame S. The rest frame of the particle is denoted by S'. We assume that origins of both frames S and S' coincide at $t' = 0 = t$. The field of the charge q in S' is the static Colulomb field given by the zero-component of A'^μ

Electric point-like charge at rest in S'

$$A'^\mu(x') \to \left(\frac{q}{|\boldsymbol{x}'|}, 0, 0, 0\right), \qquad |\boldsymbol{x}'| := \sqrt{x'^i x'^i}. \tag{3.18}$$

Thus, the electric field strength is given by

$$\boldsymbol{E}'(x') = -\nabla' A'^0(x') = q\frac{\boldsymbol{x}'}{|\boldsymbol{x}'|^3}. \tag{3.19}$$

What is the form of electric and magnetic field in S? We shall solve this problem by computing the four-potential $A^\mu(x)$ in the laboratory frame S and then calculating the electromagnetic field tensor $F_{\mu\nu}$ which allows us to read $E^i(x)$ and $B^i(x)$. The four-potential of the charge in the laboratory reference frame reads

Transformation to the Laboratory reference frame S

$$A^\mu(x) = (\hat{L}^{-1}(\boldsymbol{\beta}))^\mu{}_\nu A'^\nu(x'), \tag{3.20}$$

where one has to write all the coordinates of the four-vector x'^μ in terms of components x^μ, namely

$$x'^\mu = (\hat{L}(\boldsymbol{\beta}))^\mu{}_\nu x^\nu. \tag{3.21}$$

We denote by β^i the Cartesian components of $\boldsymbol{\beta}$ in S. In absence of rotation the inverse Lorentz boost is just a boost with velocity $-\beta^i$

$$(\hat{L}^{-1}(\boldsymbol{\beta}))^\mu{}_\nu = (\hat{L}(-\boldsymbol{\beta}))^\mu{}_\nu.$$

We shall write explicitly the argument of Lorentz transformations only for "$-\boldsymbol{\beta}$". Otherwise, it is assumed that $L^\mu{}_\nu \equiv (\hat{L}(\boldsymbol{\beta}))^\mu{}_\nu$. The four-potential has components

Four potential in the laboratory reference frame S depends on a single function $A^0(x)$

$$A^\mu(x) \to (\gamma A'^0(x'), \boldsymbol{\beta}\gamma A'^0(x')) = (A^0(x), \boldsymbol{\beta} A^0(x)). \tag{3.22}$$

The components of $\boldsymbol{x}'$ transform according to

$$\begin{aligned} x'^i &= L^i{}_\nu x^\nu = L^i{}_0 x^0 + L^i{}_j x^j = -\gamma\beta^i x^0 + \left(\delta_{ij} + \frac{\gamma-1}{\beta^2}\beta^i\beta^j\right) x^j \\ &= x^i - \gamma\beta^i x^0 + \frac{\gamma-1}{\beta^2}(\boldsymbol{\beta}\cdot\boldsymbol{x})\beta^i = x^i - \gamma\beta^i x^0 + (\gamma-1)x^i_\parallel \end{aligned} \tag{3.23}$$

where $x^i_\parallel := \boldsymbol{x}_\parallel \cdot \hat{\boldsymbol{e}}_i$. The expression (3.23) can be cast in a slightly different form, namely

Spatial components of the position vector

$$\begin{aligned} x'^i &= x^i_\parallel + x^i_\perp - \gamma\beta^i x^0 + (\gamma-1)x^i_\parallel \\ &= \gamma\left(x^i_\parallel - \beta^i x^0 + \frac{1}{\gamma}x^i_\perp\right). \end{aligned} \tag{3.24}$$

We define the vector

$$\boldsymbol{r}^* := \boldsymbol{x}_\parallel - \boldsymbol{\beta}x^0 + \frac{1}{\gamma}\boldsymbol{x}_\perp \tag{3.25}$$

$$= \boldsymbol{x} - \boldsymbol{\beta}x^0 - \frac{\gamma-1}{\gamma}\boldsymbol{x}_\perp$$

$$= \boldsymbol{R} - \frac{\gamma-1}{\gamma}\left(\boldsymbol{x} - \frac{\boldsymbol{\beta}\cdot\boldsymbol{x}}{\beta^2}\boldsymbol{\beta}\right) \tag{3.26}$$

where

The relative position vector of the particle at instant of time t

$$\boldsymbol{R} := \boldsymbol{x} - \boldsymbol{\beta}x^0. \tag{3.27}$$

The length of the vector $\boldsymbol{r}^*$ reads

$$r^* \equiv |\boldsymbol{r}^*| = \sqrt{(\boldsymbol{x}_\parallel - \boldsymbol{\beta}x^0)^2 + \frac{1}{\gamma^2}\boldsymbol{x}^2_\perp} \tag{3.28}$$

$$= \sqrt{(\boldsymbol{x} - \boldsymbol{\beta}x^0)^2 - \beta^2\boldsymbol{x}^2_\perp} \tag{3.29}$$

$$= \sqrt{\boldsymbol{R}^2 - \beta^2\left(\boldsymbol{x}^2 - \frac{1}{\beta^4}(\boldsymbol{\beta}\cdot\boldsymbol{x})^2\boldsymbol{\beta}^2\right)}$$

$$= \sqrt{\boldsymbol{R}^2 + (\boldsymbol{\beta}\cdot\boldsymbol{x})^2 - \beta^2\boldsymbol{x}^2} \tag{3.30}$$

The four-potential is given in terms of its zero-component

Final form of $A^0(x)$

$$A^0(x) = \gamma\frac{q}{\sqrt{x'^i x'^i}} = \frac{q}{r^*}. \tag{3.31}$$

Now, we calculate Cartesian components of the electric field

$$\begin{aligned} E^i &= -\partial_i A^0 - \partial_0 A^i = -q\left(\partial_i\frac{1}{r^*} + \beta^i\partial_0\frac{1}{r^*}\right) \\ &= \frac{q}{r^{*2}}(\partial_i r^* + \beta^i\partial_0 r^*). \end{aligned} \tag{3.32}$$

Making use of the relation $\partial_i \boldsymbol{x} = \hat{\boldsymbol{e}}_i$ we get

$$\begin{aligned}\partial_i r^* &= \frac{1}{2r^*}\left[2\boldsymbol{R}\cdot\hat{\boldsymbol{e}}_i + 2(\boldsymbol{\beta}\cdot\boldsymbol{x})(\boldsymbol{\beta}\cdot\hat{\boldsymbol{e}}_i) - 2\beta^2\boldsymbol{x}\cdot\hat{\boldsymbol{e}}_i\right] \\ &= \frac{1}{r^*}\left[(1-\beta^2)x^i - \beta^i x^0 + (\boldsymbol{\beta}\cdot\boldsymbol{x})\beta^i\right]. \end{aligned} \tag{3.33}$$

Similarly, the expression $\partial_0 r^*$ reads

$$\partial_0 r^* = \frac{1}{r^*}\left[\beta^2 x^0 - (\boldsymbol{\beta}\cdot\boldsymbol{x})\right]. \tag{3.34}$$

Combining the above formulas we obtain expression

$$\partial_i r^* + \beta^i \partial_0 r^* = \frac{x^i - \beta^i x^0}{\gamma^2 r^*} = \frac{R^i}{\gamma^2 r^*}. \tag{3.35}$$

The electric field (3.32) reads

$$\boldsymbol{E} = \frac{q\boldsymbol{R}}{\gamma^2 r^{*3}} \tag{3.36}$$

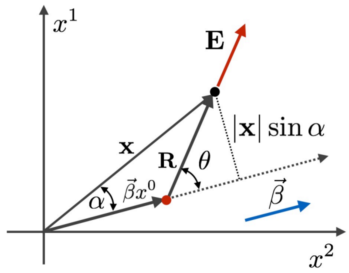


Figure 3.1: Relations between vectors in the laboratory eaference frame S.

We can write the electric field in slightly different form. Thus, we parametrize (3.29) by angle θ that form vectors $\boldsymbol{R}$ and $\boldsymbol{\beta}$

$$r^* = \sqrt{\boldsymbol{R}^2 - \beta^2\boldsymbol{x}_\perp^2} = R\sqrt{1-\beta^2\frac{|\boldsymbol{x}_\perp|^2}{R^2}} = R\sqrt{1-\beta^2\sin^2\theta} \tag{3.37}$$

It gives the electric field in the laboratory reference frame

Electric field of a charge in uniform motion

$$\boxed{\boldsymbol{E} = \frac{1-\beta^2}{(1-\beta^2\sin^2\theta)^{3/2}}\frac{q\boldsymbol{R}}{R^3}.} \tag{3.38}$$

The magnetic field can be obtained in similar way. Taking $\nabla\times\boldsymbol{A}$ one gets

$$\begin{aligned} B^i &= \epsilon_{ijk}\partial_j A^k = \epsilon_{ijk}\beta^k\partial_j A^0(x) = \epsilon_{ijk}\beta^k\partial_j\frac{q}{r^*} \\ &= -\epsilon_{ijk}\beta^k\frac{q}{r^{*2}}\partial_j r^* = \epsilon_{ikj}\beta^k\frac{q}{r^{*3}}\left[(1-\beta^2)x^j - \beta^j x^0 + (\boldsymbol{\beta}\cdot\boldsymbol{x})\beta^j\right] \\ &= \epsilon_{ikj}\beta^k\left(\frac{qx^j}{\gamma^2 r^{*3}}\right) = \epsilon_{ikj}\beta^k\left(\frac{qR^j}{\gamma^2 r^{*3}}\right). \end{aligned}$$

Thus

Magnetic field of a charge in uniform motion

$$\boxed{\boldsymbol{B} = \boldsymbol{\beta}\times\boldsymbol{E}.} \tag{3.39}$$

1. The electric field (3.38) has direction of the vector $\boldsymbol{R}$, *i.e* the vector that connects the point $\boldsymbol{\beta}x^0$ in which the charged particle encounters at x^0 and the point where the field is evaluated, see Fig.3.1. It should be stressed that the field at position $\boldsymbol{x}$ and at time x^0 *i.e.* $\boldsymbol{E}(x^0,\boldsymbol{x})$ was generated *earlier i.e.* not at x^0 but at certain instant of time determined by the intersection of the world-line of the particle with the past light cone of the event $(x^0,\boldsymbol{x})$.

2. The electric field *is not radial* because its radial component depends on the angle θ. In particular, it means that

$$E(\theta = 0) = \frac{q}{r^2}(1-\beta^2) < \frac{q}{r},$$
$$E(\theta = \frac{\pi}{2}) = \frac{q}{r^2}\frac{1}{\sqrt{1-\beta^2}} > \frac{q}{r}. \qquad (3.40)$$

This result shows that the field is "flattened" in the direction of motion of the particle.

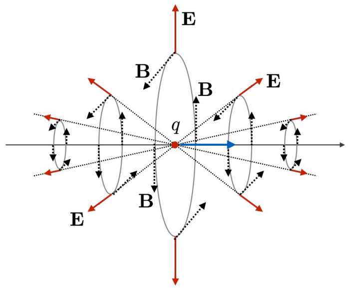


Figure 3.2: Electric and magnetic field of the uniformly moving charge in S.

3. One can conclude from comment 2 that circulation of the electric field along the closed loop cannot vanish $\oint \boldsymbol{E}\cdot d\boldsymbol{l} \neq 0$. For instance, one can take a closed loop that consists of two radial segments with non-equivalent angles θ and two circular segments. Only the radial segments contribute to the integral. Since the radial integrations are performed in two regions that differ by intensity of the electric field then they cannot cancel out. It means that the electric field of uniformly moving electric charge *is not static*. According to Faraday's law there exists *magnetic field* associated with such electric field.

Electric field is not static

3.6 *The electromagnetic field of electric dipole in uniform motion*

We shall consider an electric dipole with the moment $\boldsymbol{p}_0$ (measured in the dipole rest frame S'). The four-potential of the dipole in S' reads

Four potential of electric dipole in rest frame of dipole

$$A'^{\mu}(x') = (A'^0(x'), \boldsymbol{0}), \qquad A'^0(x') = \frac{\boldsymbol{p}_0\cdot\boldsymbol{x}'}{|\boldsymbol{x}'|^3}. \qquad (3.41)$$

In S' there is only electric field

$$E'^i = \frac{3(\boldsymbol{p}_0\cdot\boldsymbol{x}')x'^i - |\boldsymbol{x}'|^2 p_0^i}{|\boldsymbol{x}'|^5}, \qquad B'^i = 0. \qquad (3.42)$$

The four-potential in the laboratory reference frame reads

$$A^{\mu}(x) = \left(\gamma A'^0(x'), \boldsymbol{\beta}\gamma A'^0(x')\right) = \left(A^0(x), \boldsymbol{\beta}A^0(x)\right) \qquad (3.43)$$

where

A^0 is the only one nontrivial function

$$A^0(x) = \gamma\frac{\boldsymbol{p}_0\cdot\boldsymbol{x}'}{|\boldsymbol{x}'|^3} = \gamma\frac{\boldsymbol{p}_0\cdot(\gamma\boldsymbol{r}^*)}{\gamma^3 r^{*3}} = \frac{\boldsymbol{p}_0\cdot\boldsymbol{r}^*}{\gamma r^{*3}}. \qquad (3.44)$$

The electric field has components E^i in S which read

$$\begin{aligned}
E^i(x) &= -\partial_i A^0(x) - \partial_0 A^i(x)\\
&= -\frac{\boldsymbol{p}_0}{\gamma}\cdot\left(\partial_i\frac{\boldsymbol{r}^*}{r^{*3}} + \beta^i\partial_0\frac{\boldsymbol{r}^*}{r^{*3}}\right)\\
&= \frac{\boldsymbol{p}_0}{\gamma}\cdot\left[\frac{3r^{*2}\boldsymbol{r}^*(\partial_i r^* + \beta^i\partial_0 r^*) - r^{*3}(\partial_i\boldsymbol{r}^* + \beta^i\partial_0\boldsymbol{r}^*)}{r^{*6}}\right]\\
&= \frac{3(\boldsymbol{p}_0\cdot\boldsymbol{r}^*)r^*(\partial_i r^* + \beta^i\partial_0 r^*) - r^{*2}\boldsymbol{p}_0\cdot(\partial_i\boldsymbol{r}^* + \beta^i\partial_0\boldsymbol{r}^*)}{\gamma r^{*5}}
\end{aligned} \qquad (3.45)$$

where $\partial_i r^* + \beta^i \partial_0 r^* = \frac{R^i}{\gamma^2 r^*}$ follows from (3.35) and $\partial_i \boldsymbol{r}^* + \beta^i \partial_0 \boldsymbol{r}^*$ reads

$$\begin{aligned}
\partial_i \boldsymbol{r}^* + \beta^i \partial_0 \boldsymbol{r}^* &= (\partial_i + \beta^i \partial_0) \left[\boldsymbol{x} - \boldsymbol{\beta} x^0 - \frac{\gamma - 1}{\gamma} \left(\boldsymbol{x} - \frac{\boldsymbol{\beta} \cdot \boldsymbol{x}}{\beta^2} \boldsymbol{\beta} \right) \right] \\
&= \left(1 - \frac{\gamma - 1}{\gamma} \right) \hat{\boldsymbol{e}}_i + \frac{\gamma - 1}{\gamma} \frac{\boldsymbol{\beta} \beta^i}{\beta^2} - \boldsymbol{\beta} \beta^i \\
&= \frac{1}{\gamma} \hat{\boldsymbol{e}}_i + \left(\frac{\gamma - 1}{\gamma} \frac{\gamma^2}{\gamma^2 - 1} - 1 \right) \boldsymbol{\beta} \beta^i \\
&= \frac{1}{\gamma} \hat{\boldsymbol{e}}_i - \frac{1}{\gamma + 1} \boldsymbol{\beta} \beta^i. \qquad (3.46)
\end{aligned}$$

Plugging this result into (3.45) we find

$$E^i = \frac{3(\boldsymbol{p}_0 \cdot \boldsymbol{r}^*) R^i}{\gamma^3 r^{*5}} - \frac{\boldsymbol{p}_0}{\gamma r^{*3}} \cdot \left[\frac{1}{\gamma} \hat{\boldsymbol{e}}_i - \frac{1}{\gamma + 1} \boldsymbol{\beta} \beta^i \right], \qquad (3.47)$$

where $\frac{1}{\gamma+1} = \frac{\gamma - 1}{\gamma^2 \beta^2}$. The last term can be cast in the form

$$\begin{aligned}
\frac{\boldsymbol{p}_0}{\gamma} \cdot \left[\frac{1}{\gamma} \hat{\boldsymbol{e}}_i - \frac{\gamma - 1}{\gamma^2} \frac{\boldsymbol{\beta} \beta^i}{\beta^2} \right] = \frac{1}{\gamma^3} \left[\gamma p_0^i - (\gamma - 1) p_{0\|}^i \right] &= \frac{1}{\gamma^3} [p_{0\|}^i + \gamma p_{0\perp}^i] \\
&= \frac{1}{\gamma^3} p_0^i + \frac{\gamma - 1}{\gamma^3} p_{0\perp}^i \qquad (3.48)
\end{aligned}$$

which gives

Electric field of electric dipole

$$\boxed{\boldsymbol{E} = \frac{3(\boldsymbol{p}_0 \cdot \boldsymbol{r}^*) \boldsymbol{R} - r^{*2} \boldsymbol{p}_0}{\gamma^3 r^{*5}} - \frac{\gamma - 1}{\gamma^3} \frac{\boldsymbol{p}_{0\perp}}{r^{*3}}.} \qquad (3.49)$$

The magnetic field components read

$$\begin{aligned}
B^i &= \epsilon_{ijk} \beta^k \partial_j A^0 = \epsilon_{ijk} \beta^k \frac{\boldsymbol{p}_0}{\gamma} \cdot \partial_j \frac{\boldsymbol{r}^*}{r^{*3}} \\
&= \epsilon_{ikj} \beta^k \frac{3(\boldsymbol{p}_0 \cdot \boldsymbol{r}^*)(r^* \partial_j r^*) - r^{*2} (\boldsymbol{p}_0 \cdot \partial_j \boldsymbol{r}^*)}{\gamma r^{*5}} \qquad (3.50)
\end{aligned}$$

where

$$\partial_j \boldsymbol{r}^* = \frac{1}{\gamma} \hat{\boldsymbol{e}}_j + \frac{\gamma}{\gamma + 1} \boldsymbol{\beta} \beta^j$$

and

$$r^* \partial_j r^* = (1 - \beta^2) x^j - \beta^j x^0 + (\boldsymbol{\beta} \cdot \boldsymbol{x}) \beta^j.$$

The terms proportional to β^j do not contribute to B^i. It reads

$$B^i = \epsilon_{ikj} \beta^k \left[\frac{3(\boldsymbol{p}_0 \cdot \boldsymbol{r}^*) x^j}{\gamma^3 r^{*5}} - \frac{p_0^j}{\gamma^2 r^{*3}} \right] = \epsilon_{ikj} \beta^k \left[\frac{3(\boldsymbol{p}_0 \cdot \boldsymbol{r}^*) R^j}{\gamma^3 r^{*3}} - \frac{p_0^j}{\gamma^2 r^{*3}} \right]$$

Note that the vector product of $\boldsymbol{\beta}$ and $\boldsymbol{E}$, given by (3.49), reads

$$\begin{aligned}\boldsymbol{\beta}\times\boldsymbol{E} &= \boldsymbol{\beta}\times\left[\frac{3(\boldsymbol{p}_0\cdot\boldsymbol{r}^*)\boldsymbol{R}-r^{*2}\boldsymbol{p}_0}{\gamma^3 r^{*5}}-\frac{\gamma-1}{\gamma^3}\frac{\boldsymbol{p}_{0\perp}}{r^{*3}}\right]\\ &= \boldsymbol{\beta}\times\frac{3(\boldsymbol{p}_0\cdot\boldsymbol{r}^*)\boldsymbol{R}}{\gamma^3 r^{*5}}+\boldsymbol{\beta}\times\left[-\boldsymbol{p}_0-(\gamma-1)(\boldsymbol{p}_0-\boldsymbol{p}_{0\|})\right]\frac{1}{\gamma^3 r^{*3}}\\ &= \boldsymbol{\beta}\times\left[\frac{3(\boldsymbol{p}_0\cdot\boldsymbol{r}^*)\boldsymbol{R}}{\gamma^3 r^{*5}}-\frac{\boldsymbol{p}_0}{\gamma^2 r^{*3}}\right].\end{aligned}$$

It means that the magnetic field is just the product of the electric field and the velocity

Magnetic field of electric dipole in the laboratory reference frame

$$\boxed{\boldsymbol{B}=\boldsymbol{\beta}\times\boldsymbol{E}.} \tag{3.51}$$

3.7 *Electrodynamics of media in motion*

Covariant form of material equations

In this section we shall discus transformation of the electromagnetic field in continuous media. This problem is physically sound. We shall discuss only the case of material media which are characterized by constant permittivity ε and constant magnetic permeability μ.Such media are *homogeneous* and *isotropic.*

Maxwell's equations constitute a complete description of electromagnetic phenomena in material media only if *constitutive relations* are known. Constitutive relations are some additional equations which relate fields $\boldsymbol{E}$ and $\boldsymbol{D}$ and also $\boldsymbol{H}$ and $\boldsymbol{B}$. In the *medium rest frame* S_0 the constitutive relations read

Constitutive relations in the medium rest frame

$$\boldsymbol{D}_0=\varepsilon\boldsymbol{E}_0,\qquad \boldsymbol{B}_0=\mu\boldsymbol{H}_0, \tag{3.52}$$

where subscript "0" stands for the values of the fields in the frame stiffly attached to the medium.

Now we consider transformation to another inertial reference frame S such that the medium has velocity $\mathbf{V}$ in this frame. In such a frame fields have new components E^i, D^i, B^i and H^i. We have seen that transformations that relate E_0^i, B_0^i and E^i, B^i follow directly from the transformation law for the tensor $F^{\mu\nu}$. First pair of Maxwell's equations in the region of space where there are no sources, $\rho=0$ and $\boldsymbol{J}=0$, reads

In the laboratory reference frame material medium has certain velocity

$$\nabla\times\boldsymbol{H}-\partial_0\boldsymbol{D}=0,\qquad \nabla\cdot\boldsymbol{D}=0. \tag{3.53}$$

Equations (3.53) can be written in tensor form provided that there is given anti-symmetric tensor with components

Tensor of electric induction and magnetic strength

$$H^{\mu\nu} := \left(\begin{array}{c|ccc} 0 & -D^1 & -D^2 & -D^3 \\ \hline D^1 & 0 & -H^3 & H^2 \\ D^2 & H^3 & 0 & -H^1 \\ D^3 & -H^2 & H^1 & 0 \end{array}\right) \tag{3.54}$$

which has the same structure as $F^{\mu\nu}$ with E^i replaced by D^i and B^i replaced by H^i. Hence, equations (3.53) take the from

Sourceless first pair of Maxwell's equations, $J^\mu = 0$.

$$\partial_\mu H^{\mu\nu} = 0. \tag{3.55}$$

The second pair of Maxwell's equations in material have the same form as in empty sppace

Electromagnetic Bianchi identities

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0 \quad \Leftrightarrow \quad \partial_\mu {}^*F^{\mu\nu} = 0. \tag{3.56}$$

The Lorentz symmetry of electrodynamics manifests itself in covariance of Maxwell's equations. It means that they have the same form in any reference frame. In particular, in S_0 they have the form

Maxwell's equations in the rest frame of the medium S_0

$$\partial_\mu \overset{(0)}{H}{}^{\mu\nu} = 0, \qquad \partial_\mu {}^*\overset{(0)}{F}{}^{\mu\nu} = 0 \tag{3.57}$$

where $\overset{(0)}{F}{}^{\mu\nu}$ and $\overset{(0)}{H}{}^{\mu\nu}$ have components D_0^i, H_0^i, E_0^i and B_0^i whereas in S they are given by (3.55) and (3.56). What about constitutive relations? Do they preserve their form? It turns out that the presence of dielectric media modifies constitutive relations in a nontrivial way. The solution of this problem was proposed by H. Minkowski in 1908. He shown that *Einstein's principle of equivalence* applied to constitutive relations in the rest frame of the material medium leads to general tensor form of the relations (3.52). The four-velocity of the medium in its own rest frame S_0 reads

Minkowski's solution

$$\overset{(0)}{u}{}^\mu \to (c, \mathbf{0}). \tag{3.58}$$

The fields can be cast in the form

$$D_0^i = \overset{(0)}{H}{}^{i0} = \frac{1}{c}\overset{(0)}{H}{}^{i\nu}\overset{(0)}{u}_\nu,$$
$$E_0^i = \overset{(0)}{F}{}^{i0} = \frac{1}{c}\overset{(0)}{F}{}^{i\nu}\overset{(0)}{u}_\nu.$$

Thus, first constitutive relation (3.52) can be cast in tensor form what gives

First constitutive relation in tensor notation

$$\overset{(0)}{H}{}^{i\nu}\overset{(0)}{u}_\nu = \varepsilon \overset{(0)}{F}{}^{i\nu}\overset{(0)}{u}_\nu. \tag{3.59}$$

In order to write second relation (3.52) in tensor form we consider the following combinations

$$\begin{aligned}\overset{(0)}{Z}_{\mu\nu\lambda} &:= \overset{(0)}{F}_{\mu\nu}\overset{(0)}{u}_{\lambda} + \overset{(0)}{F}_{\nu\lambda}\overset{(0)}{u}_{\mu} + \overset{(0)}{F}_{\lambda\mu}\overset{(0)}{u}_{\nu},\\ \overset{(0)}{W}_{\mu\nu\lambda} &:= \overset{(0)}{H}_{\mu\nu}\overset{(0)}{u}_{\lambda} + \overset{(0)}{H}_{\nu\lambda}\overset{(0)}{u}_{\mu} + \overset{(0)}{H}_{\lambda\mu}\overset{(0)}{u}_{\nu}\end{aligned}$$

which do not vanish only if two indices are spatial and one is temporal *e.g.*

$$\overset{(0)}{Z}_{0ij} = \overset{(0)}{Z}_{ij0} = \overset{(0)}{Z}_{j0i} = c\overset{(0)}{F}_{ij}, \qquad \overset{(0)}{W}_{0ij} = \overset{(0)}{W}_{ij0} = \overset{(0)}{W}_{j0i} = c\overset{(0)}{H}_{ij}.$$

The second constitutive equation (3.52) has thus the form

Second constitutive relation in tensor notation

$$\overset{(0)}{Z}_{\mu\nu\lambda} = \mu\overset{(0)}{W}_{\mu\nu\lambda} \qquad \Leftrightarrow \qquad {}^*\overset{(0)}{F}{}^{\sigma\lambda}\overset{(0)}{u}_{\lambda} = \mu\,{}^*\overset{(0)}{H}{}^{\sigma\lambda}\overset{(0)}{u}_{\lambda} \tag{3.60}$$

where the second expression uses dual tensors ${}^*H_{\mu\nu}$ and ${}^*F_{\mu\nu}$

$${}^*F^{\mu\nu} := \left(\begin{array}{c|ccc} 0 & B^1 & B^2 & B^3 \\ \hline -B^1 & 0 & -E^3 & E^2 \\ -B^2 & E^3 & 0 & -E^1 \\ -B^3 & -E^2 & E^1 & 0 \end{array}\right), \tag{3.61}$$

$${}^*H^{\mu\nu} := \left(\begin{array}{c|ccc} 0 & H^1 & H^2 & H^3 \\ \hline -H^1 & 0 & -D^3 & D^2 \\ -H^2 & D^3 & 0 & -D^1 \\ -H^3 & -D^2 & D^1 & 0 \end{array}\right).$$

It it is obtained taking contractions

$$\begin{aligned}\frac{1}{3!}\varepsilon^{\mu\nu\sigma\lambda}\overset{(0)}{Z}_{\mu\nu\lambda} &= \frac{1}{3!}(\varepsilon^{\mu\nu\sigma\lambda}\overset{(0)}{F}_{\mu\nu}\overset{(0)}{u}_{\lambda} + \varepsilon^{\mu\nu\sigma\lambda}\overset{(0)}{F}_{\nu\lambda}\overset{(0)}{u}_{\mu} + \varepsilon^{\mu\nu\sigma\lambda}\overset{(0)}{F}_{\lambda\mu}\overset{(0)}{u}_{\nu}\\ &= \left(\frac{1}{2}\varepsilon^{\mu\nu\sigma\lambda}\overset{(0)}{F}_{\mu\nu}\right)\overset{(0)}{u}_{\lambda} = {}^*\overset{(0)}{F}{}^{\sigma\lambda}\overset{(0)}{u}_{\lambda}\end{aligned}$$

and

$$\frac{1}{3!}\varepsilon^{\mu\nu\sigma\lambda}\overset{(0)}{W}_{\mu\nu\lambda} = {}^*\overset{(0)}{H}{}^{\sigma\lambda}\overset{(0)}{u}_{\lambda}.$$

Now the point is that the equations (3.59) and (3.60) have tensor character so they must have the same form in any reference frame S in which the material medium has the four-velocity

Tensor equations must preserve its form in all inertial reference frame

$$u^{\mu} = (\gamma c, \gamma c\boldsymbol{\beta}). \tag{3.62}$$

The constitutive equations have the covariant form

First constitutive relation

$$\boxed{H^{\mu\nu}u_{\nu} = \varepsilon\, F^{\mu\nu}u_{\nu}} \tag{3.63}$$

and

$$\boxed{F_{\mu\nu}u_\lambda + F_{\nu\lambda}u_\mu + F_{\lambda\mu}u_\nu = \mu(H_{\mu\nu}u_\lambda + H_{\nu\lambda}u_\mu + H_{\lambda\mu}u_\nu)} \tag{3.64}$$

or equivalently

Second constitutive relation

$$\boxed{{}^*F^{\mu\nu}u_\nu = \mu\,{}^*H^{\mu\nu}u_\nu.} \tag{3.65}$$

The parameters ε and μ *take the same numerical values* in all inertial reference frames. It is consistent with the fact that first pair of Maxwell's equations

$$\nabla \cdot (\varepsilon \boldsymbol{E}_0) = 0, \qquad \nabla \times \left(\frac{1}{\mu}\boldsymbol{B}_0\right) - \partial_0(\varepsilon \boldsymbol{E}_0) = 0$$

in S_0 has the same form in S with $\boldsymbol{B}_0$ replaced by $\boldsymbol{B}$ and $\boldsymbol{E}_0$ replaced by $\boldsymbol{E}$. Equations (3.63) and (3.65) written in terms of electric and magnetic fields have the following form

- $\mu = 0$:

$$\boldsymbol{D} \cdot \boldsymbol{\beta} = \varepsilon(\boldsymbol{E} \cdot \boldsymbol{\beta}), \qquad \boldsymbol{B} \cdot \boldsymbol{\beta} = \mu(\boldsymbol{H} \cdot \boldsymbol{\beta}) \tag{3.66}$$

- $\mu = i$:

$$\boldsymbol{D} + \boldsymbol{\beta} \times \boldsymbol{H} = \varepsilon(\boldsymbol{E} + \boldsymbol{\beta} \times \boldsymbol{B}), \tag{3.67}$$

$$\boldsymbol{B} - \boldsymbol{\beta} \times \boldsymbol{E} = \mu(\boldsymbol{H} - \boldsymbol{\beta} \times \boldsymbol{D}). \tag{3.68}$$

Note that (3.66) do not contain new information. It can be seen from transformation formulas that relate fields in S_0 and in S. The reference frame S moves with the velocity $-\boldsymbol{\beta}$ with respect to S_0 so the transformation formulas have the form

$$\begin{aligned}
\boldsymbol{E} &= \gamma(\boldsymbol{E}_0 - \boldsymbol{\beta} \times \boldsymbol{B}_0) - \frac{\gamma - 1}{\beta^2}(\boldsymbol{\beta} \cdot \boldsymbol{E}_0)\boldsymbol{\beta},\\
\boldsymbol{B} &= \gamma(\boldsymbol{B}_0 + \boldsymbol{\beta} \times \boldsymbol{E}_0) - \frac{\gamma - 1}{\beta^2}(\boldsymbol{\beta} \cdot \boldsymbol{B}_0)\boldsymbol{\beta},\\
\boldsymbol{D} &= \gamma(\boldsymbol{D}_0 - \boldsymbol{\beta} \times \boldsymbol{H}_0) - \frac{\gamma - 1}{\beta^2}(\boldsymbol{\beta} \cdot \boldsymbol{D}_0)\boldsymbol{\beta},\\
\boldsymbol{H} &= \gamma(\boldsymbol{H}_0 + \boldsymbol{\beta} \times \boldsymbol{D}_0) - \frac{\gamma - 1}{\beta^2}(\boldsymbol{\beta} \cdot \boldsymbol{H}_0)\boldsymbol{\beta}.
\end{aligned}$$

It leads to $\boldsymbol{D} \cdot \boldsymbol{\beta} = \boldsymbol{D}_0 \cdot \boldsymbol{\beta}$ and similarly for other fields. When plugging these results to (3.66) then they result in equations $\boldsymbol{D}_0 \cdot \boldsymbol{\beta} = \varepsilon \boldsymbol{E}_0 \cdot \boldsymbol{\beta}$ and $\boldsymbol{B}_0 \cdot \boldsymbol{\beta} = \mu \boldsymbol{H}_0 \cdot \boldsymbol{\beta}$ which are projection on $\boldsymbol{\beta}$ of constitutive relations in the rest frame of the medium. On the other hand, equations (3.67) and (3.68) are the constitutive relations for electric and magnetic field in S. Multiplying by $\boldsymbol{\beta}$ both sides of relations (3.67) and (3.68) we get equalities of parallel parts $\boldsymbol{D} \cdot \boldsymbol{\beta} = \varepsilon(\boldsymbol{\beta} \cdot \boldsymbol{E})$ and $\boldsymbol{B} \cdot \boldsymbol{\beta} = \mu(\boldsymbol{\beta} \cdot \boldsymbol{H})$ which are satisfied. It means that parallel components in constitutive relations drop out and the relations read

Constitutive relations in the laboratory reference frame in which the medium has certain velocity

$$\boldsymbol{D}_\perp + \boldsymbol{\beta}\times\boldsymbol{H}_\perp = \varepsilon(\boldsymbol{E}_\perp + \boldsymbol{\beta}\times\boldsymbol{B}_\perp), \tag{3.69}$$

$$\boldsymbol{B}_\perp - \boldsymbol{\beta}\times\boldsymbol{E}_\perp = \mu(\boldsymbol{H}_\perp - \boldsymbol{\beta}\times\boldsymbol{D}_\perp). \tag{3.70}$$

Eliminating $\boldsymbol{B}_\perp$ from (3.69) we get

$$(1-\varepsilon\mu\beta^2)\boldsymbol{D}_\perp = \varepsilon(1-\beta^2)\boldsymbol{E}_\perp + (\varepsilon\mu - 1)\boldsymbol{\beta}\times\boldsymbol{H}_\perp.$$

Similarly, eliminating $\boldsymbol{D}_\perp$ from (3.70) we get

$$(1-\varepsilon\mu\beta^2)\boldsymbol{B}_\perp = \mu(1-\beta^2)\boldsymbol{H}_\perp - (\varepsilon\mu - 1)\boldsymbol{\beta}\times\boldsymbol{E}_\perp.$$

Left and right hand side of (3.69) read

$$\boldsymbol{D}_\perp + \boldsymbol{\beta}\times\boldsymbol{H}_\perp = \gamma(\boldsymbol{D}_{0\perp} - \boldsymbol{\beta}\times\boldsymbol{H}_{0\perp}) + \gamma\boldsymbol{\beta}\times(\boldsymbol{H}_{0\perp} + \boldsymbol{\beta}\times\boldsymbol{D}_{0\perp}) = \frac{1}{\gamma}\boldsymbol{D}_{0\perp},$$

$$\boldsymbol{E}_\perp + \boldsymbol{\beta}\times\boldsymbol{B}_\perp = \gamma(\boldsymbol{E}_{0\perp} - \boldsymbol{\beta}\times\boldsymbol{B}_{0\perp}) + \gamma\boldsymbol{\beta}\times(\boldsymbol{B}_{0\perp} + \boldsymbol{\beta}\times\boldsymbol{E}_{0\perp}) = \frac{1}{\gamma}\boldsymbol{E}_{0\perp}.$$

Plugging these formulas to (3.69) we obtain expression $\boldsymbol{D}_{0\perp} = \varepsilon\boldsymbol{E}_{0\perp}$. In a similar way expressions $\boldsymbol{B}_\perp - \boldsymbol{\beta}\times\boldsymbol{E}_\perp = \frac{1}{\gamma}\boldsymbol{B}_{0\perp}$ and $\boldsymbol{H}_\perp - \boldsymbol{\beta}\times\boldsymbol{D}_\perp = \frac{1}{\gamma}\boldsymbol{H}_{0\perp}$ allow to put the equation (3.70) in the form $\boldsymbol{B}_{0\perp} = \mu\boldsymbol{H}_{0\perp}$ (up to overall factor γ^{-1}).

Alternative form of tensorial constitutive relations

Let us go back to covariant form of constitutive equations. The fact that formulas (3.63) and (3.65) have different proportionality constants means that $H^{\mu\nu}$ is not proportional to $F^{\mu\nu}$. It means that tensor components $H^{\mu\nu}$ can be obtained from $F^{\mu\nu}$ by contraction with another tensor. We shall determinate the form of this tensor. Contracting relation (3.64) with u^λ, and using $u^\lambda u_\lambda = c^2$ we get

$$\mu(c^2 H_{\mu\nu} + \underbrace{H_{\nu\lambda}u^\lambda}_{\varepsilon F_{\nu\lambda}u^\lambda} u_\mu + \underbrace{H_{\lambda\mu}u^\lambda}_{\varepsilon F_{\lambda\mu}u^\lambda} u_\nu) = c^2 F_{\mu\nu} + F_{\nu\lambda}u^\lambda u_\mu + F_{\lambda\mu}u^\lambda u_\nu$$

and then

$$\mu H_{\mu\nu} = F_{\mu\nu} + \frac{1-\varepsilon\mu}{c^2}(F_{\nu\lambda}u^\lambda u_\mu + F_{\lambda\mu}u^\lambda u_\nu).$$

For expression with raised indices μ and ν we get

$$\begin{aligned}
\mu H^{\mu\nu} &= F^{\mu\nu} + \frac{\varepsilon\mu - 1}{c^2}(F^{\lambda\nu}u_\lambda u^\mu + F^{\mu\lambda}u_\lambda u^\nu) \\
&= \left(\eta^{\mu\alpha}\eta^{\nu\beta} + \frac{\varepsilon\mu-1}{c^2}(\eta^{\alpha\lambda}\eta^{\nu\beta}u_\lambda u^\mu + \eta^{\mu\alpha}\eta^{\lambda\beta}u_\lambda u^\nu)\right)F_{\alpha\beta} \\
&= \left(\eta^{\mu\alpha}\eta^{\nu\beta} + \frac{\varepsilon\mu-1}{c^2}(\eta^{\nu\beta}u^\alpha u^\mu + \eta^{\mu\alpha}u^\beta u^\nu)\right)F_{\alpha\beta} \\
&\quad + \underbrace{\frac{(\varepsilon\mu-1)^2}{c^4}u^\mu u^\nu u^\alpha u^\beta F_{\alpha\beta}}_{\equiv 0} \\
&= \left(\eta^{\mu\alpha} + \frac{\varepsilon\mu-1}{c^2}u^\mu u^\alpha\right)\left(\eta^{\nu\beta} + \frac{\varepsilon\mu-1}{c^2}u^\nu u^\beta\right)F_{\alpha\beta}
\end{aligned} \tag{3.71}$$

We define *effective metric tensor*

Contravariant components of effective metric tensor

$$\boxed{g^{\mu\nu} := \eta^{\mu\nu} + \frac{\varepsilon\mu - 1}{c^2} u^\mu u^\nu} \tag{3.72}$$

which allows us to write the last expression in the form

Tensor form of constitutive relations

$$\boxed{\mu H^{\mu\nu} = F^{(\mu)(\nu)} \qquad \text{where} \qquad F^{(\mu)(\nu)} := g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta}.} \tag{3.73}$$

Covariant components of the effective metric tensor can be obtained from condition of inverebility $g_{\mu\alpha} g^{\alpha\nu} = \delta^\nu_\mu$. Plugging $g_{\mu\alpha} = \eta_{\mu\alpha} + \rho u_\mu u_\alpha$ we get $\rho = -\frac{1}{c^2}\left(1 - \frac{1}{\varepsilon\mu}\right)$. Hence

Contravariant components of effective metric tensor

$$\boxed{g_{\mu\nu} = \eta_{\mu\nu} - \frac{1}{c^2}\left(1 - \frac{1}{\varepsilon\mu}\right) u_\mu u_\nu.} \tag{3.74}$$

This approach allows us to describe the electromagnetic field in moving media as it would be the electromagnetic field in an empty curved space.[2]

[2] This fact was spotted by W. Gordon in 1923 in the paper *Zur Lichtfortpflanzung in der Relativitätstheorie*.

Arbitrary media

The relations $\boldsymbol{D} = \boldsymbol{E} + 4\pi\boldsymbol{P}$ and $\boldsymbol{H} = \boldsymbol{B} - 4\pi\boldsymbol{M}$ can be cast in the form

Tensor of polarization and magnetization

$$H^{\mu\nu} = F^{\mu\nu} - 4\pi M^{\mu\nu} \tag{3.75}$$

where all three tensors are anti-symmetric and given by expressions

$$F^{0j} = -E^j, \qquad F^{ij} = -\epsilon_{ijk} B^k, \tag{3.76}$$

$$H^{0j} = -D^j, \qquad H^{ij} = -\epsilon_{ijk} H^k, \tag{3.77}$$

and

$$M^{0j} = +P^j, \qquad M^{ij} = -\epsilon_{ijk} M^k. \tag{3.78}$$

The tensorial notation suggest that we can easily derive transformation laws for polarization $\boldsymbol{P}$ and magnetization $\boldsymbol{M}$. Let us consider a material medium which moves with velocity $\boldsymbol{\beta}$ with respect to laboratory reference frame S. In the rest frame of the medium S' the polarization and magnetization have values $\boldsymbol{P}'$ and $\boldsymbol{M}'$. The laboratory reference frame has velocity $-\boldsymbol{\beta}$ with respect to S' so

$$M^{\mu\nu} = (\hat{L}(-\boldsymbol{\beta}))^\mu{}_\alpha (\hat{L}(-\boldsymbol{\beta}))^\nu{}_\beta M'^{\alpha\beta} \tag{3.79}$$

where

$$L^0{}_0 = \gamma, \quad L^0{}_i = L^i{}_0 = +\gamma\beta^i, \quad L^i{}_j = \delta_{ij} + \frac{\gamma - 1}{\beta^2}\beta^i\beta^j.$$

It gives

$$\begin{aligned}
P^i = M^{0i} &= (\hat{L}(-\boldsymbol{\beta}))^0{}_\alpha (\hat{L}(-\boldsymbol{\beta}))^i{}_\beta M'^{\alpha\beta} \\
&= (L^0{}_0 L^i{}_j - L^0{}_j L^i{}_0) M'^{0j} + L^0{}_j L^i{}_k M'^{jk} \\
&= \left(\gamma\delta_{ij} + \frac{\gamma(\gamma-1)}{\beta^2}\beta^i\beta^j\right) P'^j - \gamma^2\beta^i\beta^j P'^j \\
&+ \gamma\beta^j \left(\delta_{ik} + \frac{\gamma-1}{\beta^2}\beta^i\beta^k\right)(-\epsilon_{jkl}M'^l).
\end{aligned}$$

After some organisation of terms we get

Transformation of polarization vector

$$\boxed{\boldsymbol{P} = \gamma(\boldsymbol{P}' + \boldsymbol{\beta}\times\boldsymbol{M}') - \frac{\gamma-1}{\beta^2}(\boldsymbol{\beta}\cdot\boldsymbol{P}')\boldsymbol{\beta}.} \tag{3.80}$$

Similarly, we get

$$\begin{aligned}
M^i &= -\frac{1}{2}\epsilon_{ijk}M^{jk} = -\frac{1}{2}\epsilon_{ijk}(\hat{L}(-\boldsymbol{\beta}))^j{}_\alpha (\hat{L}(-\boldsymbol{\beta}))^k{}_\beta M'^{\alpha\beta} \\
&= -\frac{1}{2}\epsilon_{ijk}(L^j{}_0 L^k{}_l - L^j{}_l L^k{}_0)M'^{0l} - \frac{1}{2}\epsilon_{ijk}L^j{}_l L^k{}_m M'^{lm} \\
&= -\epsilon_{ijk}L^j{}_0 L^k{}_l M'^{0l} - \frac{1}{2}\epsilon_{ijk}L^j{}_l L^k{}_m M'^{lm}
\end{aligned}$$

Plugging elements of the Lorentz matrix we obtain

$$\begin{aligned}
M^i &= -\epsilon_{ijk}\gamma\beta^j \left(\delta_{kl} + \cancel{\frac{\gamma-1}{\beta^2}\beta^k\beta^l}\right) P'^l \\
&- \frac{1}{2}\epsilon_{ijk}\left(\delta_{jl} + \frac{\gamma-1}{\beta^2}\beta^j\beta^l\right)\left(\delta_{km} + \frac{\gamma-1}{\beta^2}\beta^k\beta^m\right)(-\epsilon_{lmn}M'^n) \\
&= -\gamma\epsilon_{ijk}\beta^j P'^k + \frac{1}{2}\underbrace{\epsilon_{ijk}\epsilon_{lmn}\delta_{jl}\delta_{km}}_{2\delta_{in}} M'^n \\
&+ \frac{1}{2}\epsilon_{ijk}\epsilon_{lmn}(\delta_{jl}\beta^k\beta^m + \delta_{km}\beta^j\beta^l)\frac{\gamma-1}{\beta^2}M'^n \\
&+ \frac{1}{2}\frac{(\gamma-1)^2}{\beta^4}\underbrace{\epsilon_{ijk}\epsilon_{lmn}\beta^j\beta^l\beta^k\beta^m}_{0} \\
&= M'^i - \gamma\epsilon_{ijk}\beta^j P'^k + \frac{\gamma-1}{2\beta^2}(\underbrace{\epsilon_{ijk}}_{\epsilon_{jki}}\epsilon_{jmn}\beta^k\beta^m + \epsilon_{ijk}\underbrace{\epsilon_{lkn}}_{\epsilon_{nlk}}\beta^j\beta^l)M'^n \\
&= M'^i - \gamma\epsilon_{ijk}\beta^j P'^k \\
&+ \frac{\gamma-1}{2\beta^2}[(\delta_{km}\delta_{in} - \delta_{kn}\delta_{im})\beta^k\beta^m + (\delta_{km}\delta_{in} - \delta_{kn}\delta_{im})\beta^k\beta^m]M'^n \\
&= M'^i - \gamma\epsilon_{ijk}\beta^j P'^k + \frac{\gamma-1}{2\beta^2}(2\beta^2 M'^i - 2(\boldsymbol{\beta}\cdot\boldsymbol{M}')\beta^i)
\end{aligned}$$

and finally

Transformation of magnetization vector

$$\boxed{\boldsymbol{M} = \gamma(\boldsymbol{M}' - \boldsymbol{\beta} \times \boldsymbol{P}') - \frac{\gamma - 1}{\beta^2}(\boldsymbol{\beta} \cdot \boldsymbol{M}')\boldsymbol{\beta}.} \tag{3.81}$$

Transformation of electric and magnetic dipole moments

These formulas allows us to get expressions for transformation of total momenta: electric $\boldsymbol{p}$ and magnetic $\boldsymbol{m}$, where

$$\boldsymbol{p} := \int_\Omega d^3x \boldsymbol{P}, \qquad \boldsymbol{m} := \int_\Omega d^3x \boldsymbol{M}. \tag{3.82}$$

In rest frame of the medium S'

$$\boldsymbol{p}' := \int_{\Omega'} d^3x' \boldsymbol{P}', \qquad \boldsymbol{m}' := \int_{\Omega'} d^3x' \boldsymbol{M}'. \tag{3.83}$$

Note that the Lorentz contraction gives $d^3x = \gamma^{-1} d^3x'$. Integrating the formulas (3.80) and (3.81) over the region occupied by the medium we get

$$\boldsymbol{p} = \boldsymbol{p}' + \boldsymbol{\beta} \times \boldsymbol{m}' - \frac{\gamma - 1}{\beta^2 \gamma}(\boldsymbol{\beta} \cdot \boldsymbol{p}')\boldsymbol{\beta}, \tag{3.84}$$

$$\boldsymbol{m} = \boldsymbol{m}' - \boldsymbol{\beta} \times \boldsymbol{p}' - \frac{\gamma - 1}{\beta^2 \gamma}(\boldsymbol{\beta} \cdot \boldsymbol{m}')\boldsymbol{\beta}. \tag{3.85}$$

Chapter 4
Conservation laws

4.1 Local conservation laws

Noether identity – simplified approach for undergraduate courses

Conservation laws associated with the electromagnetic field are consequences of *symmetries* of the electromagnetic action. This subject is discusses using the Lagrangian formalism which is a natural framework for such analysis. Unfortunately, in some basic courses on electromagnetism the Lagrangian formalism is absent. For this reason we shall present an alternative way for obtaining Noether identities. It is based on some mathematical manipulations involving Maxwell's equations which leads to formulas for local conservation laws. Although this approach enable us to discuss the physical content of conservation laws it certainly skips physically sound relation between conservation laws and the underlying symmetries. The discussion of conservation laws in the context of Noether theorem will be given in subsequent section.

The electromagnetic field is a material object

The electromagnetic field is a *material object* because it possesses energy, linear momentum and angular momentum. These quantities can be obtained from the electromagnetic field tensor $F_{\mu\nu}$. Energy has dimension [1] $[E] = [F]\,L = M\ L^2T^{-2}$, and *energy density*

$$[E]L^{-3} = ML^{-1}T^{-2}.$$

The dimension of electric charge in Gaussian units reads

$$[e] = [F]^{1/2}L = M^{1/2}L^{3/2}T^{-1}.$$

Squares of the electric field and the magnetic induction have dimension

$$[E^2] = [B^2] = [e^2]L^{-4} = ML^{-1}T^{-2}$$

i.e. the same as dimension of energy density. It suggests that energy density should be quadratic function of $\boldsymbol{E}$ and $\boldsymbol{B}$, or equivalently, quadratic function of $F_{\mu\nu}$. In the region of spacetime where four-currents vanish, $J^\mu = 0$, the electromagnetic field is *isolated* (it does not interact with other forms of matter). In such integrals of motion of the field can be

[1] In Gaussian units $M = 1g$, $L = 1cm$, $T = 1s$, $F = 1dyne$

obtained integrating local conservation laws. The existence of integrals of motion can be associated with symmetries of action and it is the subject of Noether theorem. In this section we obtain local conservation laws in different way. The ideia of the calculus is to derive an equation which contains a four-divergence of some tensor quadratic in fields.

Derivation of local conservation laws

In the first step we perform contraction of the first pair of Maxwell's equations

$$\partial_\mu F^{\mu\alpha} = \frac{4\pi}{c} J^\alpha \tag{4.1}$$

with components of the electromagnetic field tensor $F_{\nu\alpha}$. The result of contraction can be cast in the form

$$\partial_\mu (F^{\mu\alpha} F_{\nu\alpha}) - \partial_\mu F_{\nu\alpha} F^{\mu\alpha} = \frac{4\pi}{c} F_{\nu\alpha} J^\alpha, \tag{4.2}$$

where the second term on l.h.s. can be written as four-divergence

$$\begin{aligned}
\partial_\mu F_{\nu\alpha} F^{\mu\alpha} &= \frac{1}{2}(\partial_\mu F_{\nu\alpha} F^{\mu\alpha} + \partial_\alpha \underbrace{F_{\nu\mu}}_{-F_{\mu\nu}} \overbrace{F^{\alpha\mu}}^{-F^{\mu\alpha}}) \\
&= \frac{1}{2}(\partial_\mu F_{\nu\alpha} F^{\mu\alpha} + \partial_\alpha F_{\mu\nu} F^{\mu\alpha}) = \frac{1}{2} \underbrace{(\partial_\mu F_{\nu\alpha} + \partial_\alpha F_{\mu\nu})}_{-\partial_\nu F_{\alpha\mu}} F^{\mu\alpha} \\
&= \frac{1}{2}\partial_\nu F_{\mu\alpha} F^{\mu\alpha} = \frac{1}{4}\partial_\nu (F_{\alpha\beta} F^{\alpha\beta}) = \partial_\mu \left[\frac{1}{4}\delta^\mu_\nu F_{\alpha\beta} F^{\alpha\beta}\right].
\end{aligned}$$

Both terms on l.h.s. of (4.2) are total divergences so they can be put together

$$\partial_\mu \left[F^{\mu\alpha} F^\nu{}_\alpha - \frac{1}{4}\eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}\right] = \frac{4\pi}{c} F^\nu{}_\alpha J^\alpha$$

where the index ν has been raised. Thus we have equation

$$\boxed{\partial_\mu T^{\mu\nu} + \frac{1}{c} F^\nu{}_\alpha J^\alpha = 0.} \tag{4.3}$$

In fact, this equation is an example of Noether identity which follows from Noether theorem. The expression

Energy-momentum tensor

$$T^{\mu\nu} := \frac{1}{4\pi}\left[-F^{\mu\alpha} F^\nu{}_\alpha + \frac{1}{4}\eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}\right]$$

is called *tensor energy-momentum of electromagnetic field*.

4.2 *Noether identity and the spacetime symmetries*

General comments

Lagrangian formalism[2] is a very useful tool that allows to obtain some quantities that characterize physical properties of electromagnetic field. Such quantities are conserved when the electromagnetic field form an isolated system.

[2] Content of graduate course

According to *first Noether theorem* (1918) there is a mathematical relationship between conserved quantities (or underlying conserved currents) and symmetries of the action. Noether (on-shell) identities leads to conservation laws for field configurations satisfying Euler-Lagrange equations .

A particularly important group of symmetries are spacetime symmetries. According Einstein's theory, spacetime curvature depends on mass-energy distribution. A generic (without special assumptions) spacetime has no symmetries. However, for some special mass distributions it can have certain symmetries. For instance, spacetime around spherical stationary objects reads

Linear element in Schwarzchild spacetime

$$ds^2 = \left(1 - \frac{r_s}{r}\right) c^2 dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 - r^2 d\Omega^2$$

where $r_s = \frac{2GM}{c^2}$ is Schwarzschild radius. The frequency of a photon that moves in the radial direction decreases together with increasing of its distance from the mass center. Its frequency at spatial infinity is function of the frequency ω^* of the photon at the moment of emission, the mass of the object M and the radial coordinate R (describing position of the point at which the photon was emmited)

Frequency of the photon at $r \to \infty$

$$\omega = \omega^* \sqrt{1 - \frac{r_s}{R}}.$$

The observed variation of frequency $\Delta\omega = \omega^* - \omega$ is associated with photon energy loss $\Delta E = \hbar\Delta\omega$ (or equivalently its linear momentum). This effect has been confirmed experimentally providing one of important tests of Einstein's General Theory of Relativity. On the other hand it shows that energy of electromagnetic field is *not conserved* in spacetime which is not maximally symmetric. The energy of photons, equal to $c|\boldsymbol{p}|$, is not conserved in generality. Thus, conservation of linear momentum, associated with translational symmetries of the action, has *origin in symmetries of spacetime*.

Infinitesimal Poincaré transformation

Spacetime of special relativity (Minkowski spacetime) is *maximally symmetric*. The most general group of transformations that preserve its metric tensor is *Poincaré group*. This group consists on transformations containing ten generators, where six of them are Lorentz group generators and remaining four are translation generators. It means that there exist ten corresponding conserved quantities. In order to obtain pertinent Noether identity it is sufficient to look at infinitesimal Poincaré transformations.

In present section we shall mainly discuss *active transformations i.e.* transformations that map four-vectors onto four-vectors. Let G be a group of transformations (f, F),

Active transformations

$$G \ni (f, F) \tag{4.4}$$

parametrized by continuous and real-valued parameters $\omega = \{\omega^\alpha\}$, $\alpha = 1, 2, \ldots, s$. Such parameters are coordinates on the group G in vicinity of unit element $g = e$. The function f maps points of spacetime (or equivalently on position four-vectors) onto points

Transformation of position four vectors

$$f: \qquad x \to x' = f(x; \omega),$$

where x' and x are two different four-vectors that have components x'^μ and x^μ in certain Cartesian basis $\{e_\mu\}$. The function F acts on physical fields

Transformation of physical fields

$$F: \qquad u_b(x) \to u'_a(x') = F_a(u_b(x); \omega).$$

Since the unit element e has coordinates $\omega^\alpha = 0$ then

$$x = f(x; 0), \qquad u_a(x) = F_a(u_b(x), 0).$$

We are interested in symmetry transformation of electromagnetic action generated by infinitesimal Poincaré transformation. Thus

- $f(x; \omega)$ is an infinitesimal Poincaré transformation parametrized by six Lorentz parameters $\omega_{\alpha\beta} \ll 1$ that form an anti-symmetric matrix $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$ and four translation parameters $\varepsilon^\mu \ll 1$,

Here $\omega = \{\omega_{\alpha\beta}, \varepsilon^\mu\}$

- $F(A; \omega)$ is an infinitesimal Lorentz transformation parametrized by $\omega_{\alpha\beta} \ll 1$. $A = (A^\mu)$ is certain vector field which enters as argument of F.

Here $\omega = \{\omega_{\alpha\beta}\}$

The infinitesimal Lorentz transformation and its inverse have the form

$$L^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu, \qquad (L^{-1})^\mu{}_\nu = \delta^\mu{}_\nu - \omega^\mu{}_\nu$$

where $(L^{-1})^\mu{}_\alpha L^\alpha{}_\nu = \delta^\mu{}_\nu + \mathcal{O}(\omega^2)$. Components of the four-vector x' which is the result of transformation of x read

$$x'^\mu = (\delta^\mu{}_\nu + \omega^\mu{}_\nu)x^\nu + \varepsilon^\mu = x^\mu + \omega^\mu{}_\nu x^\nu + \varepsilon^\nu. \tag{4.5}$$

We shall denote by δx^μ the infinitesimal net change of components of

Expression $\delta x^\mu := x'^\mu - x^\mu$

the position four-vector

$$x'^{\mu} = x^{\mu} + \delta x^{\mu} \qquad \text{where} \qquad \delta x^{\mu} = \omega^{\mu}{}_{\nu} x^{\nu} + \varepsilon^{\mu}. \tag{4.6}$$

In matrix notation the last formula reads

$$x' = \hat{L}x + \varepsilon = x + \delta x$$

which gives

$$\hat{L}x - \delta x = x - \varepsilon. \tag{4.7}$$

Acting on both sides of (4.7) with inverse Lorentz transformation we get

$$\hat{L}^{-1}(x-\varepsilon) = \hat{L}^{-1}(\hat{L}x - \delta x) \quad \Rightarrow \quad \hat{L}^{-1}(x-\varepsilon) = x - \underbrace{(\mathbb{1} - \hat{\omega})\delta x}_{\delta x + \mathcal{O}(\omega^2) + \mathcal{O}(\omega\varepsilon)} .$$

Summarising the above results:

$$\boxed{\hat{L}x + \varepsilon = x + \delta x} \qquad \text{and} \qquad \boxed{\hat{L}^{-1}(x-\varepsilon) = x - \delta x.} \tag{4.8}$$

The change of components of any tensor field[3] $\boldsymbol{T}(x)$ generated by infinitesimal Poincaré transformation $x^{\mu} \to x^{\mu} + \delta x^{\mu}$ is defined as

[3] Here $A^{\mu}(x)$, $F^{\mu\nu}(x)$ e.t.c.

Change of components of generic Lorentz tensor under infinitesimal Poincaré transformation

$$\delta T^{\mu\ldots}{}_{\nu\ldots}(x) := T'^{\mu\ldots}{}_{\nu\ldots}(x) - T^{\mu\ldots}{}_{\nu\ldots}(x). \tag{4.9}$$

Note that both sides of (4.9) are taken at the same point of space-time. The tensor field transform under active Poincaré transformation[4] according to

[4] Translation that acts on arguments of tensor field

$$T'^{\mu\ldots}{}_{\nu\ldots}(x') = L^{\mu}{}_{\alpha}(L^{-1})^{\beta}{}_{\nu} \cdots T^{\alpha\ldots}{}_{\beta\ldots}(x) \tag{4.10}$$

where $x' = x + \delta x$. Shifting arguments[5] on both sides of (4.10)

[5] Often passive coordinate transformation is used to mask the effect of active transformation.

$$x \to x - \delta x$$

one gets

$$T'^{\mu\ldots}{}_{\nu\ldots}(x) = L^{\mu}{}_{\alpha} \cdots (L^{-1})^{\beta}{}_{\nu} \cdots T^{\alpha\ldots}{}_{\beta\ldots}(x - \delta x). \tag{4.11}$$

It gives the following expression describing variation of tensor components

$$\boxed{\delta T^{\mu\ldots}{}_{\nu\ldots}(x) = (\delta^{\mu}{}_{\alpha} + \omega^{\mu}{}_{\alpha}) \cdots (\delta^{\beta}{}_{\nu} - \omega^{\beta}{}_{\nu}) \cdots T^{\alpha\ldots}{}_{\beta\ldots}(x - \delta x) - T^{\mu\ldots}{}_{\nu\ldots}(x).} \tag{4.12}$$

In particular, for the scalar field $T(x)$ this expression simplifies to the following one

$$\delta T(x) = T'(x) - T(x) = T(x - \delta x) - T(x). \tag{4.13}$$

The Noether identity for the electromagnetic field

Let Ω be some region of spacetime

$$\Omega = \{(ct, \boldsymbol{x}), t \in [t', t''], \boldsymbol{x} \in \mathbb{R}^3\}$$

and Ω' its image under f

$$f : \Omega \to \Omega' = f(\Omega, \omega).$$

We consider the electromagnetic action being functional of A_μ

$$S_\Omega[A_\mu] = \frac{1}{c}\int_\Omega d^4x \mathcal{L}(A_\mu(x), \partial_\nu A_\mu(x); x). \tag{4.14}$$

The set of transformations (4.4) is *group symmetry* of the model[6] (4.14) providing that

$$S_{\Omega'}[A'_\mu] = S_\Omega[A_\mu] + \int_{\partial\Omega} d\Sigma_\mu K^\mu(A; x; \omega) \tag{4.15}$$

for all t', t'', where K^μ is *surface term*[7] equal to

$$\int_{\partial\Omega} d\Sigma_\mu K^\mu(A; x; \omega) = \int_\Omega d^4x \partial_\mu K^\mu.$$

Note that (4.15) gives the relationship between arbitrary functions $A_\mu(x)$ and $A'_\mu(x')$ even though they do not satisfy the Euler-Lagrange equations (*off-shell condition*). When A_μ represents physical field configurations (solutions of the Euler-Lagrange equations) than (4.15) is called *on-shell condition*. In such a case the symmetry condition leads to (on-shell) Noether's identity

$$\int_\Omega d^4x\, \partial_\mu j^\mu_\alpha = 0,$$

where j^μ_α are currents densities (Noether currents). We shall not present derivation of j^μ_α neither give their explicit form because we shall use different approach. The reason is that direct application of this formalism to electromagnetic field gives energy-momentum tensor which is *not gauge invariant*.

In order to get gauge-invariant form of currents and gauge-invariant energy-momentum tensor we adopt another strategy. Namely, we shall perform the symmetry transformation of the Lagrangian density[8]

$$\mathcal{L}(x) = -\frac{1}{16\pi}F_{\mu\nu}(x)F^{\mu\nu}(x)$$

generated by an infinitesimal Poincaré transformation taking into consideration formulas (4.9) and (4.13). We shall apply (4.13) to left hand side of

$$\delta\mathcal{L}(x) = -\frac{1}{8\pi}F^{\mu\nu}(x)\delta F_{\mu\nu}(x). \tag{4.16}$$

The symmetry condition

[6] Symmetry of a model does cannot be confused with symmetry of its solutions. In fact models invariant under rotations have also solutions which are not invariant under rotations.

[7] The surface term is admissible because in quantum mechanics such a term is related to a change of phase factor of state vectors. It can be shown that on classical levels the postulate (4.15) correctly captures the idea of symmetry of a model.

On-shell Noether's identity

Problem with gauge invariance

Alternative derivation resulting in gauge-invariant energy momentum tensor

[8] Note that variation of the action with respect to metric tensor gives also gauge-invariant the energy-momentum tensor.

and (4.9) to its right hand side. The left hand side of (4.16) reads

Symmetry transformation of scalar expression $\mathcal{L}(x)$

$$
\begin{aligned}
\delta\mathcal{L}(x) &= \mathcal{L}(x-\delta x) - \mathcal{L}(x) \\
&= \cancel{\mathcal{L}(x)} - \delta x^{\mu}\partial_{\mu}\mathcal{L}(x) - \cancel{\mathcal{L}(x)} \\
&= \frac{1}{16\pi}\delta x^{\mu}\partial_{\mu}\left[F_{\alpha\beta}(x)F^{\alpha\beta}(x)\right] = \frac{1}{16\pi}\delta x_{\nu}\eta^{\nu\mu}\partial_{\mu}\left[F_{\alpha\beta}F^{\alpha\beta}\right] \\
&= \frac{1}{16\pi}\partial_{\mu}\left[\delta x_{\nu}\eta^{\nu\mu}F_{\alpha\beta}F^{\alpha\beta}\right] - \frac{1}{16\pi}\overbrace{\underbrace{\partial_{\mu}\delta x_{\nu}}_{\omega_{\nu\mu}}\eta^{\nu\mu}}^{0}F_{\alpha\beta}F^{\alpha\beta}
\end{aligned} \tag{4.17}
$$

where

$$\partial_{\mu}\delta x_{\nu} = \partial_{\mu}(\omega_{\nu\alpha}x^{\alpha} + \varepsilon_{\nu}) = \omega_{\nu\mu}.$$

The last term vanishes because it contains contraction of symmetric and anti-symmetric tensors $\omega_{\nu\mu}\eta^{\nu\mu} \equiv 0$.

On the other hand, applying (4.9) and (4.12) to the right hand side of (4.16) one gets

$$
\begin{aligned}
F^{\mu\nu}\delta F_{\mu\nu} &= F^{\mu\nu}(x)[F'_{\mu\nu}(x) - F_{\mu\nu}(x)] \\
&= F^{\mu\nu}(x)[(\delta^{\alpha}{}_{\mu} - \omega^{\alpha}{}_{\mu})(\delta^{\beta}{}_{\nu} - \omega^{\beta}{}_{\nu})F_{\alpha\beta}(x-\delta x) - F_{\mu\nu}(x)] \\
&= F^{\mu\nu}(x)[\underbrace{F_{\mu\nu}(x-\delta x)}_{\cancel{F_{\mu\nu}(x)}-\delta x^{\alpha}\partial_{\alpha}F_{\mu\nu}(x)+\ldots} - \omega^{\alpha}{}_{\mu}\underbrace{F_{\alpha\nu}(x-\delta x)}_{F_{\alpha\nu}(x)+\ldots} - \omega^{\beta}{}_{\nu}\underbrace{F_{\mu\beta}(x-\delta x)}_{F_{\mu\beta}(x)+\ldots} - \cancel{F_{\mu\nu}(x)}] \\
&= F^{\mu\nu}[-\delta x^{\alpha}\partial_{\alpha}\overbrace{F_{\mu\nu}}^{-F_{\nu\mu}} - (\partial_{\mu}\delta x^{\alpha})F_{\alpha\nu} - \overbrace{(\partial_{\nu}\delta x^{\beta})F_{\mu\beta}}^{(\partial_{\nu}\delta x^{\alpha})F_{\mu\alpha}}] \\
&= F^{\mu\nu}[\delta x^{\alpha}\partial_{\alpha}F_{\nu\mu} - \partial_{\mu}(\delta x^{\alpha}F_{\alpha\nu}) + \delta x^{\alpha}\partial_{\mu}F_{\alpha\nu} - \partial_{\nu}(\delta x^{\alpha}F_{\mu\alpha}) + \delta x^{\alpha}\partial_{\nu}F_{\mu\alpha}] \\
&= F^{\mu\nu}[\delta x^{\alpha}\underbrace{(\partial_{\alpha}F_{\nu\mu} + \partial_{\nu}F_{\mu\alpha} + \partial_{\mu}F_{\alpha\nu})}_{0} - \partial_{\mu}(\delta x^{\alpha}F_{\alpha\nu}) - \partial_{\nu}(\delta x^{\alpha}F_{\mu\alpha})] \\
&= -F^{\mu\nu}[\partial_{\mu}(\delta x^{\alpha}F_{\alpha\nu}) + \partial_{\nu}(\delta x^{\alpha}F_{\mu\alpha})] = -2F^{\mu\nu}\partial_{\mu}(\delta x^{\alpha}F_{\alpha\nu}) \\
&= -2\partial_{\mu}(F^{\mu\nu}F_{\alpha\nu}\delta x^{\alpha}) + 2\underbrace{\partial_{\mu}F^{\mu\nu}}_{\frac{4\pi}{c}J^{\nu}}(F_{\alpha\nu}\delta x^{\alpha}).
\end{aligned} \tag{4.18}
$$

Hence the right hand side of (4.16) is of the form

$$-\frac{1}{8\pi}F^{\mu\nu}(x)\delta F_{\mu\nu}(x) = -\frac{1}{4\pi}\left[-\partial_{\mu}(F^{\mu\nu}F_{\alpha\nu}\delta x^{\alpha}) + \frac{4\pi}{c}J^{\nu}(F_{\alpha\nu}\delta x^{\alpha})\right]. \tag{4.19}$$

The identity (4.16) reads

$LHS = RHS$

$$\frac{1}{16\pi}\partial_{\mu}\left[\delta x_{\nu}\eta^{\nu\mu}F_{\alpha\beta}F^{\alpha\beta}\right] = -\frac{1}{4\pi}\left[-\partial_{\mu}(F^{\mu\nu}F_{\alpha\nu}\delta x^{\alpha}) + \frac{4\pi}{c}J^{\nu}(F_{\alpha\nu}\delta x^{\alpha})\right].$$

It can be written in the form

$$\frac{1}{4\pi}\left[-\partial_{\mu}\underbrace{(F^{\mu\nu}F_{\alpha\nu}\delta x^{\alpha})}_{F^{\mu\alpha}F^{\nu}{}_{\alpha}\delta x_{\nu}} + \frac{1}{4}\partial_{\mu}\left[\delta x_{\nu}\eta^{\nu\mu}F_{\alpha\beta}F^{\alpha\beta}\right]\right] + \frac{1}{c}\underbrace{J^{\nu}(F_{\alpha\nu}\delta x^{\alpha})}_{F^{\nu}{}_{\alpha}J^{\alpha}\delta x_{\nu}} = 0.$$

Defining the energy-momentum tensor

$T^{\mu\nu} = T^{\nu\mu}$

$$T^{\mu\nu} := \frac{1}{4\pi}\left[-F^{\mu\alpha}F^{\nu}{}_{\alpha} + \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right] \tag{4.20}$$

one gets

$$\underbrace{\partial_\mu (T^{\mu\nu}\delta x_\nu)}_{\partial_\mu T^{\mu\nu}\delta x_\nu + (\partial_\mu \delta x_\nu)T^{\mu\nu}} + \frac{1}{c}F^{\nu}{}_{\alpha}J^{\alpha}\delta x_\nu = 0. \tag{4.21}$$

The symmetry of the energy-momentum tensor under exchanging of its indices allows for further simplification. Remembering that expression $\partial_\mu x_\nu$ is anti-symmetric *i.e.* $\partial_\mu x_\nu = \omega_{\nu\mu} = -\omega_{\mu\nu}$ we get that $(\partial_\mu \delta x_\nu)T^{\mu\nu} \equiv 0$. Thus, Noether identity takes the following final form

$\partial_\mu \delta x_\nu T^{\mu\nu} = \omega_{\nu\mu}T^{\mu\nu} = 0$

Noether identity

$$\boxed{\partial_\nu T^{\mu\nu} + \frac{1}{c}F^{\mu}{}_{\nu}J^{\nu} = 0.} \tag{4.22}$$

The equation (4.22) is on-shell identity because it was derived using Maxwell's equations.

In absence of external currents $J^\nu = 0$ the electromagnetic field is an isolated object. Thus equation (4.22) simplifies to the following one

$$\partial_\mu T^{\mu\nu} = 0 \tag{4.23}$$

The equation (4.23) are local conservation laws for any physical (*i.e.* obeying Maxwell's equation) and isolated electromagnetic field configuration.

$T^{\mu\nu}$ in curvilinear coordintes

Note, similar considerations are possible in *curvilinear coordinates*. In such a case the ordinary partial derivatives in first pair of Maxwell's equations must be replaced by covariant ones

$$\nabla_\mu F^{\mu\nu} = \frac{4\pi}{c}J^\nu.$$

The same is true for second pair of Maxwell's equations (Bianchi identities)

$$\nabla_\mu F_{\nu\alpha} + \nabla_\alpha F_{\mu\nu} = -\nabla_\nu F_{\alpha\mu}.$$

Rising indices is imediate because the metric tensor is *covariantly constant* for choice of Christoffel symbols as connection coefficients. Thus covariant version of Noether identity reads

$$\nabla_\nu T^{\mu\nu} + \frac{1}{c}F^{\mu}{}_{\nu}J^{\nu} = 0$$

where

$$\boxed{T^{\mu\nu} = \frac{1}{4\pi}\left[-F^{\mu\alpha}F^{\nu}{}_{\alpha} + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right].} \tag{4.24}$$

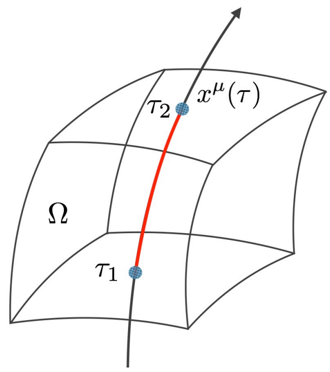


Figure 4.1: The world-line of the charged particle.

Balance of four-momenta

Although we announced (4.24) as energy-momentum tensor, its relation with energy and momentum has not been shown yet. In order to justify this name we consider a point-like charged particle in the external electromagnetic field. The trajectory of the particle is a solution of the equation

$$\frac{dp^\mu}{d\tau} = \frac{e}{c}F^{\mu\nu}u_\nu. \tag{4.25}$$

The interaction of the particle with electromagnetic field results in change of its four-momentum. In order to get the difference of particle four-momenta we look at spacetime region Ω such that the world-line of the particle pass through it, see Figure 4.1. Two instants τ_1 and τ_2 correspond with events of intersection of the world-line of the particle with the boundary $\partial\Omega$ of the region. The four-momenta take values $p^\mu(\tau_1)$ and $p^\mu(\tau_2)$ at two points at this world-line. The four-current density, associated with the motion of a point-like particle, reads

$$J^\nu(x') = ec \int_{-\infty}^{\infty} d\tau u^\nu(\tau)\delta^4\left(x' - x(\tau)\right). \tag{4.26}$$

The integral of (4.3) over the region Ω is given by

$$\int_\Omega d^4x'\ \partial_\nu T^{\mu\nu} + e\int_\Omega d^4x' \int_{-\infty}^{\infty} d\tau\, F^\mu{}_\nu(x')u^\nu\delta^4\left(x' - x(\tau)\right) = 0.$$

It can be written in the form

$$\oint_{\partial\Omega} T^{\mu\nu} d^3\Sigma_\nu + e\int_{\tau_1}^{\tau_2} d\tau\, F^\mu{}_\nu(x(\tau))\, u^\nu = 0. \tag{4.27}$$

Multiplying both sides of (4.27) by $\frac{1}{c}$ and using equation of motion (4.25) one gets

Difference of four-momenta of a charged particle $p^\mu(\tau_1) - p^\mu(\tau_2)$ is equal to variation of four-momentum of electromagnetic field

$$\boxed{\frac{1}{c}\oint_{\partial\Omega} T^{\mu\nu} d^3\Sigma_\nu + p^\mu(\tau_2) - p^\mu(\tau_1) = 0.} \tag{4.28}$$

This equality allows us to interpret the term

$$\frac{1}{c}\oint_{\partial\Omega} T^{\mu\nu} d^3\Sigma_\nu$$

as the variation of four-momentum of the electromagnetic field due to interaction with the charged particle.

4.3 *The energy-momentum tensor*

The energy-momentum tensor[9] possesses the following algebraic properties:

[9] Called also stress-energy tensor

- it is *symmetric* $T^{\mu\nu} = T^{\nu\mu}$,
- it is *traceless* $T^\mu{}_\mu = 0$.

In present section we shall analyse components of this tensor. In order to analyse the physical content of $T^{\mu\nu}$ we fix the inertial reference frame S and calculate its components in this frame. Moreover, for simplicity, we look at its Cartesian components. Its curvilinear components x'^μ can be easily obtained from the Cartesian ones through the transformation $x^\mu \to x'^\mu$ which gives

$$T'^{\mu\nu}(x') = \frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x'^\nu}{\partial x^\beta}T^{\alpha\beta}(x).$$

The component T^{00} is given by the expression

T^{00} : energy density

$$T^{00} = \frac{1}{4\pi}\left[-F^{0k}F^{0}{}_{k} + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\right] = \frac{1}{8\pi}(\boldsymbol{B}^2 + \boldsymbol{E}^2). \tag{4.29}$$

It has interpretation of *energy density of electromagnetic field*. The components of tensor $T^{\mu\nu}$ have dimension of energy density because Lagrangian density $\mathcal{L} \propto F^{\mu\nu}F_{\mu\nu}$ has this dimension $[\mathcal{L}] = ML^{-1}T^{-2}$. Thus

$$[T^{00}] = ML^{-1}T^{-2} = L^{-3}\underbrace{\left(ML^2T^{-2}\right)}_{\text{dim. of energy}},$$

where "density" refers to *volume density*. The integral over bulk of the box V (at rest in the reference frame S) gives total energy of electromagnetic field in this region.

The mixed components T^{i0} have the form

Components $T^{i0} = T^{0i}$

$$T^{i0} = -\frac{1}{4\pi}F^{ij}F^{0}{}_{j} = \frac{1}{4\pi}\epsilon_{ijk}B^kE^j = \frac{1}{4\pi}(\boldsymbol{E}\times\boldsymbol{B})^i. \tag{4.30}$$

The expression $\frac{1}{c}T^{i0}$ has dimension of momentum density (amount of linear momentum confined in unit volume)

$\frac{1}{c}T^{i0} = \mathcal{P}^i$: momentum density

$$\left[\frac{1}{c}T^{i0}\right] = L^{-3}\underbrace{\left(MLT^{-1}\right)}_{\text{dim. of momentum}} = L^{-1}T\underbrace{\left(ML^{-1}T^{-2}\right)}_{\text{dim. of energy density}}.$$

If the integration region V is some region on the hyperplane $x^0 = const$ then amount of linear momentum in V is given by

$$P^i = \frac{1}{c}\int_V T^{i0}d^3\Sigma_0.$$

Expression $T^{0j} = T^{j0}$ is proportional to *Poynting vector*

$cT^{0j} = S^j$: Poynting vector - energy flux density

$$S^j := \frac{c}{4\pi}(\boldsymbol{E}\times\boldsymbol{B})^j = cT^{0j} \tag{4.31}$$

which has interpretation[10] of *energy flux density* (amount of energy passing through unit area in unit time)

[10] Interpretation of components T^{i0} and T^{0i} depends on adopted convention of contraction of the tensor $T^{\mu\nu}$ with three-surface element $d^3\Sigma_\nu$. Here we have adopted convention $T^{\mu\nu}d^3\Sigma_\nu$ *i.e.* the second index of $T^{\mu\nu}$ is contracted with three-area element.

$$[\boldsymbol{S}] = L^{-2}T^{-1}\underbrace{\left(ML^2T^{-2}\right)}_{\text{dim. of energy}} = LT^{-1}\left(ML^{-1}T^{-2}\right).$$

In order to justify interpretation of this quantity we consider the integral[11]

[11] This integral has dimension of energy because $[T^{\mu\nu}] = [E]L^{-3}$ and $[d^3\Sigma_j] = L^3$

$$\int_\Sigma T^{0j}d^3\Sigma_j \equiv \frac{1}{c}\int_\Sigma S^jd^3\Sigma_j \tag{4.32}$$

where the region of spacetime Σ given by

$$\Sigma := [ct_1, ct_2] \times \partial V \tag{4.33}$$

and where $[ct_1, ct_2]$ is time interval and ∂V stands for border of the cuboid-shape region. The quantity $T^{01}d^3\Sigma_1$ represents the amount of energy which crosses an infinitesimal area dx^2dx^3 in time interval dx^0 *e.t.c.* Thus, the integral (4.32) expresses the total amount of energy that flows out/in (positive/negative value of the integral) the box V during the time interval $t_2 - t_1$.

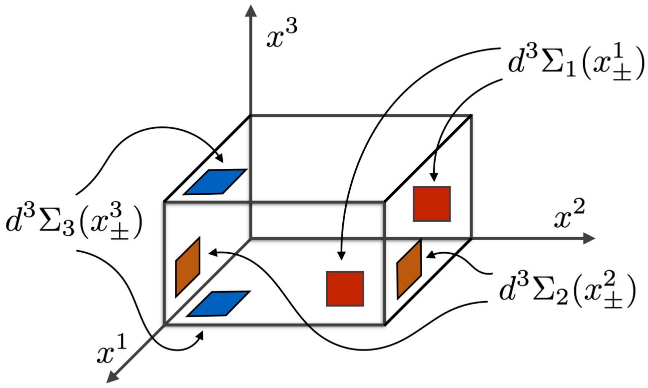


Figure 4.2: The box $\partial V = \Sigma|_{t=const}$ and its border. Three-volume integration elements on the region Σ have the form:

$$d^3\Sigma_1 = \pm dx^0dx^2dx^3,$$
$$d^3\Sigma_2 = \pm dx^0dx^1dx^3,$$
$$d^3\Sigma_3 = \pm dx^0dx^1dx^2.$$

The symmetry of energy-momentum tensor gives relation between the momentum density carried by electromagnetic field and energy flux density

$$c\mathcal{P}^i = T^{i0} = T^{0i} = \frac{1}{c}S^i. \tag{4.34}$$

Energy flux density is proportional to momentum density by factor c^2

$$S^i = \mathcal{P}^i c^2$$

Note, that the proportionality relation $\vec{S} = \vec{\mathcal{P}}c^2$ resembles the relativistic relation for particles $E = mc^2$.

Components of the tensor $T^{\mu\nu}$ containing spatial indices $\mu = i$ and $\nu = j$ read

T^{ij} : Stress tensor

$$\begin{aligned} T^{ij} &= -\frac{1}{4\pi}\left[F^{i\alpha}F^{j}{}_{\alpha} + \frac{1}{4}\delta_{ij}F_{\alpha\beta}F^{\alpha\beta}\right] \\ &= -\frac{1}{4\pi}\left[F_{i0}F_{j0} - F_{ik}F_{jk} + \frac{1}{4}\delta_{ij}F_{\alpha\beta}F^{\alpha\beta}\right] \\ &= -\frac{1}{4\pi}\left[E^iE^j - (-\epsilon_{ikl}B^l)(-\epsilon_{jkm}B^m) + \frac{1}{2}\delta_{ij}(\boldsymbol{B}^2 - \boldsymbol{E}^2)\right] \\ &= -\frac{1}{4\pi}\left[E^iE^j - (\delta_{ij}\boldsymbol{B}^2 - B^iB^j) + \frac{1}{2}\delta_{ij}(\boldsymbol{B}^2 - \boldsymbol{E}^2)\right] \\ &= -\frac{1}{4\pi}\left[E^iE^j + B^iB^j - \frac{1}{2}\delta_{ij}(\boldsymbol{E}^2 + \boldsymbol{B}^2)\right] \equiv -\sigma_{ij}. \end{aligned} \tag{4.35}$$

It is called *stress tensor*. We express its physical dimension in terms of dimension of momentum[12] and it reads

$$[T^{ij}] = ML^{-1}T^{-2} = L^{-2}T^{-1}\underbrace{\left(MLT^{-1}\right)}_{\text{dim. of momentum}}.$$

[12] It is suggested by spatial character of indices

T^{ij} : momentum flux density

Tensor $\sigma_{ij} := -T^{ij}$ has interpretation of *momentum flux density* (amount of "i"-th component of the momentum that crosses unit area perpendicular to "j"-th axis in unit of time).[13] When integrated on three-surface Σ, defined by (4.33), it gives expression with dimension of momentum multiplied by LT^{-1}

$$(ML^{-1}T^{-2})L^3 = (MLT^{-1})LT^{-1}.$$

[13] In convention $d^3\Sigma_\mu T^{\mu\nu}$ interpretation of indices "i" and "j" is exchanged.

The component "i" of linear momentum that flows through the two-surface dx^2dx^3 during the time interval dx^0 is given by

$$\pm\frac{1}{c}T^{i1}dx^0dx^2dx^3 \equiv \pm T^{i1}dtdx^2dx^3$$

The integral

$$\begin{aligned}
\frac{1}{c}\int_\Sigma T^{ij}d^3\Sigma_j &= \frac{1}{c}\int_\Sigma T^{ij}\left(\frac{1}{3!}\epsilon_{j\alpha\beta\gamma}\frac{\partial(x^\alpha x^\beta x^\gamma)}{\partial(\lambda^0\lambda^1\lambda^2)}d\lambda^0d\lambda^1d\lambda^2\right)\\
&= \frac{1}{c}\int_\Sigma T^{ij}\,dx^0\left(-\frac{1}{2}\epsilon_{0jkl}\frac{\partial(x^kx^l)}{\partial(\lambda^1\lambda^2)}d\lambda^1d\lambda^2\right)\\
&= \frac{1}{c}\int_\Sigma (-T^{ij})\,dx^0\left(\frac{1}{2}\epsilon_{jkl}\frac{\partial(x^kx^l)}{\partial(\lambda^1\lambda^2)}d\lambda^1d\lambda^2\right)\\
&= \int_\Sigma \sigma_{ij}\,dt\,da^j = \int_{t_1}^{t_2}dt\oint_{\partial V}\sigma_{ij}da^j \qquad (4.36)
\end{aligned}$$

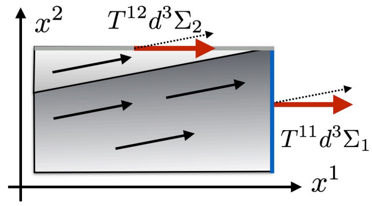


Figure 4.3: Flux of component $i=1$ of T^{ij} through "area" elements $d^3\Sigma_1$ and $d^3\Sigma_2$.

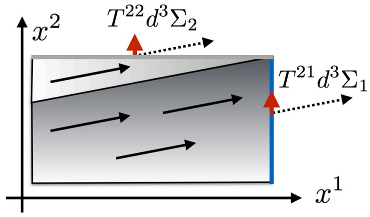


Figure 4.4: Flux of component $i=2$ of T^{ij} through "area" elements $d^3\Sigma_1$ and $d^3\Sigma_2$.

taken on the surface Σ, defined in (4.33), has interpretation of "i"-th component of total momentum that crosses the border of the box V during the time interval $t_2 - t_1$. Note, that the integral (4.36) can be transformed into three-volume integral evaluated on the interior of V

Relation with force density

$$\oint_{\partial V}\sigma_{ij}da^j = \int_V \partial_j\sigma_{ij}d^3x = \int_V f^i\,d^3x. \qquad (4.37)$$

This integral represent three components of net force acting on the volume V.

The analogy with material media

A formally identical expressions appear in theory of elasticity in description of tensions that are present in deformed physical bodies.[14] The net force, acting on small volume element of a deformed body, can be transmitted only across its border. It cannot be caused by internal forces because such forces must cancel out due to the action-reaction equality. Consequently, the total force, obtained integratng force density f^i, has origin in the surface integrals. It is possible if the force density is given as divergence of certain tensor, called *stress tensor*, σ_{ij}, *i.e.* $f^i = \partial_j\sigma_{ij}$.

[14] Landau, Lifshitz "Theory of Elasticity"

Summarizing our results, we have found that the energy-momentum tensor is of the form

$$T^{\mu\nu} \to \left(\begin{array}{c|c} u & \frac{1}{c} S^i \\ \hline c\mathcal{P}^j & -\sigma_{ij} \end{array} \right) \tag{4.38}$$

where $\frac{1}{c} S^i = c\mathcal{P}^i$ according to $T^{\mu\nu} = T^{\nu\mu}$.

4.4 *Integral form of conservation laws*

Conservation of electric charge

Before discussing conserved quantities that are consequences of Noether identity $\partial_\nu T^{\mu\nu} = 0$ we shall analyse the continuity equation involving electromagnetic four-current. The continuity equation follows directly from first pair of Maxwell's equations. Taking four-divergence of its both sides one gets

Local conservation law of the electric charge

$$\underbrace{\partial_\mu \partial_\nu F^{\mu\nu}}_{\equiv 0} = \frac{4\pi}{c} \partial_\nu J^\nu \qquad \Rightarrow \qquad \partial_\nu J^\nu = 0. \tag{4.39}$$

The equation (4.39) is similar to Noether identity $\partial_\nu T^{\mu\nu}$, however, simpler than it. This is the main reason why we discuss first the problem of integration of such equation considering (4.39). The integral of continuity equation over some four-volume Ω gives quantity of electric charge inside this region. This can be seen as follows. We fix the four-volume Ω whose border $\partial\Omega$ consists on three-dimensional regular surfaces. The integral over the region Ω can be converted into three-dimensional integral over its border

Global version of conservation law

$$\int_\Omega d^4\Omega \partial_\nu J^\nu = \int_{\partial\Omega} d^3\Sigma_\nu J^\nu = 0. \tag{4.40}$$

The three-dimensional volume element $d^3\Sigma_\nu$ reads

Three-surfaces $x^\nu = const$ in the Minkowski spacetime

$$d^3\Sigma_\nu = \frac{1}{3!} \sqrt{-g} \epsilon_{\nu\alpha\beta\gamma} \frac{\partial(x^\alpha x^\beta x^\gamma)}{\partial(t^1 t^2 t^3)} dt^1 dt^2 dt^3 \tag{4.41}$$

where $x^\mu = x^\mu(t^1, t^2, t^3)$.

We shall take the hyper-surface $\partial\Omega$ which has the form presented in Figure 4.5. It consists of two purely spatial three-surfaces at t_1 and t_2, denoted by Σ_1 i Σ_2 (in a certain inertial reference frame), and the three-surface Σ_∞ such that $\partial\Omega = \Sigma_1 \cup \Sigma_2 \cup \Sigma_\infty$. The integral over $\partial\Omega$ takes the form

$$\int_{\Sigma_2} d^3\Sigma_\nu J^\nu - \int_{\Sigma_1} d^3\Sigma_\nu J^\nu + \int_{\Sigma_\infty} d^3\Sigma_\nu J^\nu = 0, \tag{4.42}$$

where the minus sign reflects the border orientation ($\partial\Omega$ is outwardly oriented). Thus orientation of Σ_1 is given by $-e_0$. In addition, the spatial character of the surfaces Σ_1 and Σ_2 implies that the only non-vanishing volume element $d^3\Sigma_\nu$ is spatial volume element $d^3\Sigma_0 \equiv dV$. Assuming that the current J^ν is localized *i.e.* it vanishes currents at spatial infinity one gets (after multiplying by $\frac{1}{c}$) that electric charge preserve its value on hyper-surfaces parametrised by t

$$\underbrace{\frac{1}{c}\int_{\Sigma_2} d^3\Sigma_0 J^0}_{Q(t_2)} = \underbrace{\frac{1}{c}\int_{\Sigma_1} d^3\Sigma_0 J^0}_{Q(t_1)}. \tag{4.43}$$

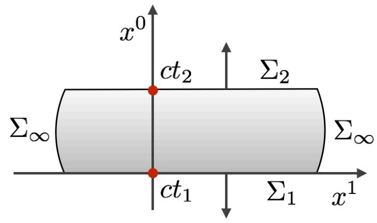


Figure 4.5: The region Ω and its border $\partial\Omega$ that consists on two surfaces simultaneity Σ_1 and Σ_2 and the surface $\Sigma_\infty \times [ct_1, ct_2]$.

It shows that differential equations, like $\partial_\nu J^\nu = 0$, represent local conservation laws.

Note, that demonstration of electric charge conservation does not requires purely spatial character of surfaces Σ_1 and Σ_2. In fact, it is sufficient have them as *space-like surfaces*. The example of such surfaces is presented in Figure 4.6 and Figure 4.7. The electric charges obtained by integration over these hyper-surfaces read

$$\underbrace{\frac{1}{c}\int_{\Sigma_2} d^3\Sigma_\nu J^\nu}_{Q_2} = \underbrace{\frac{1}{c}\int_{\Sigma_1} d^3\Sigma_\nu J^\nu}_{Q_1}, \tag{4.44}$$

providing that the four-current vanishes at spatial infinity. Both terms in (4.44) are invariant expressions representing the amount of electric charge.

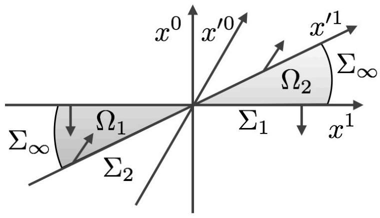


Figure 4.6: The region $\partial\Omega$ which consists of two surfaces of simultaneity in two different inertial reference frames an the sphere at spatial infinity.

Conservation of four-momentum

The electromagnetic field is isolated in the absence of electromagnetic four-currents, $J^\mu = 0$. In such a case the equation (4.3) simplifies to the following one

$$\partial_\nu T^{\mu\nu} = 0.$$

In fact we have here four continuity equations - each one for each value of ν. The integral of the equation $\partial_\nu T^{\mu\nu} = 0$ taken over the region Ω can be replaced by integral over the border $\partial\Omega$ of that region

$$\int_\Omega d^4\Omega\, \partial_\nu T^{\mu\nu} = 0 \qquad \Rightarrow \qquad \int_{\partial\Omega} d^3\Sigma_\nu T^{\mu\nu} = 0. \tag{4.45}$$

Assuming that the energy-momentum tensor vanishes sufficiently quickly[15] at spatial infinity $|\boldsymbol{x}| \to 0$ (quicker than $|\boldsymbol{x}|^{-2}$)

$$T^{\mu\nu} \to 0 \qquad \text{for} \qquad |\boldsymbol{x}| \to \infty,$$

and taking the region of integration as in Figure 4.5 that one gets

Conservation of four-momentum for region presented in Figure 4.5

$$\underbrace{\int_{\Sigma_2} d^3\Sigma_0 T^{\mu 0}}_{c\,P^\mu(t_2)} - \underbrace{\int_{\Sigma_1} d^3\Sigma_0 T^{\mu 0}}_{c\,P^\mu(t_1)} + \underbrace{\int_{\Sigma_\infty} d^3\Sigma_\nu T^{\mu\nu}}_{0} = 0, \tag{4.46}$$

[15] It excludes the radiation field which behaves at spatial infinity as $F^{\mu\nu} \sim |\boldsymbol{x}|^{-1}$ so $T^{\mu\nu} \sim |\boldsymbol{x}|^{-2}$.

where $P^\mu(t_1)$ i $P^\mu(t_2)$ are two four-momenta of electromagnetic field at Σ_1 and Σ_2. This relation shows that total four-momentum of electromagnetic field is conserved. In particular, conservation of the component P^0 expresses energy conservation of the field, whereas conservation of P^i expresses three-momentum conservation.

The precise meaning of the integral at spatial infinity:

$$\int_{\Sigma_\infty} d^3\Sigma_\nu T^{\mu\nu} := \lim_{|\boldsymbol{x}|\to\infty} \int_{\Sigma_{|\boldsymbol{x}|}} d^3\Sigma_\nu T^{\mu\nu}$$

Taking Σ_1 and Σ_2 as two space-like surfaces, shown in Figure 4.6 and Figure 4.7, and assuming that the energy-momentum tensor vanishes sufficiently quickly at spatial infinity one gets *four-momemtum conservation*

Four-momentum conservation for the regions presented in Figure 4.6 and Figure 4.7

$$\frac{1}{c}\int_{\Sigma_1} d^3\Sigma_\nu T^{\mu\nu} = \frac{1}{c}\int_{\Sigma_2} d^3\Sigma_\nu T^{\mu\nu}, \tag{4.47}$$

where the total four-momentum P^μ is defined as the integral over the hyper-surface Σ

$$\boxed{P^\mu(\Sigma) := \frac{1}{c}\int_{\Sigma} d^3\Sigma_\nu T^{\mu\nu}.} \tag{4.48}$$

This result has the following interpretation: the total four-momentum of the field is conserved quantity which means that two different inertial observers, shown in Figure 4.6, on the simultaneity surfaces Σ_1 and Σ_2 measure *equal amounts of total four-momentum of the field*

Interpretation of conservation of the four-momentum

$$P = P^\mu e_\mu = P'^\alpha e'_\alpha.$$

The respective *components* of total four-momentum in different inertial frames do not coincide, $P^\mu \neq P'^\mu$, because $\{e_\mu\}$ and $\{e'_\alpha\}$ are different. Both sides of the equality (4.47) are expressed in *the same coordinates* (*e.g.* of the observer S), however, the integrals are taken over *different hyper-surfaces*.

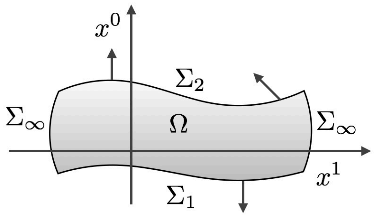


Figure 4.7: The region $\partial\Omega$ which consists on two arbitrary space-like surfaces and the sphere at spatial infinity.

The right hand side of (4.47) can be equivalently expressed in coordinates of the observer S'. Taking Σ_1 as hyper-surface of simultaneity in S and Σ_2 as hyper-surface of simultaneity in S' we get

$$\underbrace{\frac{1}{c}\int_{\Sigma_1} d^3\Sigma_0 T^{\mu 0}}_{P^\mu(\Sigma_1)} = (L^{-1})^\mu{}_\alpha \underbrace{\frac{1}{c}\int_{\Sigma_2} d^3\Sigma'_0 T'^{\alpha 0}}_{P'^\alpha(\Sigma_2)}, \tag{4.49}$$

where the only non-vanishing three-volume element on the surface Σ_2 (in coordinates of the observer S') is $d^3\Sigma'_0$. Similarly, the only non-vanishing component of the volume element at Σ_1 in coordinates of the observer S is $d^3\Sigma_0$.

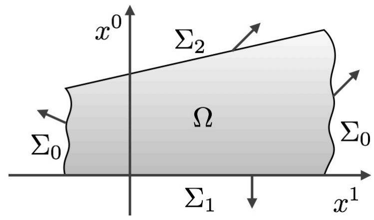


Figure 4.8: The compact region Ω with time-varying border.

In the example shown in Figure 4.7 there is no global inertial reference frame such that the surface Σ_1 (or Σ_2) would be a simultaneity surface (one cannot reduce an integration over such surfaces to a purely spatial integration with the volume element $d^3\Sigma'_0$).

Another example, shown in Figure 4.8, presents the case when integration region Ω is finite. The difference of four-momenta at Σ_1

and Σ_2 depends on the four-momentum flow through the time-like surface Σ_0

$$P^\mu(\Sigma_2) - P^\mu(\Sigma_1) = -\frac{1}{c}\int_{\Sigma_0} d^3\Sigma_\nu T^{\mu\nu}. \qquad (4.50)$$

The integral over Σ_0 represents flow of four-momentum from/to the region Ω. The formula (4.50) expresses balance of the field four-momenta and flow of four-momentum through the border.

Angular momentum of the field

Before discussing the problem of angular momentum stored in electromagnetic field, it is very instructive to recall the mechanical example illustrating the concept of torque and angular momentum in the case of continuous media.

Example: continuous media

The torque acting on some infinitesimal volume element dV containing matter and localised at x^i reads [16]

[16] Here $g_{ij} = \delta_{ij}$ so $x^k = x_k$, $f^k = f_k$ *e.t.c.*

$$d\tau_{jk} = (x^j dF^k - x^k dF^j) = (x^j f^k - x^k f^j)dV,$$

where f^k is force density at the point $\boldsymbol{r} = x^j \hat{\boldsymbol{e}}_j$. The usual form of torque is obtained using duality transformation

$$d\tau^i = \frac{1}{2}\epsilon_{ijk} d\tau_{jk} = \epsilon_{ijk} x^j f^k dV = (\boldsymbol{r} \times \boldsymbol{f})^i dV.$$

Similarly, matter inside some infinitesimal element dV at $\boldsymbol{r}$ has angular momentum

Angular momentum in material media

$$dM^{jk} = x^j dp^k - x^k dp^j, \qquad (4.51)$$

where dp^k is linear momentum of matter inside the region dV. The most frequently used form of angular momentum

$$dJ^i = \frac{1}{2}\epsilon_{ijk} dM^{jk} = \epsilon_{ijk} x^j dp^k = (\boldsymbol{r} \times d\boldsymbol{p})^i$$

is formally the dual tensor of dM^{jk} in three-dimensional space $\mathbb{E}^3$. The expression dM^{ij} captures correctly the idea of angular momentum for continuous material object.

The electromagnetic field has angular momentum. The amount of angular momentum in certain region of space is given as integral if its density over this region. This density is a rank-three tensor built up of $T^{\mu\nu}$ and x^α

Definition of the field angular momentum density

$$\boxed{M^{\mu\nu\alpha} := x^\mu T^{\nu\alpha} - x^\nu T^{\mu\alpha}.} \qquad (4.52)$$

Since the energy-momentum tensor $T^{\mu\nu}$ and the currents J^μ obey

$$\partial_\alpha T^{\nu\alpha} = -\frac{1}{c} F^{\nu\alpha} J_\alpha,$$

then the four-divergence $\partial_\alpha M^{\mu\nu\alpha}$ can be cast in the form

$$\begin{aligned}\partial_\alpha M^{\mu\nu\alpha} &= x^\mu \partial_\alpha T^{\nu\alpha} - x^\nu \partial_\alpha T^{\mu\alpha} + \overbrace{\delta^\mu_\alpha T^{\nu\alpha} - \delta^\nu_\alpha T^{\mu\alpha}}^{0} \\ &= -\frac{1}{c}(x^\mu F^{\nu\alpha} - x^\nu F^{\mu\alpha}) J_\alpha .\end{aligned}$$

Hence, $M^{\mu\nu\alpha}$ satisfies the following differential equation

$$\boxed{\partial_\alpha M^{\mu\nu\alpha} + \frac{2}{c} x^{[\mu} F^{\nu]\alpha} J_\alpha = 0.} \tag{4.53}$$

In the absence of currents, $J^\alpha = 0$, the electromagnetic field is an isolated object. It leads to continuity equations for $M^{\mu\nu\alpha}$, namely

Local conservation law of angular momentum of electromagnetic field

$$\partial_\alpha M^{\mu\nu\alpha} = 0. \tag{4.54}$$

Total quantity of angular momentum at the space-like three-surface Σ reads

$$\begin{aligned}M^{\mu\nu} &:= \frac{1}{c}\int_\Sigma (x^\mu T^{\nu\alpha} - x^\nu T^{\mu\alpha}) d^3\Sigma_\alpha . \\ &= \int_\Sigma (x^\mu dP^\nu - x^\nu dP^\mu) \equiv \int_\Sigma dM^{\mu\nu}\end{aligned} \tag{4.55}$$

where

$$dP^\mu = \frac{1}{c} T^{\mu\alpha} d^3\Sigma_\alpha \tag{4.56}$$

stands for infinitesimal amount of four-momentum and $dM^{\mu\nu}$ for infinitesimal amount of angular momentum of the field. Comparing the integrand of (4.55) and the expression (4.51) we see similarity between these two expressions.

The integral (4.56) does not depend on the choice of the surface Σ. Integrating the equation (4.54) over the four-dimensional region Ω (like this one presented in Figure 4.7) with the border[17] $\partial\Omega = \Sigma_1 \cup \Sigma_2 \cup \Sigma_\infty$ and multiplying it by $1/c$ we get the equality of integrals[18] on Σ_1 and Σ_2

Independence of $M^{\mu\nu}$ on the choice of Σ

[17] Σ_1 and Σ_2 are spatial-like surfaces and Σ_∞ is a time-like surface at spatial infinity

[18] We assume that $T^{\mu\nu}$ vanishes sufficiently quickly at spatial infinity (absence of radiation field).

$$\underbrace{\frac{1}{c}\int_{\Sigma_1} M^{\mu\nu\alpha} d^3\Sigma_\alpha}_{M^{\mu\nu}(\Sigma_1)} = \underbrace{\frac{1}{c}\int_{\Sigma_2} M^{\mu\nu\alpha} d^3\Sigma_\alpha}_{M^{\mu\nu}(\Sigma_2)} . \tag{4.57}$$

In particular, when the surface Σ is purely spatial (surface $x^0 = const$ in certain inertial reference frame) then only $d^3\Sigma_0 = d^3x$ does not vanish. In such a case, one gets

Angular momentum of the electromagnetic field at the surface $x^0 = const$

$$M^{\mu\nu} = \frac{1}{c}\int_{\mathbb{R}^3} d^3x\, (x^\mu T^{\nu 0} - x^\nu T^{\mu 0}).$$

Since the expression $\frac{1}{c}T^{\mu 0}$ is density of four-momentum then the electromagnetic field inside the volume element d^3x at x^μ has four-momentum $\frac{1}{c}T^{\mu 0} d^3x$.

Poynting theorem

What is the physical meaning of equations

$$\partial_\nu T^{\mu\nu} + \frac{1}{c} F^{\mu}{}_{\nu} J^\nu = 0$$

integrated over some (purely spatial) region V at $t = const$? The current density J^μ represents contributions from many charged particles.

Integrating the $\mu = 0$ equation over V one gets

Integral on three-volume V of component $\mu = 0$

$$\frac{1}{c}\int_V d^3x\, \partial_t T^{00} + \int_V d^3x\, \partial_j T^{0j} + \frac{1}{c}\int_V d^3x\, \boldsymbol{E}\cdot\boldsymbol{J} = 0.$$

Since the border of V (by assumption) does not depend on time, then the integral of $\partial_t T^{00}$ over V is equal to the total time derivative of the integral of T^{00} over V. Applying the divergence theorem to the second term, containing divergence of T^{0j}, one gets

$$\frac{d}{dt}\underbrace{\int_V d^3x\, T^{00}}_{E} + \underbrace{c\oint_{\partial V} da^j \overbrace{T^{0j}}^{\frac{1}{c}S^j}}_{\oint d\boldsymbol{a}\cdot\boldsymbol{S}} + \int_V d^3x\, \boldsymbol{E}\cdot\boldsymbol{J} = 0. \tag{4.58}$$

The first term in (4.58) describes variation of the energy of the electromagnetic field in V and the second term expresses flux of the energy through the border ∂V. Finally, the third term has interpretation of power associated with transfer of energy to charged particles. It can be seen taking the four-current density

$$\boldsymbol{J}(t,\boldsymbol{x}) = \sum_{k=1}^{N} q_k v_k \delta^3[\boldsymbol{x} - \boldsymbol{x}_k(t)], \tag{4.59}$$

where $\boldsymbol{x}_k(t)$ are trajectories of the particles and plugging (4.59) into the third term. It gives

$$\begin{aligned}\int_V d^3x\, \boldsymbol{E}(t,\boldsymbol{x})\cdot\boldsymbol{J}(t,\boldsymbol{x}) &= \sum_{k=1}^{N} q_k \boldsymbol{E}(t,\boldsymbol{x}_k(t))\cdot\boldsymbol{v}_k \\ &= \sum_{k=1}^{N} \boldsymbol{F}(t,\boldsymbol{x}_k(t))\cdot\frac{d\boldsymbol{x}_k}{dt}. \end{aligned}\tag{4.60}$$

The above expression has interpretation of power *i.e* work done by the electromagnetic field in unit of time. The equation (4.58) is known as *Poynting theorem*.

Poynting theorem

Theorem. *The rate of change of energy of the electromagnetic field in the region V is equal to minus energy flux through the border ∂V and the work done by the field on charged particles inside V.*

The loss of energy per unit of time $\frac{dE}{dt} < 0$ is related with flow of energy outside of the region, $\oint d\boldsymbol{a}\cdot\boldsymbol{S} > 0$, and/or work done by the field on particles, $\int_V d^3x\, \boldsymbol{E}\cdot\boldsymbol{J} > 0$.

Similarly, integrating equations with $\mu = i$ one gets

Integral on three-volume V of components $\mu = i$

$$\int_V d^3x\, \partial_t \left(\frac{1}{c}T^{i0}\right) + \int_V d^3x\, \partial_j T^{ij} + \int_V d^3x \left[\rho E^i + \frac{1}{c}(\boldsymbol{J}\times\boldsymbol{B})^i\right] = 0.$$

This equality can be cast in the form

$$\frac{d}{dt}\underbrace{\left[\frac{1}{c}\int_V d^3x\, \partial_t T^{i0}\right]}_{P^i} + \underbrace{\oint_{\partial V} T^{ij} da^j}_{\text{flux of momentum}} + \int_V d^3x \left[\rho E^i + \frac{1}{c}(\boldsymbol{J}\times\boldsymbol{B})^i\right] = 0. \tag{4.61}$$

One can expresses the last term in the following form

$$\begin{aligned}
&\int_V d^3x \left[\rho E^i + \frac{1}{c}(\boldsymbol{J}\times\boldsymbol{B})^i\right] = \\
&= \int_V d^3x \sum_{k=1}^{N} q_k \left[E^i(t,\boldsymbol{x}) + \frac{1}{c}(\boldsymbol{v}\times\boldsymbol{B}(t,\boldsymbol{x}))^i\right] \delta^3[\boldsymbol{x}-\boldsymbol{x}_k(t)] = \\
&= \sum_{k=1}^{N} q_k \left[E^i(t,\boldsymbol{x}_k) + \frac{1}{c}\left(\boldsymbol{v}_k\times\boldsymbol{B}(t,\boldsymbol{x}_k)\right)^i\right].
\end{aligned}$$

The above expression represents sum of Coulomb's and Lorentz's forces. The electromagnetic field acting on the charged particles changes their linear momenta and the rate of change of these momenta is given by the sum of forces. Equation (4.61) describes *balance of liner momenta*.

The balance of three-momenta

Theorem. *The rate of change of the linear momentum of the electromagnetic field in the region V is equal to the momentum flow of the field through the border ∂V (flux of the momentum per unit of time) and the rate of change of linear momentum caused by its transfer to the particles in V.*

The loss of the momentum per unit of time, $\frac{dP^i}{dt} < 0$, is associated with the momentum which flows out of the region and/or the momentum transferred to the charged particles

$$\int_V d^3x \left[\rho E^i + \frac{1}{c}(\boldsymbol{J}\times\boldsymbol{B})^i\right] > 0.$$

Chapter 5
Electromagnetic waves

5.1 Electromagnetic waves in non-dispersive dielectric media

In this section we discuss some basic properties of electromagnetic waves.[1] Maxwell noticed that sourceless electromagnetic field equations possess wave solutions. This theoretical prediction was experimentally confirmed by H. Hertz in November 1886. The most important Hertz's results were published in 1888.[2]

[1] The radiation mechanism is discussed in a separate chapter.

[2] Ann. Phys. **34** 610, Ann. Phys. **36** 769, Ann. Phys. **36** 1

Wave equations

Macroscopic *sourceless* Maxwell's equations in non-conducting continuous media read

$$-\partial_0 \mathbf{D} + \nabla \times \mathbf{H} = 0, \tag{5.1}$$

$$\nabla \cdot \mathbf{D} = 0, \tag{5.2}$$

$$\partial_0 \mathbf{B} + \nabla \times \mathbf{E} = 0, \tag{5.3}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{5.4}$$

where $\partial_0 \equiv \frac{1}{c}\partial_t$. They can be solved only if there are also given *constitutive relations i.e.* relations between fields forming pairs $(\mathbf{E}, \mathbf{B})$ and $(\mathbf{D}, \mathbf{H})$. In the simplest case of isotropic linear media those relations have the form

Constitutive relations in linear and isotropic media

$$\boldsymbol{D} = \varepsilon \boldsymbol{E}, \qquad \boldsymbol{H} = \frac{1}{\mu}\boldsymbol{B} \tag{5.5}$$

where $\varepsilon = const$ and $\mu = const$. Acting with the curl operator "$\nabla\times$" on equations (5.1) and (5.3) one gets

$$-\varepsilon\mu\partial_0 \underbrace{\nabla \times \mathbf{E}}_{-\partial_0 \mathbf{B}} + \underbrace{\nabla \times \nabla \times \mathbf{B}}_{\nabla(\nabla\cdot\mathbf{B}) - \nabla^2\mathbf{B}} = 0, \tag{5.6}$$

$$\partial_0 \underbrace{\nabla \times \mathbf{B}}_{\varepsilon\mu\partial_0 \mathbf{E}} + \underbrace{\nabla \times \nabla \times \mathbf{E}}_{\nabla(\nabla\cdot\mathbf{E}) - \nabla^2\mathbf{E}} = 0. \tag{5.7}$$

It gives

$$\partial^2 \mathbf{E} = 0, \qquad \text{and} \qquad \partial^2 \mathbf{B} = 0 \tag{5.8}$$

where ∂^2 is the d'Alembert operator

$$\boxed{\partial^2 \equiv \frac{\varepsilon\mu}{c^2}\,\partial_t^2 - \nabla^2.} \tag{5.9}$$

Equations (5.8) are just wave equations. They indicate that Maxwell's theory may support the existence of electromagnetic waves.[3] The parameter (characteristic speed)

$$v := \frac{c}{\sqrt{\varepsilon\mu}} \tag{5.10}$$

which appears in the d'Alembert operator depends on properties of continuous media. Note that for $\varepsilon = 1 = \mu$ the continuous medium is replaced by an empty space where $v = c$. The fact that the speed of electromagnetic waves in dielectric media is lower than the speed of light in empty space does not violate the Einstein's postulate. The slowing effect originates in effective description of dielectric media. On quantum level photons are absorbed end emitted by atoms. Thus, speed v cannot be interpreted as real speed of photons – it is rather a sort of phenomenological parameter.

[3] Wave equations are resultant equations and thus not all their solutions are solutions of the set of Maxwell's equations. In order to get the electromagnetic wave solution one has to show that solution of wave equation solves Maxwell's equations.

Longitudinal and transverse components

The electromagnetic potentials $(\varphi, \mathbf{A})$ also satisfy the wave equation. Indeed, plugging expressions

$$\boldsymbol{E} = -\partial_0 \boldsymbol{A} - \nabla\varphi, \qquad \boldsymbol{B} = \nabla \times \boldsymbol{A} \tag{5.11}$$

into the first pair of Maxwell's equations (5.1) and (5.2) one gets

$$(\varepsilon\mu\,\partial_0^2 - \nabla^2)\mathbf{A} + \nabla(\varepsilon\mu\,\partial_0\varphi + \nabla\cdot\boldsymbol{A}) = 0, \tag{5.12}$$

$$-\partial_0(\nabla\cdot\mathbf{A}) - \nabla^2\varphi = 0. \tag{5.13}$$

Furthermore, imposing *the Lorenz gauge condition*

Lorenz gauge condition

$$\boxed{\varepsilon\mu\,\partial_0\varphi + \nabla\cdot\boldsymbol{A} = 0} \tag{5.14}$$

one gets that potentials φ and $\boldsymbol{A}$ satisfy the wave equation

$$\boxed{\partial^2\varphi = 0, \qquad \partial^2\boldsymbol{A} = 0.} \tag{5.15}$$

The approach based on the electromagnetic potentials requires further comments due to some redundancy in description based on electromagnetic potentials. The gauge transformations of the potentials

$$\varphi'(x) = \varphi(x) - \partial_0\chi(x) \qquad \text{and} \qquad \boldsymbol{A}'(x) = \boldsymbol{A}(x) + \nabla\chi(x)$$

preserve the fields $\boldsymbol{E}$ and $\boldsymbol{B}$. It means that *there exists a whole class of electromagnetic potentials that describe the same physical situation*.

We shall split the solution of the equation (5.15) into two components

Decomposition on two sets of potentials

$$(\varphi, \boldsymbol{A}) = \underbrace{(\varphi, \boldsymbol{0})}_{(\varphi_1, \boldsymbol{A}_1)} + \underbrace{(0, \boldsymbol{A})}_{(\varphi_2, \boldsymbol{A}_2)} \tag{5.16}$$

and assume that each pair of components satisfies the Lorenz condition

$$\varepsilon\mu\,\partial_0 \overbrace{\varphi_1}^{\varphi} + \nabla\cdot \overbrace{\boldsymbol{A}_1}^{\boldsymbol{0}} = 0 \qquad \Rightarrow \qquad \partial_0\varphi = 0 \tag{5.17}$$

$$\varepsilon\mu\,\partial_0 \underbrace{\varphi_2}_{0} + \nabla\cdot \underbrace{\boldsymbol{A}_2}_{\boldsymbol{A}} = 0 \qquad \Rightarrow \qquad \nabla\cdot\boldsymbol{A} = 0 \tag{5.18}$$

Clearly, the potentials $(\varphi, \boldsymbol{A})$ being a superposition of $(\varphi_1, \boldsymbol{A}_1)$ and $(\varphi_2, \boldsymbol{A}_2)$ satisfy the Lorenz condition as well. We shall assume that each pair of potentials (5.16) satisfies the wave equation. Potentials $(\varphi_1, \boldsymbol{A}_1)$ give

$$\boldsymbol{E}_1 = -\nabla\varphi, \qquad \boldsymbol{B}_1 = 0, \tag{5.19}$$

whereas potentials $(\varphi_2, \boldsymbol{A}_2)$ lead to

$$\boldsymbol{E}_2 = -\partial_0\boldsymbol{A}, \qquad \boldsymbol{B}_2 = \nabla\times\boldsymbol{A}. \tag{5.20}$$

According to (5.19) the electric field obeys

$$\nabla\times\boldsymbol{E}_1 = 0.$$

Thus, waves described by $\partial^2\varphi = 0$ would be longitudinal [4]. On the other hand, from (5.20) and the Lorenz condition (5.18) we get

$$\nabla\cdot\mathbf{E}_2 = 0.$$

[4] First, note that we do not claim that such waves do exist in the nature. We just analyse some consequences of description of the field in terms of electromagnetic potentials. Second, note also that for electric field given by expression $\boldsymbol{E}_1 = \boldsymbol{n}E_1(\boldsymbol{n}\cdot\boldsymbol{x}, t)$ one gets

$$\begin{aligned}\nabla\times\boldsymbol{E}_1 &= (\hat{\boldsymbol{e}}_i\times\boldsymbol{n})\partial_i E_1(\boldsymbol{n}\cdot\boldsymbol{x}, t)\\ &= (\hat{\boldsymbol{e}}_i\times\boldsymbol{n})n^i\partial_s E_1(s,t)\\ &= (\boldsymbol{n}\times\boldsymbol{n})\partial_s E_1(s,t) = 0,\end{aligned}$$

where $s := \boldsymbol{n}\cdot\boldsymbol{x}$.

Hence, the electromagnetic waves being solutions of $\partial^2\boldsymbol{A} = 0$ would be transversal. For instance, taking $\boldsymbol{E}_2 = \boldsymbol{n}'E_2(\boldsymbol{n}\cdot\boldsymbol{x}, t)$ with $\boldsymbol{n}'\cdot\boldsymbol{n} = 0$, one gets $\nabla\cdot\boldsymbol{E}_2 = 0$. Thus, the potentials $(\varphi_1, \boldsymbol{A}_1)$ represent *longitudinal* degrees of freedom whereas the potentials $(\varphi_2, \boldsymbol{A}_2)$ represent *transverse* degrees of freedom. Note, that distinguishing on longitudinal and transverse time dependent degrees of freedom is meaningful *only for the electromagnetic potentials*. The longitudinal degrees of freedom can be eliminated by properly chosen gauge transformation. In order to see it we consider the gauge transformation

Elimination of longitudinal degrees of freedom

$$\varphi' = \varphi - \partial_0\chi, \qquad \mathbf{A}' = \mathbf{A} + \nabla\chi. \tag{5.21}$$

which leaves unchanged the electric field $\boldsymbol{E} = \boldsymbol{E}'$. Transformation (5.21) can be seen as combination of two gauge transformations $(\varphi_1, \boldsymbol{A}_1) \to (\varphi_1', \boldsymbol{A}_1')$ and $(\varphi_2, \boldsymbol{A}_2) \to (\varphi_2', \boldsymbol{A}_2')$ with $\chi_1 \equiv \chi \equiv \chi_2$. The fields $\boldsymbol{E}_1$ and

$\boldsymbol{E}_2$ given in terms of $(\varphi_1, \boldsymbol{A}_1)$ and $(\varphi_2, \boldsymbol{A}_2)$ can be replaced by $\boldsymbol{E}_1'$ and $\boldsymbol{E}_2'$ expressed by new potentials $(\varphi_1', \boldsymbol{A}_1')$ and $(\varphi_2', \boldsymbol{A}_2')$. One gets

$$\begin{aligned}
\boldsymbol{E} &= \overbrace{[-\nabla\varphi - \partial_0\boldsymbol{0}]}^{\boldsymbol{E}_1} + \overbrace{[-\partial_0\boldsymbol{A} - \nabla 0]}^{\boldsymbol{E}_2} \\
&= \overbrace{[-\nabla(\varphi - \partial_0\chi) - \partial_0(\boldsymbol{0} + \cancel{\nabla\chi})]}^{\boldsymbol{E}_1'} + \overbrace{[-\partial_0(\boldsymbol{A} + \nabla\chi) - \nabla(0 - \cancel{\partial_0\chi})]}^{\boldsymbol{E}_2'} \\
&= -\nabla\varphi' - \partial_0\boldsymbol{A}'.
\end{aligned} \tag{5.22}$$

An appropriate choice of the gauge transformation

The appropriate choice of $\chi(t, \boldsymbol{x})$, namely

$$\varphi = \partial_0\chi, \qquad \partial^2\chi = 0 \tag{5.23}$$

allows to eliminate longitudinal degrees of freedom: $-\nabla\varphi'$. The condition $\partial^2\chi = 0$ assures that new potentials $(\varphi', \mathbf{A}')$ satisfy also the Lorenz condition. Indeed, one gets

$$\begin{aligned}
\varepsilon\mu\partial_0\varphi' + \nabla\cdot\boldsymbol{A}' &= \varepsilon\mu\partial_0(\varphi - \partial_0\chi) + \nabla\cdot(\boldsymbol{A} + \nabla\chi) \\
&= \underbrace{\varepsilon\mu\partial_0\varphi + \nabla\cdot\boldsymbol{A}}_{0} - \underbrace{\partial^2\chi}_{0}.
\end{aligned}$$

Note that the choice $\varphi = \partial_0\chi$ is compatible with the fact that $\partial^2\varphi = 0$. The Lorenz condition (5.14) imposed on the potentials $(\varphi', \mathbf{A}')$ reduces to the form

Coulomb gauge condition: gauge potentials giving transverse fields

$$\boxed{\varphi' = 0, \qquad \nabla\cdot\mathbf{A}' = 0} \tag{5.24}$$

for χ given by (5.23).

It restricts wave solutions to *transverse waves*. It is important to stress that the gauge fixing (5.23) exist only for some *free* and *time-dependent* fields. For electromagnetic field which is not free the condition (5.24) is not gauge condition anymore. Instead it is a kind of *restriction* which allows to separate out the transversal (radiation) part of the electromagnetic field. The longitudinal part represents static fields that do not satisfy (5.24). Note that such decomposition is *not invariant under Lorentz transformations*.

5.2 *Plane waves*

Phase velocity.

A special group of electromagnetic waves is given by waves characterized by *constant phase surfaces*. Such surfaces are solutions of condition

$$\psi(t, \boldsymbol{x}) = const. \tag{5.25}$$

The form of function ψ determines geometric character of constant phase surfaces. For instance, any constant phase surface describing *plane wave* that propagates in homogeneous dielectric media is given by solution of equation

Plane surface of the constancy of a phase of a wavefront

$$\psi \equiv \boldsymbol{n} \cdot \boldsymbol{x} - vt = const, \tag{5.26}$$

where $\boldsymbol{n}$ is a *constant unit vector* that points out in direction of propagation of the wave. For *spherical wave* the vector $\boldsymbol{n}$ is replaced by a *radial spherical versor* $\hat{\boldsymbol{r}}$ and for *cylindrical wave* by a *radial versor* in cylindrical coordinates $\hat{\boldsymbol{\rho}}$.

The phase velocity v_p of a wave is defined as *velocity of translocation of its phase surface*. It can be obtained from equation $d\psi = 0$ which can be written in the form

$$\partial_t \psi \, dt + \nabla \psi \cdot d\boldsymbol{x} = 0.$$

Dividing by $|\nabla \psi|$ one gets

$$\boxed{\mathrm{v}_p = \frac{\nabla \psi}{|\nabla \psi|} \cdot \frac{d\boldsymbol{x}}{dt} = -\frac{\partial_t \psi}{|\nabla \psi|}.} \tag{5.27}$$

The phase velocity is just a projection of $\frac{d\boldsymbol{x}}{dt}$ on a unit vector $\frac{\nabla\psi}{|\nabla\psi|}$, where $\frac{d\boldsymbol{x}}{dt}$ is the velocity of the point $\boldsymbol{x}$ belonging to the surface of constant phase and $\frac{\nabla\psi}{|\nabla\psi|}$ is a vector normal to this surface. The phase velocity of a plane wave with the surface $\psi = const$ given by (5.26) reads

$$\mathrm{v}_p = v = \frac{c}{\sqrt{\varepsilon\mu}}. \tag{5.28}$$

Solution of wave equation in 1+1 dimensions

Equations $\partial^2 \mathbf{E} = 0$ and $\partial^2 \mathbf{B} = 0$ can be represented by a single equation $\partial^2 \Phi = 0$ where $\Phi = \{E^1, E^2, E^3, B^1, B^2, B^3\}$. In the case of *plane waves* the surface of constant phase depends on a single coordinate. Aligning one of the axes (e.g. x) of the Cartesian reference frame in direction of propagation one gets Φ as function of only two variables (t, x). Defining two light-cone coordinates

Light-cone coordinates

$$x_\pm := x \pm vt \tag{5.29}$$

one gets

$$\partial_t = \frac{\partial x_+}{\partial t}\partial_+ + \frac{\partial x_-}{\partial t}\partial_- = v(\partial_+ - \partial_-), \tag{5.30}$$

$$\partial_x = \frac{\partial x_+}{\partial x}\partial_+ + \frac{\partial x_-}{\partial x}\partial_- = \partial_+ + \partial_-, \tag{5.31}$$

$$\partial^2 = -4\partial_+\partial_-. \tag{5.32}$$

The equation $\partial_+\partial_-\Phi(x_+, x_-) = 0$ has general solution which is a sum of two arbitrary functions, each depending on a single light-cone variable $\Phi = \Phi_+(x_+) + \Phi_-(x_-)$. In our original coordinates

Superposition of two waves

$$\boxed{\Phi(t,x) = \Phi_+(x+vt) + \Phi_-(x-vt).} \tag{5.33}$$

This solution describes superposition of two waves $\Phi_-(x-vt)$ and $\Phi_+(x+vt)$ that propagate forward and backward along the x-axis.

Let us observe that for *spherical waves*[5] one can also get an explicit form of the solution. The wave equation in this case reads

[5] Thy are so-called scalar spherical waves. The spherical electromagnetic wave must satisfy the Maxwell's equations and not only the wave equation.

$$\frac{1}{v^2}\partial_t^2\Phi - \frac{1}{r^2}\partial_r\left(r^2\partial_r\Phi\right) = 0. \tag{5.34}$$

Plugging $\Phi(t,r) = \frac{1}{r}\chi(t,r)$ into the above equation one gets

$$\frac{1}{r}\left[\frac{1}{v^2}\partial_t^2\chi - \partial_r^2\chi\right] = 0, \tag{5.35}$$

where $r^2\partial_r\Phi(t,r) = r\partial_r\chi(t,r) - \chi(t,r)$. The function $\chi(t,r)$ satisfies equation $\partial_+\partial_-\chi = 0$, where $x_\pm := r \pm vt$. Thus the spherical scalar wave reads

Spherical wave for scalar field

$$\boxed{\Phi(t,r) = \frac{\chi_+(r+vt)}{r} + \frac{\chi_-(r-vt)}{r}.} \tag{5.36}$$

This solution contains superposition of two spherical waves: the ingoing χ_+ and the outgoing χ_- one.

Spectral decomposition

In this section we study a generic plane wave which can be decomposed on many plane waves with different frequencies ω. The frequency of a single component enters to the solution through one of the factors $\cos\omega t$, $\sin\omega t$, $\exp(-i\omega t)$. Here we shall not focus on polarization aspects. This subject will be discussed in further part of the present chapter. We assume that all vectors $\boldsymbol{A}$ (or $\boldsymbol{E}$) assigned to different frequencies oscillate in *single common direction*. Mathematically, such superposition of waves can be represented by Fourier transform.

Any function $f(x)$ of class $\mathcal{L}_1$ *i.e.*

$$\int_{-\infty}^{\infty} |f(x)|dx < \infty \tag{5.37}$$

which satisfies

$$f(x) = \frac{1}{2}[f(x-0) + f(x+0)] \tag{5.38}$$

at the discontinuity points can be represented by Fourier integral

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, F(k)e^{ikx}. \tag{5.39}$$

The expansion coefficients are given by Fourier transform of $f(x)$

$$F(k) \equiv \mathcal{F}[f(x)](k) := \int_{-\infty}^{\infty} dx\, f(x)e^{-ikx}. \tag{5.40}$$

If the function $f(x)$ depends on more spacetime coordinates $\boldsymbol{x}$ then

$$f(\boldsymbol{x}) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} d^3x\, F(\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{5.41}$$

$$F(\boldsymbol{k}) = \int_{\mathbb{R}^3} d^3k\, f(\boldsymbol{x})e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{5.42}$$

A very convenient approach to electromagnetic field is based on its complex-valued representation. The complex version of the electromagnetic field may be seen as a sort of *auxiliary field*. The physical content of the field is encoded in its real or imaginary part. The electromagnetic plane wave can be represented by Fourier decomposition on monochromatic plane waves

$$\boldsymbol{A}(t,\boldsymbol{x}) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} d^3k\, \boldsymbol{a}(t,\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \tag{5.43}$$

where

$$\boldsymbol{a}(t,\boldsymbol{k}) = \int_{\mathbb{R}^3} d^3x\, \boldsymbol{A}(t,\boldsymbol{x})e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{5.44}$$

Here $\boldsymbol{A}$ is a complex-valued vector function.[6] Since the generic wave is given by superposition of many waves with different frequencies then one can factorize the time dependence in expansion coefficients[7]

$$\boldsymbol{a}(t,\boldsymbol{k}) = \boldsymbol{a}(\boldsymbol{k})e^{-i\omega(k)t}, \tag{5.45}$$

which depend on *wave number* $k := |\boldsymbol{k}|$ and *wave vector* $\boldsymbol{k}$. The electromagnetic potential takes the form

$$\boldsymbol{A}(t,\boldsymbol{x}) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} d^3k\, \boldsymbol{a}(\boldsymbol{k})e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega(k)t)}. \tag{5.46}$$

Let us observe that each monochromatic component must be a solution of wave equation

$$\left(\frac{\varepsilon\mu}{c^2}\partial_t^2 - \nabla^2\right)e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega(k)t)} = 0 \qquad \Rightarrow \qquad \boxed{\varepsilon\mu\frac{\omega^2}{c^2} - \boldsymbol{k}^2 = 0.} \tag{5.47}$$

The algebraic equality in (5.47) is called *dispersion relation* and it can be written in terms of wave number $ck = \sqrt{\varepsilon\mu}\,\omega(k)$. The characteristic expression $\sqrt{\varepsilon\mu}$ is called *refraction coefficient* and it is usually denoted by

$$n := \sqrt{\varepsilon\mu}.$$

If the $x-$axis is parallel to the vector $\boldsymbol{k}$ then

$$\boldsymbol{a}(\boldsymbol{k}) = \boldsymbol{a}(k^1)\,(2\pi)^2\delta(k^2)\delta(k^3).$$

Consequently, Fourier integral can be written in the form

$$\boldsymbol{A}(t,\boldsymbol{x}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk^1\, \boldsymbol{a}(k^1)e^{i(k^1x-\omega(|k^1|)t)}. \tag{5.48}$$

[6] Note, the Fourier coefficients must obey relation $a^*(t,-\boldsymbol{k}) = a(t,\boldsymbol{k})$ for a real-valued electromagnetic potential.

[7] A factor $e^{-i\omega(k)t}$ can be introduced only for complex-valued fields. For real-valued fields it must be replaced by either $\cos(\omega(k)t)$ or $\cos(\omega(k)t)$.

Suppose that the coefficients $\boldsymbol{a}(k^1)$ vanish outside the interval $|k^1 - k_0^1| < \epsilon$, where ϵ is certain small number. If $k_0^1 > 0$ then the integral contains only contributions from $k^1 > 0$. In such a case $k^1 := k \equiv |\boldsymbol{k}|$.[8]

[8] For $k_0^1 < 0$ analysis is very similar with $k^1 = -k$.

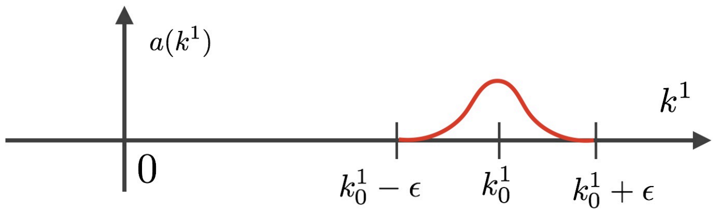


Figure 5.1: The amplitude coefficient $|\boldsymbol{a}(k^1)|$.

First few terms of expansion of $\omega(k)$ in neighborhood of k_0 have the form

$$\omega(k) = \omega(k_0) + (k - k_0)\left(\frac{d\omega}{dk}\right)_{k=k_0} + \dots \tag{5.49}$$

Group velocity

Then, defining $\omega_0 := \omega(k_0)$ and

$$\boxed{\mathrm{v}_g := \left(\frac{d\omega}{dk}\right)_{k=k_0}} \tag{5.50}$$

one gets

$$\begin{aligned} \boldsymbol{A}(t, \boldsymbol{x}) &= \frac{1}{2\pi}\int_{k_0-\epsilon}^{k_0+\epsilon} dk\, \boldsymbol{a}(k) e^{i(kx - \omega_0 t - (k-k_0)\mathrm{v}_g t)} && (5.51)\\ &= e^{i(k_0 x - \omega_0 t)} \underbrace{\frac{1}{2\pi}\int_{k_0-\epsilon}^{k_0+\epsilon} dk\, \boldsymbol{a}(k) e^{i(k-k_0)(x - \mathrm{v}_g t)}}_{\boldsymbol{A}_0(t,x)}. && (5.52)\end{aligned}$$

The expression $e^{i(k_0 x - \omega_0 t)}$ contains the dominant frequency term and therefore it is the fastest oscillating function. The amplitude term $\boldsymbol{A}_0(t, x)$ assumes constant values on the planes $x - \mathrm{v}_g t = \text{const}$. This expression defines the profile (envelope) of a wave packet. The velocity with which the envelope moves is given by (5.50). This velocity is termed *group velocity*.

Linear dispersion

When frequency is a certain linear function of wave number, i.e. $\omega = vk$, then the phase factor ψ has the form $\psi = kx - vkt$. In such a case the group velocity is equal to the phase velocity

$$\mathrm{v}_g = \frac{d\omega}{dk} = v, \qquad \mathrm{v}_p = -\frac{\partial_t \psi}{|\nabla\psi|} = \frac{\omega}{k} = v. \tag{5.53}$$

Let us observe that relation $\omega = k\,\mathrm{v}_p$, where $\mathrm{v}_p = \mathrm{v}_p(k) \neq const$, leads to

$$\mathrm{v}_g = \frac{d(k\,\mathrm{v}_p)}{dk} = \mathrm{v}_p + k\frac{d\mathrm{v}_p}{dk} = \mathrm{v}_p - \lambda\frac{d\mathrm{v}_p}{d\lambda} \tag{5.54}$$

Wavelength

where *wavelength* is denoted by $\lambda := \frac{2\pi}{k}$.

Monochromatic wave in homogeneous dielectrics

We consider an electromagnetic wave with a single value of angular frequency – monochromatic wave. The wave propagates in homogeneous dielectric medium with constant permittivity ε and constant permeability μ. Such wave is described by *auxiliary complex fields*

$$\boldsymbol{E} = \boldsymbol{E}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}, \qquad \boldsymbol{B} = \boldsymbol{B}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}, \tag{5.55}$$

where $\boldsymbol{E}_0$ and $\boldsymbol{B}_0$ are some constant complex amplitude vectors. The fields (5.55) are solutions of the wave equation provided that

$$n^2 \frac{\omega^2}{c^2} = \boldsymbol{k}^2. \tag{5.56}$$

It is enough to consider $\boldsymbol{k}$ as real-valued vector. In order to establish what is the mutual orientation of vectors $\boldsymbol{E}$, $\boldsymbol{B}$ and $\boldsymbol{k}$ one has to plug solutions (5.55) into Maxwell's equations and solve the resulting algebraic equations. Considering that

$$\nabla e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)} = i\boldsymbol{k}\, e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}$$

one gets

$$\nabla\cdot\boldsymbol{E} = i\,\boldsymbol{k}\cdot\boldsymbol{E}, \qquad \nabla\times\boldsymbol{E} = i\,\boldsymbol{k}\times\boldsymbol{E}, \qquad \partial_t\boldsymbol{E} = -i\omega\boldsymbol{E} \tag{5.57}$$

and similarly for $\boldsymbol{B}$. Electric and magnetic Gauss laws give

$$\nabla\cdot\boldsymbol{E} = 0 \quad\Rightarrow\quad \boldsymbol{k}\cdot\boldsymbol{E}_0 = 0, \tag{5.58}$$

$$\nabla\cdot\boldsymbol{B} = 0 \quad\Rightarrow\quad \boldsymbol{k}\cdot\boldsymbol{B}_0 = 0, \tag{5.59}$$

whereas from Ampere-Maxwell's law and Faraday's law one gets

$$\nabla\times\boldsymbol{B} - \frac{n^2}{c}\partial_t\boldsymbol{E} = 0 \quad\Rightarrow\quad \boldsymbol{k}\times\boldsymbol{B}_0 + n^2\frac{\omega}{c}\boldsymbol{E}_0 = 0, \tag{5.60}$$

$$\nabla\times\boldsymbol{E} + \frac{1}{c}\partial_t\boldsymbol{B} = 0 \quad\Rightarrow\quad \boldsymbol{k}\times\boldsymbol{E}_0 - \frac{\omega}{c}\boldsymbol{B}_0 = 0. \tag{5.61}$$

Equations (5.58) and (5.59) imply that vectors $\boldsymbol{E}$ and $\boldsymbol{B}$ are orthogonal to the wave vector $\boldsymbol{k}$. Taking into account that the wave number $k = n\frac{\omega}{c}$ is given by dispersion relation (5.164), one gets from (5.60) and (5.61) that

$$\boxed{\boldsymbol{E}_0 = -\frac{1}{n}\hat{\boldsymbol{k}}\times\boldsymbol{B}_0, \qquad \boldsymbol{B}_0 = n\,\hat{\boldsymbol{k}}\times\boldsymbol{E}_0, \qquad \hat{\boldsymbol{k}} \equiv \frac{\boldsymbol{k}}{k}.} \tag{5.62}$$

From scalar product of amplitudes one gets

$$\boldsymbol{E}_0\cdot\boldsymbol{B}_0 = -[\hat{\boldsymbol{k}}^2\boldsymbol{E}_0\cdot\boldsymbol{B}_0 - (\hat{\boldsymbol{k}}\cdot\boldsymbol{E}_0)(\hat{\boldsymbol{k}}\cdot\boldsymbol{B}_0)] = -\boldsymbol{E}_0\cdot\boldsymbol{B}_0 \tag{5.63}$$

which allows us to conclude that

$$\boxed{\boldsymbol{E}_0\cdot\boldsymbol{B}_0 = 0.} \tag{5.64}$$

Thus electric and magnetic field vectors are mutually perpendicular. The amplitudes of both fields are proportional. It can be seen from

$$\boldsymbol{E}_0 \cdot \boldsymbol{E}_0^* = \frac{1}{n^2}[\hat{\boldsymbol{k}} \times \boldsymbol{B}_0] \cdot [\hat{\boldsymbol{k}} \times \boldsymbol{B}_0^*] = \frac{1}{n^2}[\hat{\boldsymbol{k}}^2(\boldsymbol{B}_0 \cdot \boldsymbol{B}_0^*) - (\hat{\boldsymbol{k}} \cdot \boldsymbol{B}_0)(\hat{\boldsymbol{k}} \cdot \boldsymbol{B}_0^*)]$$

which gives

$$\boxed{|\boldsymbol{E}_0|^2 = \frac{1}{n^2}|\boldsymbol{B}_0|^2.} \tag{5.65}$$

Moreover, the electric and magnetic fields *have the same phase*. It is clear from the fact that the amplitude of a given field is a vector product of two vectors: the amplitude of the other field and the real-valued vector $n\,\hat{\boldsymbol{k}}$. The square of $\boldsymbol{E}_0$ is a complex number and thus it can be represented in the form $\boldsymbol{E}_0^2 = |\boldsymbol{E}_0|^2 e^{-2i\varphi}$. The choice of the phase "-2φ" is motivated by convenience. Taking the square of the complex magnetic field vector one gets

$$(\boldsymbol{B}_0)^2 = n^2(\hat{\boldsymbol{k}} \times \boldsymbol{E}_0)^2 = n^2|\boldsymbol{E}_0|^2 e^{-2i\varphi} \qquad \Rightarrow \qquad \boldsymbol{B}_0^2 = n^2|\boldsymbol{E}_0|^2 e^{-2i\varphi}.$$

what proves our statement. Some of the above statements[9] do not hold in conducting media where the vector $\boldsymbol{k}$ must be replaced by a complex vector.

[9] For instance, the equality of phases of electric and magnetic field.

5.3 Polarization of electromagnetic waves

Decomposition on polarization states.

We consider a free electromagnetic field in an empty space. The field is described by solutions of sourceless Maxwell equations. In order to eliminate non-physical degrees of freedom we impose the Coulomb gauge condition

Coulomb gauge condition

$$A^0 = 0, \qquad \nabla \cdot \boldsymbol{A} = 0.$$

We shall solve explicitly the condition $\nabla \cdot \boldsymbol{A} = 0$. The vector potential $\boldsymbol{A}$ is a solution of the wave equation $(\partial_0^2 - \nabla^2)\boldsymbol{A} = 0$. Since the equation is linear then the potential can be written in the form of Fourier integral

$$\boldsymbol{A}(t, \boldsymbol{x}) = \frac{1}{(2\pi)^3} \sum_{a=1}^{3} \int d^3k\, \hat{\boldsymbol{e}}_a(\boldsymbol{k}) a_a(t, \boldsymbol{k})\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{5.66}$$

where $\hat{\boldsymbol{e}}_a(\boldsymbol{k})$ are some real-valued and constant vectors that satisfy the condition of orthogonality

$$\hat{\boldsymbol{e}}_a(\boldsymbol{k}) \cdot \hat{\boldsymbol{e}}_b(\boldsymbol{k}) = \delta_{ab}. \tag{5.67}$$

They are called *polarization vectors*. We take the vector $\hat{\boldsymbol{e}}_3(\boldsymbol{k})$ as

The polarization vectors

$$\hat{\boldsymbol{e}}_3(\boldsymbol{k}) := \frac{\boldsymbol{k}}{|\boldsymbol{k}|}, \qquad \boldsymbol{k} \neq 0. \tag{5.68}$$

The vector $\boldsymbol{k}$ in quantum field theory is proportional to linear momentum of photons. Since photons with $\boldsymbol{k} = 0$ do not contribute to total energy and momentum of the electromagnetic field one can assume that coefficients $a_a(t, \boldsymbol{k})$ vanish at $\boldsymbol{k} = 0$ *i.e.*

$$a_a(t, \boldsymbol{k} = \boldsymbol{0}) = 0. \tag{5.69}$$

Another assumption is *independence* of the polarization vectors on spatial reflection of $\boldsymbol{k}$, namely

$$\hat{\boldsymbol{e}}_a(-\boldsymbol{k}) = \hat{\boldsymbol{e}}_a(\boldsymbol{k}) \qquad \text{for} \qquad a = 1, 2. \tag{5.70}$$

The complex coefficients $a_a(t, \boldsymbol{k})$ are called *modes of electromagnetic field*. Since the electromagnetic field is described by real-valued function then the modes satisfy some additional constraints. The condition $\boldsymbol{A}^*(t, \boldsymbol{x}) = \boldsymbol{A}(t, \boldsymbol{x})$ has its explicit form

Restrictions on modes

$$\sum_{a=1}^{3} \int d^3k\, \hat{\boldsymbol{e}}_a(\boldsymbol{k}) a_a^*(t, \boldsymbol{k})\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} = \sum_{a=1}^{3} \int d^3k\, \hat{\boldsymbol{e}}_a(\boldsymbol{k}) a_a(t, \boldsymbol{k})\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}.$$

Substituting variables $k^i \to -k^i$ in the first integral one gets

$$\sum_{a=1}^{3} \int d^3k\, \hat{\boldsymbol{e}}_a(-\boldsymbol{k}) a_a^*(t, -\boldsymbol{k})\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = \sum_{a=1}^{3} \int d^3k\, \hat{\boldsymbol{e}}_a(\boldsymbol{k}) a_a(t, \boldsymbol{k})\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

where the sign change $dk^i \to -dk^i$ has been absorbed in limits of integration

$$\int_{-\infty}^{\infty} dk \to \int_{+\infty}^{-\infty} (-dk) = \int_{-\infty}^{\infty} dk.$$

It gives

$$\hat{\boldsymbol{e}}_a(-\boldsymbol{k}) a_a^*(t, -\boldsymbol{k}) = \hat{\boldsymbol{e}}_a(\boldsymbol{k}) a_a(t, \boldsymbol{k}). \tag{5.71}$$

Since polarization vectors labeled by $a = 1, 2$ do not depend on spatial reflection of the vector $\boldsymbol{k}$ *i.e.* $\hat{\boldsymbol{e}}_a(\boldsymbol{k}) = \hat{\boldsymbol{e}}_a(-\boldsymbol{k})$, then (5.71) implies that

$$\boxed{a_a^*(t, -\boldsymbol{k}) = a_a(t, \boldsymbol{k}).} \tag{5.72}$$

On the other hand, the vector (5.68) satisfies $\hat{\boldsymbol{e}}_3(-\boldsymbol{k}) \equiv -\frac{\boldsymbol{k}}{|\boldsymbol{k}|} = -\hat{\boldsymbol{e}}_3(\boldsymbol{k})$ which gives

$$\boxed{a_3^*(t, -\boldsymbol{k}) = -a_3(t, \boldsymbol{k}).} \tag{5.73}$$

The Coulomb condition $\nabla \cdot \boldsymbol{A} = 0$ takes the form

$$\sum_{a=1}^{3} \int d^3k\, i\boldsymbol{k} \cdot \hat{\boldsymbol{e}}_a(\boldsymbol{k}) a_a(t, \boldsymbol{k})\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = 0.$$

This condition is satisfied providing that

$$\boldsymbol{k} \cdot \hat{\boldsymbol{e}}_a(\boldsymbol{k}) a_a(t, \boldsymbol{k}) = 0. \tag{5.74}$$

The condition (5.74) is satisfied for $\boldsymbol{k} = 0$. Since vectors $\hat{\boldsymbol{e}}_a(\boldsymbol{k})$ are orthonormal and $\hat{\boldsymbol{e}}_3 \propto \boldsymbol{k}$ then for $\boldsymbol{k} \neq 0$ the condition (5.74) constititues constraint on $a_3(t, \boldsymbol{k})$ which is of the form $|\boldsymbol{k}| a_3(t, \boldsymbol{k}) = 0$. It has a solution

Two polarization states of electromagnetic waves

$$\boxed{a_3(t, \boldsymbol{k}) = 0.} \tag{5.75}$$

We conclude that electromagnetic potentials which are consistent with the Coulomb gauge condition are parametrized by just two Fourier amplitudes $a_1(t, \boldsymbol{k})$ and $a_2(t, \boldsymbol{k})$. It means that the electromagnetic field has only *two polarization states*. Both of them are perpendicular to the propagation vector $\boldsymbol{k}$.

Totally polarized electromagnetic wave

In order to study polarization of electromagnetic waves we choose any point P and look at the electric field $\boldsymbol{E}$ at this point.[10] The magnetic field can be discarded in this analysis because it is not independent. Its orientation is determined by (5.62).

[10] We choose $\boldsymbol{x} = 0$ in order to avoid the additional complex number $e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$.

The monochromatic wave is totally polarized – spatial orientation of constant amplitude vector $\boldsymbol{E}_0$ remains unchanged in time. In this section we shall consider totally polarized waves. We consider complex-valued[11] electric field $\boldsymbol{E} = \boldsymbol{E}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}$ describing monochromatic electromagnetic wave. The amplitude vector $\boldsymbol{E}_0$ is complex-valued and constant. The *physical electric field* is given by its real part $Re(\boldsymbol{E})$. The field $\boldsymbol{E}$ is function of time at any fixed point P.

[11] Auxiliary field

Gauss law implies that $\boldsymbol{E}_0 \cdot \boldsymbol{k} = 0$. The expression $\boldsymbol{E}_0 \cdot \boldsymbol{E}_0$ is a complex number because the amplitude vector is complex-valued. Following the previous section we shall parametrize this number as

$$\boldsymbol{E}_0 \cdot \boldsymbol{E}_0 = |\boldsymbol{E}_0|^2 e^{-2i\varphi}. \tag{5.76}$$

The amplitude vector $\boldsymbol{E}_0$ can be parametrized in terms of two *real vectors* $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ in the following way

$$\boldsymbol{E}_0 = (\boldsymbol{e}_1 + i\eta \boldsymbol{e}_2) e^{-i\varphi}, \qquad \eta = \pm 1. \tag{5.77}$$

The square of (5.77) reads $\boldsymbol{E}_0 \cdot \boldsymbol{E}_0 = (\boldsymbol{e}_1{}^2 - \boldsymbol{e}_2{}^2 + 2i\eta \boldsymbol{e}_1 \cdot \boldsymbol{e}_2) e^{-2i\varphi}$. This expression must be equal to (5.76). It means that $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are *mutually perpendicular* real vectors which, in addition, are perpendicular to the wave vector $\boldsymbol{k}$,

$$\boldsymbol{e}_1 \cdot \boldsymbol{e}_2 = 0, \qquad \boldsymbol{e}_1 \cdot \boldsymbol{k} = 0, \qquad \boldsymbol{e}_2 \cdot \boldsymbol{k} = 0. \tag{5.78}$$

Lengths of vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$, denoted by $e_a := |\boldsymbol{e}_a|$, can be expressed in terms of three parameters: *two amplitudes* of electric field in certain reference frame, measured in two orthogonal directions, and a third parameter – *phase shift*. Without loss of generality we can choose two

Cartesian versors $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ which are parallel to the plane defined by $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$. A third versor $\hat{\boldsymbol{z}}$ is defined as the vector product $\hat{\boldsymbol{z}} = \hat{\boldsymbol{x}} \times \hat{\boldsymbol{y}}$. The versors $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$ define *laboratory reference frame*. The amplitude of electric field has components

$$\boldsymbol{E}_0 = A\, e^{i\alpha} \hat{\boldsymbol{x}} + B\, e^{i\beta} \hat{\boldsymbol{y}} \tag{5.79}$$

where A, B, α, β are real-valued quantities. Then

$$\boldsymbol{E}_0^* \cdot \boldsymbol{E}_0 = A^2 + B^2, \tag{5.80}$$

$$\boldsymbol{E}_0^* \times \boldsymbol{E}_0 = AB \left[e^{i(\beta-\alpha)} - e^{-i(\beta-\alpha)} \right] \hat{\boldsymbol{z}} = 2i\, AB \sin(\delta) \hat{\boldsymbol{z}} \tag{5.81}$$

where $\delta := \beta - \alpha \in [-\pi, \pi]$. On the other hand, scalar and vector product of the electric vector and its complex conjugate can be cast in the form

$$\boldsymbol{E}_0^* \cdot \boldsymbol{E}_0 = (\boldsymbol{e}_1 - i\eta \boldsymbol{e}_2) \cdot (\boldsymbol{e}_1 + i\eta \boldsymbol{e}_2) = e_1^2 + e_2^2, \tag{5.82}$$

$$\boldsymbol{E}_0^* \times \boldsymbol{E}_0 = (\boldsymbol{e}_1 - i\eta \boldsymbol{e}_2) \times (\boldsymbol{e}_1 + i\eta \boldsymbol{e}_2) = 2i\eta\, \boldsymbol{e}_1 \times \boldsymbol{e}_2 = 2i\eta\, e_1 e_2\, \hat{\boldsymbol{e}}_1 \times \hat{\boldsymbol{e}}_2 \tag{5.83}$$

where $\hat{\boldsymbol{e}}_1 \times \hat{\boldsymbol{e}}_2$ is a unit vector. One can always choose both versors $\hat{\boldsymbol{e}}_1$ and $\hat{\boldsymbol{e}}_2$ in a way that $\hat{\boldsymbol{e}}_1 \times \hat{\boldsymbol{e}}_2 = \hat{\boldsymbol{z}}$. Comparing (5.80) with (5.82) and (5.81) with (5.83) one gets

$$e_1^2 + e_2^2 = A^2 + B^2, \qquad \eta e_1 e_2 = AB \sin\delta. \tag{5.84}$$

Note, that since $e_a \geq 0$ and $A, B \geq 0$ then $\eta = \text{sgn}\delta$. Hence, $\eta\delta \in [0, \pi]$. One gets

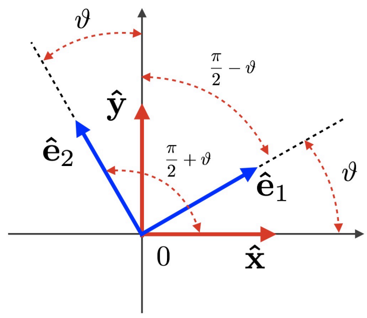


Figure 5.2: Relations between vectors

$$(e_1 \pm e_2)^2 = A^2 + B^2 \pm 2AB \sin(\eta\delta). \tag{5.85}$$

Sum and difference of square roots of (5.85) gives

$$e_1 = \frac{1}{2} \left[\sqrt{A^2 + B^2 + 2AB \sin(\eta\delta)} + \sqrt{A^2 + B^2 - 2AB \sin(\eta\delta)} \right] \tag{5.86}$$

and

$$e_2 = \frac{1}{2} \left[\sqrt{A^2 + B^2 + 2AB \sin(\eta\delta)} - \sqrt{A^2 + B^2 - 2AB \sin(\eta\delta)} \right]. \tag{5.87}$$

Expressions (5.86) and (5.87) give, respectively, lengths of major and minor semi-axes of the polarization ellipse. The orientation of the ellipse is given by the angle ϑ between vectors $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{e}}_1$

$$\hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{e}}_1 = \cos\vartheta, \qquad \hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{e}}_2 = -\sin\vartheta \tag{5.88}$$

$$\hat{\boldsymbol{y}} \cdot \hat{\boldsymbol{e}}_1 = \sin\vartheta, \qquad \hat{\boldsymbol{y}} \cdot \hat{\boldsymbol{e}}_2 = \cos\vartheta. \tag{5.89}$$

The angle ϑ can be determined from the identity

$$Re[(\boldsymbol{E}_0 \cdot \hat{\boldsymbol{e}}_1)(\boldsymbol{E}_0^* \cdot \hat{\boldsymbol{e}}_2)] \equiv 0 \tag{5.90}$$

which follows from

$$[e^{-i\varphi}(\boldsymbol{e}_1 + i\eta \boldsymbol{e}_2) \cdot \hat{\boldsymbol{e}}_1][e^{i\varphi}(\boldsymbol{e}_1 - i\eta \boldsymbol{e}_2) \cdot \hat{\boldsymbol{e}}_2] = -i\eta e_1 e_2.$$

Plugging (5.79) into (5.90) one gets

$$\begin{aligned}
& Re[(Ae^{i\alpha}\hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{e}}_1 + Be^{i\beta}\hat{\boldsymbol{y}} \cdot \hat{\boldsymbol{e}}_1)(Ae^{-i\alpha}\hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{e}}_2 + Be^{-i\beta}\hat{\boldsymbol{y}} \cdot \hat{\boldsymbol{e}}_2)] \\
= \; & Re[(A\cos\vartheta + Be^{i\delta}\sin\vartheta)(-A\sin\vartheta + Be^{-i\delta}\cos\vartheta)] \\
= \; & Re[(B^2 - A^2)\sin\vartheta\cos\vartheta + AB(e^{-i\delta}\cos^2\vartheta - e^{i\delta}\sin^2\vartheta)] \\
= \; & -\frac{1}{2}(A^2 - B^2)\sin(2\vartheta) + AB\cos\delta\,\cos(2\vartheta) \equiv 0.
\end{aligned} \tag{5.91}$$

Equation (5.91) has solution

$$\boxed{\tan(2\vartheta) = \frac{2AB}{A^2 - B^2}\cos\delta.} \tag{5.92}$$

Below we study some characteristic cases.

1. **Linear polarization**
   For $\eta\delta = \{0, \pi\}$, and consequently $\cos\delta = \pm 1$, one gets

   $$e_1 = \sqrt{A^2 + B^2}, \qquad e_2 = 0, \qquad \tan(2\vartheta) = \pm\frac{2AB}{A^2 - B^2}.$$

   The complex-valued electric field reads

   $$\boldsymbol{E} = (\boldsymbol{e}_1 + i\eta \boldsymbol{e}_2)e^{-i\chi}, \qquad \chi := \omega t - \boldsymbol{k} \cdot \boldsymbol{x} + \varphi. \tag{5.93}$$

   It gives physical electric field

   $$\boxed{Re[\boldsymbol{E}] = \sqrt{A^2 + B^2}\cos\chi\,\hat{\boldsymbol{e}}_1.} \tag{5.94}$$

   Electric field vector at any point of space is a periodic function which oscillates with frequency ω. The vector $Re[\boldsymbol{E}]$ is fixed in direction of $\hat{\boldsymbol{e}}_1$. The field oscillate in the $\hat{\boldsymbol{e}}_1 = \hat{\boldsymbol{x}}$ direction for $B = 0$ and it oscillates in the $\hat{\boldsymbol{e}}_1 = \hat{\boldsymbol{y}}$ direction for $A = 0$. In the case $A = B$ the electric field vector form an angle $\vartheta = \pi/4$ with $\hat{\boldsymbol{x}}$ providing that $\delta = 0$ and it forms angle $\vartheta = -\pi/4$ for $\delta = \pi$.

2. **Circular polarization**
   Another interesting case is $\eta\delta = \frac{\pi}{2}$ and $A = B$. In this case

   $$e_1 = A, \qquad e_2 = \pm B \equiv \pm A, \qquad \tan(2\vartheta) = \text{undetermined}. \tag{5.95}$$

   The auxiliary, complex-valued, electric field reads

   $$\boldsymbol{E} = A(\hat{\boldsymbol{e}}_1 \pm i\eta\hat{\boldsymbol{e}}_2)(\cos\chi - i\sin\chi) \tag{5.96}$$

whereas the physical electric field has the form

$$\boxed{Re[\boldsymbol{E}] = A[\cos\chi\,\hat{\boldsymbol{e}}_1 + \eta\sin\chi\,\hat{\boldsymbol{e}}_2]} \tag{5.97}$$

where $\chi := \omega t - \boldsymbol{k}\cdot\boldsymbol{x} + \varphi$. A characteristic property of circular polarization is length preserving rotation of the vector $Re[\boldsymbol{E}]$ at any point of space. The rotation has *positive helicity* (anti-clockwise) for $\delta = \pi/2$ ($\eta = +1$) and *negative helicity* (clockwise) for $\delta = -\pi/2$ ($\eta = -1$).

3. **Elliptical polarization**
   If non of cases listed above is present then the electromagnetic wave has *elliptical polarization*. The electric field vector rotates and oscillates simultaneously. Note that elliptically polarized wave

Ellipticaly polarized wave as superposition of two linearly polarized waves

$$Re[\boldsymbol{E}] = Re[(\boldsymbol{e}_1 + i\eta\boldsymbol{e}_2)(\cos\chi - i\sin\chi)] = e_1\hat{\boldsymbol{e}}_1\cos\chi + \eta e_2\hat{\boldsymbol{e}}_2\sin\chi \tag{5.98}$$

can be seen as *superposition of two linearly polarized waves* with polarization planes being mutually perpendicular. Each linear polarization can be decomposed on a combination of two circularly polarized waves. With help of two vectors

$$\hat{\boldsymbol{e}}_\pm := \hat{\boldsymbol{e}}_1\cos\chi \pm \hat{\boldsymbol{e}}_2\sin\chi, \tag{5.99}$$

one gets

$$\hat{\boldsymbol{e}}_1\cos\chi = \frac{1}{2}(\hat{\boldsymbol{e}}_+ + \hat{\boldsymbol{e}}_-), \qquad \hat{\boldsymbol{e}}_2\sin\chi = \frac{1}{2}(\hat{\boldsymbol{e}}_+ - \hat{\boldsymbol{e}}_-). \tag{5.100}$$

It allows us to cast the formula (5.98) in the form

Ellipticaly polarized wave as superposition of two circularly polarized waves

$$\boxed{Re[\boldsymbol{E}] = \frac{e_1 + \eta e_2}{2}\,\hat{\boldsymbol{e}}_+ + \frac{e_1 - \eta e_2}{2}\,\hat{\boldsymbol{e}}_-.} \tag{5.101}$$

One can conclude that each elliptically polarized electromagnetic wave can be decomposed into two *linearly polarized* waves with mutually orthogonal directions of polarization, or alternatively, into two *circularly polarized* waves with opposite helicities.

Partially polarized electromagnetic wave

A realistic electromagnetic wave is not perfectly monochromatic. Its frequencies belong to a narrow interval $\Delta\omega$ around some frequency ω. A single monochromatic wave is polarized, however, superposition of such waves with different polarizations needs a special treatment. At fixed space point the electric field of such a wave is of the form

$$\boldsymbol{E} = \boldsymbol{E}_0(t)e^{-i\omega t}, \tag{5.102}$$

where the amplitude $\boldsymbol{E}_0(t)$ is slow-varying function of time. The amplitude vector describes polarization hence polarization of electromagnetic wave changes slowly with time .

Experimental data of polarized wave contain measurement of the intensity of the light beam that passes through the polarizing filter. The intensity of light is a quadratic function of electric field. For this reason we shall consider only quadratic functions containing components $E^i(t)$ and $E^{*i}(t)$, namely

$$E^i E^j = E_0^i E_0^j \, e^{-2i\omega t}, \qquad E^{*i} E^{*j} = E_0^{*i} E_0^{*j} \, e^{2i\omega t}, \qquad E^i E^{*j} = E_0^i E_0^{*j}.$$

Actual values of such quantities are less important than theirs time average values [12]

[12] A measure process is made at a certain time interval.

$$\langle f(t) \rangle := \frac{1}{T} \int_0^T dt\, f(t). \tag{5.103}$$

The characteristic time scale in which amplitudes vary and the period of functions $e^{\pm 2i\omega t}$ are essentially different. Thus, time averaging over the interval $T \gg \frac{2\pi}{\omega}$, such that the phase factor oscillates many times whereas the amplitude remains essentially unchanged, *vanishes* for terms which depends on dominant frequency ω. On the other hand, the average values $\langle E^i E^{*j} \rangle = \left\langle E_0^i E_0^{*j} \right\rangle$ do not vanish. It means that properties of *partially polarized* electromagnetic wave are completely characterized by the tensor

$\left\langle e^{\pm i\omega t} \right\rangle = 0$

Tensor of absolute intensity

$$J_{ij} := \left\langle E_0^i E_0^{*j} \right\rangle. \tag{5.104}$$

Since the vector $\boldsymbol{E}_0$ is perpendicular to the wave vector $\boldsymbol{k}$ then J_{ij} has only four components. We choose the Cartesian x^1, x^2 axes perpendicular to the wave vector. The indices in (5.104) run over $i, j = \{1, 2\}$. The trace of this tensor represents intensity of the wave (density of the energy flux) and it reads

$$\mathrm{Tr}\hat{J} = \sum_{i=1}^{2} J_{ii} = \left\langle |E_0^1|^2 \right\rangle + \left\langle |E_0^2|^2 \right\rangle = \left\langle |\boldsymbol{E}_0|^2 \right\rangle. \tag{5.105}$$

This quantity is not related to polarization properties of the wave and therefore tensor containing *relative intensities* is more adequate than (5.104). A *polarization tensor* is defined in the following way

Polarization tensor

$$\boxed{\rho_{ij} := \frac{J_{ij}}{\mathrm{Tr}\hat{J}} = \frac{\left\langle E_0^i E_0^{*j} \right\rangle}{\langle |\boldsymbol{E}_0|^2 \rangle}.} \tag{5.106}$$

It has the following properties:

1. $\mathrm{Tr}\hat{\rho} = 1 \qquad \Leftrightarrow \qquad \rho_{11} + \rho_{22} = 1,$

2. $\hat{\rho}^{\dagger} = \hat{\rho} \qquad \Leftrightarrow \qquad \rho_{11}, \rho_{22} \in \mathbb{R}, \qquad \rho_{21} = \rho_{12}^{*}.$

The polarization tensor determinant reads

$$\det \hat{\rho} = \frac{1}{\langle |\boldsymbol{E}_0|^4 \rangle} \left[\left\langle E_0^1 E_0^{*1} \right\rangle \left\langle (E_0^2 E_0^{*2} \right\rangle - \left\langle E_0^1 E_0^{*2} \right\rangle \left\langle E_0^2 E_0^{*1} \right\rangle \right]. \qquad (5.107)$$

There are two limit cases of partially polarized waves – total polarization and absence of polarization.

For *totally polarized* electromagnetic waves the amplitude vector is constant, $\boldsymbol{E}_0 = const$, and its time averaging give the proper vector, hence

Totally polarized wave

$$\rho_{ij} = \frac{E_0^i E_0^{*j}}{|\boldsymbol{E}_0|^2}. \qquad (5.108)$$

The polarization tensor determinant vanishes in such a case

$$\det \hat{\rho} = \frac{1}{|\boldsymbol{E}_0|^4} \left[(E_0^1 E_0^{*1})(E_0^2 E_0^{*2}) - (E_0^1 E_0^{*2})(E_0^2 E_0^{*1}) \right] = 0. \qquad (5.109)$$

On the other hand, for *unpolarized* electromagnetic wave (e.g. natural light beam) the average intensity has the same value in all directions. It gives

Unpolarized wave

$$\left\langle E_0^1 E_0^{*1} \right\rangle = \left\langle E_0^2 E_0^{*2} \right\rangle = \frac{1}{2} \left\langle |\boldsymbol{E}_0|^2 \right\rangle. \qquad (5.110)$$

The components $E_0^1(t)$ and $E_0^2(t)$ are *not correlated* for totally unpolarized wave. It leads to vanishing of expressions

$$\left\langle E_0^1 E_0^{*2} \right\rangle = 0 = \left\langle E_0^2 E_0^{*1} \right\rangle. \qquad (5.111)$$

Plugging this results into (5.106) one gets the polarization tensor

$$\rho_{ij} = \frac{1}{2} \delta_{ij}. \qquad (5.112)$$

Its determinant (5.109) has value $\det \hat{\rho} = \frac{1}{4}$. Thus, the polarization tensor determinant vanishes for totally polarized electromagnetic wave and it equals to $1/4$ in absence of polarization. A *grade of polarization* $P \in [0, 1]$ is defined as follows

Grade of polarization

$$\boxed{\det \hat{\rho} = \frac{1}{4} \left(1 - P^2 \right),} \qquad (5.113)$$

where $P = 0$ and $P = 1$ represent, respectively, absence of polarization and maximal grade of polarization.

A convenient decomposition of polarization tensor consists on its symmetric $\mathcal{S}_{ij}$ and anti-symmetric $\mathcal{A}_{ij}$ part. They are defined as follows

Polarization tensor decomposition

$$\mathcal{S}_{ij} := \frac{1}{2}(\rho_{ij} + \rho_{ji}) = \frac{1}{2}(\rho_{ij} + \rho_{ij}^{*}) \in \mathbb{R}, \qquad (5.114)$$

$$\mathcal{A}_{ij} := \frac{1}{2}(\rho_{ij} - \rho_{ji}) = \frac{1}{2}(\rho_{ij} - \rho_{ij}^{*}) \equiv -\frac{i}{2} \varepsilon_{ij} \mathcal{A} \in \mathbb{I}, \qquad (5.115)$$

where $\mathcal{A} \in \mathbb{R}$, then

$$\boxed{\rho_{ij} = \mathcal{S}_{ij} - \frac{i}{2}\varepsilon_{ij}\mathcal{A}.} \tag{5.116}$$

Polarization tensor for totally polarized wave.

In this section we shall analyse what is the meaning of the polarization tensor components in the case of totally polarized electromagnetic wave

$$\boldsymbol{E} = \boldsymbol{E}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)} = (E_0^1\hat{\boldsymbol{x}} + E_0^2\hat{\boldsymbol{y}})e^{i\boldsymbol{k}\cdot\boldsymbol{x}}e^{-i\omega t},$$

where $E_0^1 = A\,e^{i\alpha}$ and $E_0^2 = B\,e^{i\beta}$. At given point P described by the position vector $\boldsymbol{x}$ the expression $e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$ is a fixed number. It is irrelevant for the tensor ρ_{ij} because it is an overall phase factor and therefore it does not contribute to $\delta = \beta - \alpha$. Plugging this components to (5.108) one gets

$$\begin{aligned}\rho_{ij} &= \frac{1}{A^2+B^2}\left[\begin{array}{c|c} A^2 & AB\,e^{-i\delta} \\ \hline AB\,e^{i\delta} & B^2 \end{array}\right] \\ &= \underbrace{\frac{1}{A^2+B^2}\left[\begin{array}{c|c} A^2 & AB\,\cos\delta \\ \hline AB\,\cos\delta & B^2 \end{array}\right]}_{\mathcal{S}_{ij}} - \frac{i}{2}\left[\begin{array}{c|c} 0 & 1 \\ \hline -1 & 0 \end{array}\right]\underbrace{\frac{2AB}{A^2+B^2}\sin\delta}_{\mathcal{A}}\end{aligned} \tag{5.117}$$

The parameter $\mathcal{A}$ vanishes for $\delta = \{0, \pi\}$. It has been shown that electromagnetic wave is linearly polarized for such values of phase shift. The corresponding polarization tensor is symmetric,

Linear polarization

$$\rho_{ij} = \frac{1}{A^2+B^2}\left[\begin{array}{c|c} A^2 & \pm AB \\ \hline \pm AB & B^2 \end{array}\right], \tag{5.118}$$

and it takes the following forms

$$\rho_{ij} = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array}\right] \qquad \rho_{ij} = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 1 \end{array}\right] \qquad \rho_{ij} = \frac{1}{2}\left[\begin{array}{c|c} 1 & \pm 1 \\ \hline \pm 1 & 1 \end{array}\right] \tag{5.119}$$

for polarizations aligned with, respectively, the x, y axes and the diagonal $y = \pm x$ direction. The circular polarization is corresponds to $\delta = \pm\frac{\pi}{2}$ and $A = B$. It gives

Circular polarization

$$\rho_{ij} = \frac{1}{2}\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right] - \frac{i}{2}(\pm)\left[\begin{array}{c|c} 0 & 1 \\ \hline -1 & 0 \end{array}\right]. \tag{5.120}$$

The coefficient $\mathcal{A}$ has interpretation of *degree of circular polarization*. Its extremal values $\mathcal{A} = -1$ and $\mathcal{A} = +1$ correspond to circularly polarized waves with, respectively, negative and positive helicity.

Stokes parameters

Going back to our observation that the polarization tensor is a 2×2 Hermitian matrix we shall make use of the fact that it can be decomposed on the Pauli matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and the identity matrix $\mathbb{1}$,

$$\hat{\rho} = \frac{1}{2}\left[\mathbb{1} + \xi_a \sigma_a\right]. \tag{5.121}$$

Coefficients ξ_a are called *Stokes parameters*. The polarization tensor determinant is related to the polarization degree P according to (5.113). Hence, the equality

$$\frac{1}{4}(1 - P^2) = \frac{1}{4}\det\begin{bmatrix} 1+\xi_3 & \xi_1 - i\xi_2 \\ \xi_1 + i\xi_2 & 1 - \xi_3 \end{bmatrix} = \frac{1}{4}[1 - (\xi_1^2 + \xi_2^2 + \xi_3^2)]$$

gives

$$\boxed{P = \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}.} \tag{5.122}$$

The expression (5.122) means that all states having the same grade of polarization form spherical surfaces in three-dimensional space of Stokes parameters. The states with maximal polarization form the sphere with $P = 1$ radius and unpolarized states correspond with the point at the origin $\xi_1 = \xi_2 = \xi_3 = 0$.

In order to understand better what is the physical significance of the Stokes parameters we consider all the states $P = 1$ and express the parameters ξ_a in terms of A, B and δ. Using properties of the Pauli matrixces

$$\mathrm{Tr}(\sigma_a \sigma_b) = 2\delta_{ab}, \qquad \mathrm{Tr}(\sigma_a) = 0$$

one gets from (5.121) the coefficients

$$\xi_a = \mathrm{Tr}(\sigma_a \hat{\rho}). \tag{5.123}$$

Next, plugging the polarization tensor parametrizd by A, B and δ,

$$\hat{\rho} = \frac{1}{A^2 + B^2}\left[\begin{array}{c|c} A^2 & AB\,(\cos\delta - i\sin\delta) \\ \hline AB\,(\cos\delta + i\sin\delta) & B^2 \end{array}\right],$$

into (5.123) one gets

$$\boxed{\xi_1 = \frac{2AB}{A^2 + B^2}\cos\delta, \qquad \xi_2 = \frac{2AB}{A^2 + B^2}\sin\delta \qquad \xi_3 = \frac{A^2 - B^2}{A^2 + B^2}.} \tag{5.124}$$

Comparing with decomposition (5.117) and (5.121) one gets $\xi_2 = \mathcal{A}$, and thus it describes *the grade of circular polarization*. For $B = 0$ one gets $\xi_1 = \xi_2 = 0$ and $\xi_3 = +1$ what correspond to polarization along the x-axis. Similarly, for $A = 0$ the Stokes parameters read $\xi_1 = \xi_2 = 0$ and $\xi_3 = -1$. This case stands for the wave polarized along the y-axis. One can conclude that the parameter ξ_3 *describes polarization along* the x and y axes. Finally, in the case $A = B$ and $\delta = 0$ the wave is polarized along the line which form the angle $\vartheta = \pi/4$ with the x axis. This case corresponds to the Stokes parameters $\xi_2 = \xi_3 = 0$ and $\xi_1 = +1$. For $\delta = 0$ replaced by $\delta = \pi$ the parameter ξ_3 became $\xi_3 = -1$ (polarization along the line which form the angle $\vartheta = -\pi/4$ with the x-axis).

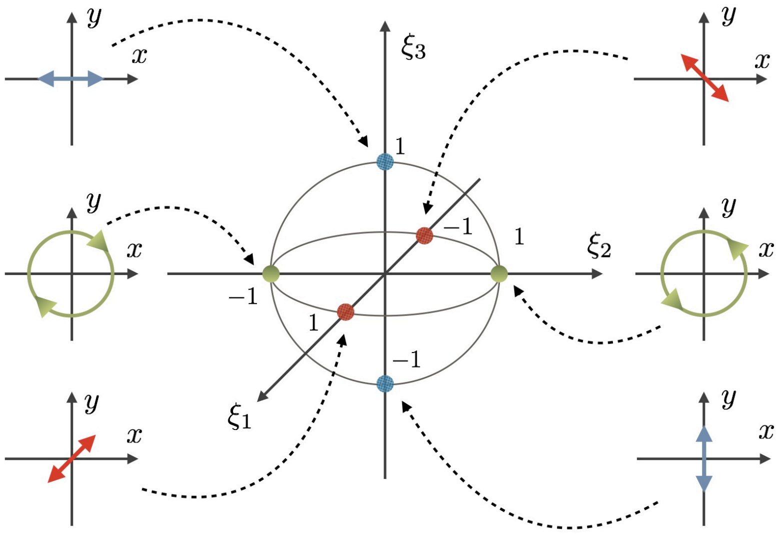


Figure 5.3: The space of Stokes parameters and the meaning of points at the surface of the sphere with the radius $P = 1$.

It has been shown that *totally polarized* electromagnetic wave, $P = 1$, is characterized by the polarization ellipse with semi-axes e_1 and e_2 that satisfy

$$e_1^2 + e_2^2 = A^2 + B^2, \qquad \eta e_1 e_2 = AB \sin\delta, \qquad \tan(2\vartheta) = \frac{2AB}{A^2 - B^2}\cos\delta.$$

These partameters allows us to express the Stokes parameters in the form

$$\boxed{\frac{\xi_1}{\xi_3} = \tan(2\vartheta), \qquad \xi_2 = \eta\frac{2e_1e_2}{e_1^2 + e_2^2}.} \tag{5.125}$$

Expressions (5.125) gives relations between parameters of the polarisation ellipse (its size and orientation) and the Stokes parameters. The circular polarization component exists if none of the parameters e_1 and e_2 is equal to zero. Otherwise, the ellipse degenerates to a segment of straight line.

The parameter ξ_2 and $\sqrt{\xi_1^2 + \xi_3^2}$ are invariant under Lorentz transformations.

Decomposition of partially polarized wave on polarized and unpolarized components

We split the tensor $J_{ij} = \langle E^i E^{*j} \rangle$ into two parts: the component $J_{ij}^{(n)}$ that represent *unpolarized* electromagnetic wave and the component $J_{ij}^{(p)}$ corresponding to *totally polarized* electromagnetic wave. It has been shown that the polarization tensor describing unpolarized wave is proportional to the Kronecker delta. Hence

$$\rho_{ij}^{(n)} := \frac{J_{ij}^{(n)}}{J^{(n)}} = \frac{1}{2}\delta_{ij} \qquad \Rightarrow \qquad \boxed{J_{ij}^{(n)} = \frac{1}{2}J^{(n)}\delta_{ij}} \tag{5.126}$$

where $J^{(n)} \equiv \mathrm{Tr}(\hat{J}^{(n)})$. Time averaging of polarized components is redundant $J_{ij}^{(p)} = E_0^{i(p)} E_0^{*j(p)}$, hence the expression $J_{ij} - J_{ij}^{(n)} = J_{ij}^{(p)}$ reads

$$J_{ij} - \frac{1}{2}J^{(n)}\delta_{ij} = E_0^{i(p)} E_0^{*j(p)}. \tag{5.127}$$

The matrix $\left[E_0^{i(p)} E_0^{*j(p)}\right]$ has null determinant, see (5.109). It leads to the equation

$$\det\left[J_{ij} - \frac{1}{2}J^{(n)}\delta_{ij}\right] = 0, \tag{5.128}$$

where $J_{ij} = J\rho_{ij}$ with $J \equiv \mathrm{Tr}(\hat{J})$. The equation (5.128) allows us to determine the intensity of unpolarized component $J^{(n)}$

$$\begin{aligned}
&\det\begin{bmatrix} J\rho_{11} - \frac{1}{2}J^{(n)} & J\rho_{12} \\ J\rho_{21} & J\rho_{22} - \frac{1}{2}J^{(n)} \end{bmatrix} = \\
&= (J\rho_{11} - \frac{1}{2}J^{(n)})(J\rho_{22} - \frac{1}{2}J^{(n)}) - J^2\rho_{12}\rho_{21} \\
&= J^2 \underbrace{[\rho_{11}\rho_{22} - \rho_{12}\rho_{21}]}_{\det\hat{\rho} = \frac{1}{4}(1-P^2)} + \frac{1}{4}(J^{(n)})^2 - \frac{1}{2}J^{(n)}J \underbrace{[\rho_{11} + \rho_{22}]}_{\mathrm{Tr}\hat{\rho} = 1} \\
&= \frac{1}{4}\left[(J^{(n)})^2 - 2JJ^{(n)} + (1 - P^2)J^2\right] = 0
\end{aligned} \tag{5.129}$$

which gives $J^{(n)} = (1 \pm P)J$. Since $J^{(n)} < J$, then the physical solution reads

$$\boxed{J^{(n)} = (1 - P)J.} \tag{5.130}$$

The intensity of the polarized component equals to $J^{(p)} = PJ$ since $J_{ij}^{(p)} = J_{ij} - J_{ij}^{(n)}$. One can easily establish the relation with the Stokes parameters. The only difference is that the polarized component intensity is certain fraction of total intensity *i.e.* $e_1^2 + e_2^2 = A^2 + B^2 = PJ$. Hence

$$\frac{\xi_1}{\xi_3} = \tan(2\vartheta), \qquad \xi_2 = \eta \frac{2e_1 e_2}{PJ}. \tag{5.131}$$

Decomposition of partially polarized wave on two incoherent elliptically polarized waves

The eigenvalues λ_a, $a = 1, 2$ of the polarisation tensor $\hat{\rho}$ are real-valued because the tensor is Hermitian. The eigenvectors $\boldsymbol{n}^{(a)}$ of the polarization tensor have the form of two complex versors $\boldsymbol{n}^{*(a)} \cdot \boldsymbol{n}^{(a)} = 1$ that satisfy equations[13]

[13] L.D. Landau, E.M. Lifshitz, The Classical Theory of Fields. Volume 2.

$$\rho_{ij} n_j^{(a)} = \lambda_a n_i^{(a)}. \tag{5.132}$$

Multiplying this equation by $n_i^{*(a)}$ and taking sum over i we get

$$\lambda_a = \rho_{ij} n_i^{*(a)} n_j^{(a)} = \frac{1}{J} \left\langle E_0^i E_0^{*j} \right\rangle n_i^{*(a)} n_j^{(a)} = \frac{1}{J} \left\langle (E_0^i n_i^{*(a)})(E_0^{*j} n_j^{(a)}) \right\rangle$$
$$= \frac{1}{J} \left\langle |E_0^i n_i^{*(a)}|^2 \right\rangle > 0.$$

Hence, both eigenvalues are *real-valued and positive*. The eigenvalues λ_a can be parametrized by the polarisation degree P. Indeed, the equation

$$\det[\hat{\rho} - \lambda \mathbb{1}] = 0 \qquad \Leftrightarrow \qquad \lambda^2 - \underbrace{\mathrm{Tr}\hat{\rho}}_{1}\, \lambda + \underbrace{\det \hat{\rho}}_{\frac{1}{4}(1-P^2)} = 0 \tag{5.133}$$

has two solutions $\lambda_{1,2} = \frac{1}{2}(1 \pm P)$.

The eigenvectors are mutually orthogonal. This can be shown as follows. Multiplying the equation with $a = 1$ by $\boldsymbol{n}^{*(2)}$ and the complex conjugated equation with $a = 2$ by $\boldsymbol{n}^{(1)}$

$$\begin{cases} \rho_{ij} n_j^{(1)} = \lambda_1 n_i^{(1)} & / n_i^{*(2)} \\ \rho_{ij}^* n_j^{*(2)} = \lambda_2 n_i^{*(2)} & / n_i^{(1)} \end{cases} \tag{5.134}$$

and subtracting the resultant equations one gets

$$\underbrace{(\rho_{ij} - \rho_{ji}^*)}_{0}\, n_j^{(1)} n_i^{*(2)} = (\lambda_1 - \lambda_2) n_i^{(1)} n_i^{*(2)}.$$

Since $\lambda_1 - \lambda_2 = P \neq 0$ (there is unpolarized cimponent) then $\boldsymbol{n}^{(1)} \cdot \boldsymbol{n}^{*(2)} = 0$. It means that the complex eigenvectors form the orthonormal set

$$\boldsymbol{n}^{(a)} \cdot \boldsymbol{n}^{*(b)} = \delta_{ab}. \tag{5.135}$$

The matrix whose columns are formed by the eigenvectors is an unitary matrix

$$U := \left[\; \boldsymbol{n}^{(1)} \; \middle| \; \boldsymbol{n}^{(2)} \; \right] \qquad \text{and} \qquad U^{\dagger} := \left[\begin{array}{c} \boldsymbol{n}^{*(1)} \\ \hline \boldsymbol{n}^{*(2)} \end{array} \right].$$

This definition and the fact that $\boldsymbol{n}^{(a)}$ are eigenvectors of $\hat{\rho}$ give

$$U^{\dagger}\hat{\rho}\, U = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \qquad \Leftrightarrow \qquad \hat{\rho} = U \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} U^{\dagger} \tag{5.136}$$

The last expression in (5.136) reads

$$\boxed{\rho_{ij} = \lambda_1\, n_i^{(1)} n_j^{*(1)} + \lambda_2\, n_i^{(2)} n_j^{*(2)}.} \tag{5.137}$$

A complex amplitude vector can always be chosen in the way that one of two mutually perpendicular components is real-valued whereas the other one is purely imaginary-valued. Let $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ be two ortogonal real vectors. We consider the following form of first eigenvector

$$\boldsymbol{n}^{(1)} := e_1\hat{\boldsymbol{e}}_1 + i\eta e_2\hat{\boldsymbol{e}}_2. \tag{5.138}$$

The normalization condition $\boldsymbol{n}^{*(1)} \cdot \boldsymbol{n}^{(1)} = 1$ leads to the condition $e_1^2 + e_2^2 = 1$. The second eigenvector $\boldsymbol{n}^{(2)}$ is orthogonal to the first one. Thus, taking $\boldsymbol{n}^{*(2)} = \alpha^*\hat{\boldsymbol{e}}_1 + \beta^*\hat{\boldsymbol{e}}_2$ one gets $\alpha^* e_1 + i\eta\beta^* e_2 = 0$. The solution of the last condition reads $\alpha^* = -ie_2$ and $\beta^* = \eta e_1$

$$\boldsymbol{n}^{(2)} := ie_2\hat{\boldsymbol{e}}_1 + \eta e_1\hat{\boldsymbol{e}}_2 \tag{5.139}$$

where we made use of the fact that $\boldsymbol{n}^{*(2)} \cdot \boldsymbol{n}^{(2)} = 1 = |\alpha|^2 + |\beta|^2$.

The vectors (5.138) and (5.139) describe two identical ellipses (the same ratio of semi-axes). The major semi-axes of these ellipses form the angle $\pi/2$. The polarization components associated which each ellipse are *incoherent i.e.* there are no cross terms in (5.137).

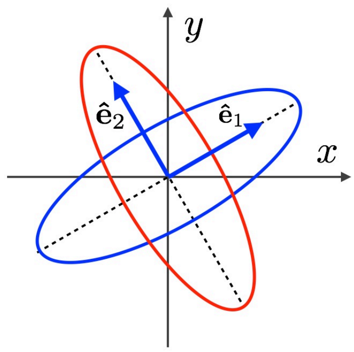


Figure 5.4: The polarization ellipses.

5.4 *Energy and momentum flux of electromagnetic waves*

The energy density u and the momentum flux density $\boldsymbol{S}$ of the electromagnetic field is given by expressions

$$u = \frac{1}{8\pi}\left[\boldsymbol{E}\cdot\boldsymbol{D} + \boldsymbol{H}\cdot\boldsymbol{B}\right], \tag{5.140}$$

$$\boldsymbol{S} = \frac{c}{4\pi}\left[\boldsymbol{E}\times\boldsymbol{H}\right], \tag{5.141}$$

where all fields are real-valued. We assume that the continuum medium in which the wave propagates is linear, homogeneous and isotropic. It leads to linear constitutive realtions $\boldsymbol{D} = \varepsilon\boldsymbol{E}$ and $\boldsymbol{B} = \mu\boldsymbol{H}$.

The physical real fields are given by real (or imaginary) parts of auxiliary complex fields $\boldsymbol{E} \in \mathbb{C}$, $\boldsymbol{B} \in \mathbb{C}$, hence

$$Re\,\boldsymbol{E} = \frac{1}{2}(\boldsymbol{E} + \boldsymbol{E}^*), \qquad Re\,\boldsymbol{B} = \frac{1}{2}(\boldsymbol{B} + \boldsymbol{B}^*). \tag{5.142}$$

Consequently, the expressions (5.140) and (5.141) must be replaced by the following ones

$$u = \frac{1}{8\pi}\left[\varepsilon(Re\,\boldsymbol{E})^2 + \frac{1}{\mu}(Re\,\boldsymbol{B})^2\right], \tag{5.143}$$

$$\boldsymbol{S} = \frac{c}{4\pi\,\mu}\left[Re\,\boldsymbol{E} \times Re\,\boldsymbol{B}\right]. \tag{5.144}$$

Instantaneous values of u and $\boldsymbol{S}$ are less relevant than theirs time average values $\langle u \rangle$ and $\langle \boldsymbol{S} \rangle$ for fields that oscillate very quickly. Taking the monochromatic wave in the representation of complex-valued auxiliary fields

$$\boldsymbol{E} = \boldsymbol{E}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}, \qquad \boldsymbol{B} = \boldsymbol{B}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}, \tag{5.145}$$

one gets

$$\left\langle (Re\,\boldsymbol{E})^2 \right\rangle = \frac{1}{4}\left[\left\langle \boldsymbol{E}^2 \right\rangle + 2\left\langle \boldsymbol{E}\cdot\boldsymbol{E}^* \right\rangle + \left\langle \boldsymbol{E}^{*2} \right\rangle\right] = \frac{1}{2}\boldsymbol{E}\cdot\boldsymbol{E}^* = \frac{1}{2}\boldsymbol{E}_0\cdot\boldsymbol{E}_0^*,$$
$$\left\langle (Re\,\boldsymbol{B})^2 \right\rangle = \frac{1}{4}\left[\left\langle \boldsymbol{B}^2 \right\rangle + 2\left\langle \boldsymbol{B}\cdot\boldsymbol{B}^* \right\rangle + \left\langle \boldsymbol{B}^{*2} \right\rangle\right] = \frac{1}{2}\boldsymbol{B}\cdot\boldsymbol{B}^* = \frac{1}{2}\boldsymbol{B}_0\cdot\boldsymbol{B}_0^*,$$

where quickly oscillating terms drop out

$$\left\langle \boldsymbol{E}^2 \right\rangle \sim \left\langle e^{-2i\omega t} \right\rangle = 0, \qquad \left\langle \boldsymbol{E}^{*2} \right\rangle \sim \left\langle e^{2i\omega t} \right\rangle = 0.$$

Similarly, the following expression

$$\begin{aligned}\langle Re\,\boldsymbol{E} \times Re\,\boldsymbol{B} \rangle &= \frac{1}{4}[\langle \boldsymbol{E}\times\boldsymbol{B}^* \rangle + \langle \boldsymbol{E}^*\times\boldsymbol{B} \rangle + \underbrace{\langle \boldsymbol{E}\times\boldsymbol{B} \rangle}_{0} + \underbrace{\langle \boldsymbol{E}^*\times\boldsymbol{B}^* \rangle}_{0}] \\ &= \frac{1}{2}Re\,[\boldsymbol{E}\times\boldsymbol{B}^*] = \frac{1}{2}Re\,[\boldsymbol{E}_0\times\boldsymbol{B}_0^*]\end{aligned} \tag{5.146}$$

does not contain terms proportional to $e^{\pm i\omega t}$. Thus, time averaged values of energy density ant the Poynting vector read

$$\boxed{\langle u \rangle = \frac{1}{16\pi}Re\,[\boldsymbol{E}\cdot\boldsymbol{D}^* + \boldsymbol{H}\cdot\boldsymbol{B}^*]} \tag{5.147}$$

$$\boxed{\langle \boldsymbol{S} \rangle = \frac{c}{8\pi}Re\,[\boldsymbol{E}\times\boldsymbol{H}^*].} \tag{5.148}$$

It has been shown that Maxwell's equations and plane wave ansatz lead to the following algebraic equations

$$\boldsymbol{E} = -\frac{1}{n}\hat{\boldsymbol{k}}\times\boldsymbol{B}, \qquad \boldsymbol{B} = n\,\hat{\boldsymbol{k}}\times\boldsymbol{E}, \qquad |\boldsymbol{E}|^2 = \frac{1}{n^2}|\boldsymbol{B}|^2, \tag{5.149}$$

where $\hat{\boldsymbol{k}} := \frac{\boldsymbol{k}}{|\boldsymbol{k}|}$. The average value of the energy density reads

$$\langle u \rangle = \frac{1}{16\pi}\left[\varepsilon|\boldsymbol{E}|^2 + \frac{1}{\mu}|\boldsymbol{B}|^2\right] = \frac{1}{16\pi\,\mu}\left[\frac{\varepsilon\mu}{n^2}+1\right]|\boldsymbol{B}|^2 = \frac{1}{8\pi\mu}|\boldsymbol{B}|^2.$$

Similarly, the average value of the Poynting vector has the form

$$\langle \boldsymbol{S} \rangle = \frac{c}{8\pi}Re\left[-\frac{1}{n}(\hat{\boldsymbol{k}}\times\boldsymbol{B})\times\left(\frac{1}{\mu}\boldsymbol{B}^*\right)\right] = \frac{c}{8\pi\mu\,n}Re(|\boldsymbol{B}|^2)\hat{\boldsymbol{k}}$$
$$= \frac{c}{8\pi\mu\,n}|\boldsymbol{B}|^2\hat{\boldsymbol{k}}.$$

The magnitude of $\langle \boldsymbol{S} \rangle$ is proportional to $\langle u \rangle$, namely

$$\boxed{\frac{\langle \boldsymbol{S} \rangle \cdot \hat{\boldsymbol{k}}}{\langle u \rangle} = \frac{c}{n} = v,} \tag{5.150}$$

where the proportionality coefficient is equal to the speed of propagation of electromagnetic wave.

5.5 *Reflection and refraction of light at the interface between two dielectrics*

In this section we shall deal with description of electromagnetic wave on the border of two different homogeneous dielectrics. Such dielectrics are characterized by electric permittivities ε_1, ε_2 and magnetic permeabilities μ_1 and μ_2. The refractive indices read $n_1 := \sqrt{\varepsilon_1\mu_1}$ and $n_2 := \sqrt{\varepsilon_2\mu_2}$. The boundary between two spatial regions containing different dielectric media is called an *interface*. One of the simplest solutions is obtained for interface in the form of infinite plane. Let $\hat{\boldsymbol{n}}$ be a unit vector, normal to the interface and pointing out from first dielectric (1) to the second dielectic (2). In absence of free charges and free currents the fields are solutions of sourceless Maxwell's equations

$$\nabla\cdot\boldsymbol{D} = 0, \qquad \nabla\times\boldsymbol{H} - \frac{1}{c}\partial_t\boldsymbol{D} = 0, \tag{5.151}$$
$$\nabla\cdot\boldsymbol{B} = 0, \qquad \nabla\times\boldsymbol{E} + \frac{1}{c}\partial_t\boldsymbol{B} = 0, \tag{5.152}$$

and the boundary conditions

$$\hat{\boldsymbol{n}}\cdot(\boldsymbol{D}_2 - \boldsymbol{D}_1) = 0, \qquad \hat{\boldsymbol{n}}\times(\boldsymbol{H}_2 - \boldsymbol{H}_1) = 0, \tag{5.153}$$
$$\hat{\boldsymbol{n}}\cdot(\boldsymbol{B}_2 - \boldsymbol{B}_1) = 0, \qquad \hat{\boldsymbol{n}}\times(\boldsymbol{E}_2 - \boldsymbol{E}_1) = 0. \tag{5.154}$$

These conditions imply continuity of normal components D_n and B_n and continuity of tangent components H_t and E_t.

We choose the $z-$axis perpendicular to the interface and oriented in direction of $\hat{\boldsymbol{n}}$, *i.e.* $\hat{\boldsymbol{z}} = \hat{\boldsymbol{n}}$. The incoming electromagnetic wave,

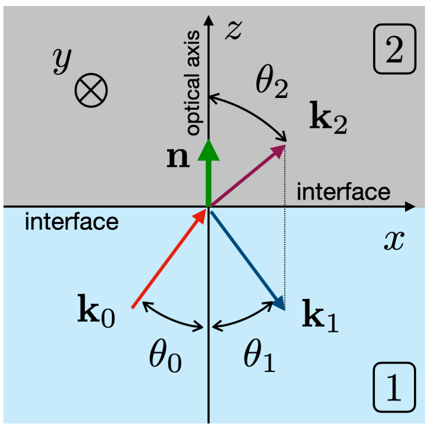


Figure 5.5: The incidence, reflection and refraction wave vectors at the interface of two optical media.

characterized by the wave vector $\boldsymbol{k}_0$, propagates in the medium (1) We shall denote by $\boldsymbol{k}_1$ the wave vector associated with the reflected electromagnetic wave in (1) and by $\boldsymbol{k}_2$ the wave vector describing the transmitted electromagnetic wave in (2). For oblique incidence the vectors $\boldsymbol{k}_0$ and $\hat{\boldsymbol{n}}$ are not parallel. In such a case they define a *plane of incidence*. The wave vectors $\boldsymbol{k}_0$, $\boldsymbol{k}_1$, $\boldsymbol{k}_2$ form angles θ_0, θ_1 and θ_2 with the vector $\hat{\boldsymbol{n}}$. Without loss of generality we choose the $\hat{\boldsymbol{x}}$ versor parallel to the plane of incidence (and perpendicular to $\hat{\boldsymbol{z}}$). Third versor $\hat{\boldsymbol{y}}$ is given by $\hat{\boldsymbol{y}} = \hat{\boldsymbol{z}} \times \hat{\boldsymbol{x}}$.

The *incident*, *reflected* and *transmitted* waves are given by fields

$$
\begin{aligned}
&\text{Incident beam} \quad && \boldsymbol{E}_0 = \boldsymbol{E}_0^0\, e^{i(\boldsymbol{k}_0\cdot\boldsymbol{r}-\omega_0 t)} \\
& && \boldsymbol{H}_0 = \frac{n_1}{\mu_1}\,\hat{\boldsymbol{k}}_0 \times \boldsymbol{E}_0 = \left(\frac{n_1}{\mu_1}\,\hat{\boldsymbol{k}}_0 \times \boldsymbol{E}_0^0\right) e^{i(\boldsymbol{k}_0\cdot\boldsymbol{r}-\omega_0 t)}, \\
&\text{Reflected beam} && \boldsymbol{E}_1 = \boldsymbol{E}_1^0\, e^{i(\boldsymbol{k}_1\cdot\boldsymbol{r}-\omega_1 t)}, \\
& && \boldsymbol{H}_1 = \frac{n_1}{\mu_1}\,\hat{\boldsymbol{k}}_1 \times \boldsymbol{E}_1 = \left(\frac{n_1}{\mu_1}\,\hat{\boldsymbol{k}}_1 \times \boldsymbol{E}_1^0\right) e^{i(\boldsymbol{k}_1\cdot\boldsymbol{r}-\omega_1 t)}, \\
&\text{Refracted beam} && \boldsymbol{E}_2 = \boldsymbol{E}_2^0\, e^{i(\boldsymbol{k}_2\cdot\boldsymbol{r}-\omega_2 t)}, \\
& && \boldsymbol{H}_2 = \frac{n_2}{\mu_2}\,\hat{\boldsymbol{k}}_2 \times \boldsymbol{E}_2 = \left(\frac{n_2}{\mu_2}\,\hat{\boldsymbol{k}}_2 \times \boldsymbol{E}_2^0\right) e^{i(\boldsymbol{k}_2\cdot\boldsymbol{r}-\omega_2 t)}.
\end{aligned}
$$

Angles of incidence, reflection and refraction

All waves must exist simultaneously at the boundary $z = 0$ which is the xy-plane. Tangential components of the fields must be equal on both sides of the interface. The fields satisfy boundary conditions at the interface $z = 0$, namely their tangential components must be continuous on both sides of the boundary

$$\hat{\boldsymbol{n}} \times [\boldsymbol{E}_0 + \boldsymbol{E}_1] = \hat{\boldsymbol{n}} \times \boldsymbol{E}_2, \tag{5.155}$$

$$\hat{\boldsymbol{n}} \times [\boldsymbol{H}_0 + \boldsymbol{H}_1] = \hat{\boldsymbol{n}} \times \boldsymbol{H}_2. \tag{5.156}$$

Plugging the complex vectors into the boundary conditions (5.155) and (5.156) we get

$$\hat{\boldsymbol{n}} \times [\boldsymbol{E}_0^0 e^{i(\boldsymbol{k}_0\cdot\boldsymbol{r}-\omega_0 t)} + \boldsymbol{E}_1^0 e^{i(\boldsymbol{k}_1\cdot\boldsymbol{r}-\omega_1 t)}] = \hat{\boldsymbol{n}} \times \boldsymbol{E}_2^0 e^{i(\boldsymbol{k}_2\cdot\boldsymbol{r}-\omega_2 t)}, \tag{5.157}$$

$$\hat{\boldsymbol{n}} \times [\boldsymbol{H}_0^0 e^{i(\boldsymbol{k}_0\cdot\boldsymbol{r}-\omega_0 t)} + \boldsymbol{H}_1^0 e^{i(\boldsymbol{k}_1\cdot\boldsymbol{r}-\omega_1 t)}] = \hat{\boldsymbol{n}} \times \boldsymbol{H}_2^0 e^{i(\boldsymbol{k}_2\cdot\boldsymbol{r}-\omega_2 t)}. \tag{5.158}$$

If the wave amplitudes are constant, the only way that the conditions (5.157) and (5.158) can be true is phase matching

$$\boldsymbol{k}_0 \cdot \boldsymbol{r} - \omega_0 t = \boldsymbol{k}_1 \cdot \boldsymbol{r} - \omega_1 t = \boldsymbol{k}_2 \cdot \boldsymbol{r} - \omega_2 t. \tag{5.159}$$

For $\boldsymbol{r} = 0$ the condition (5.159) gives $\omega_0 t = \omega_1 t = \omega_2 t$ which imply equality of frequencies $\omega_0 = \omega_1 = \omega_2$. We shall denote this frequency

The incident, relected and refracted waves have the same frequency $\omega := \omega_0 = \omega_1 = \omega_2$

by ω. Consequently, (5.159) gives

$$\boldsymbol{k}_0 \cdot \boldsymbol{r} = \boldsymbol{k}_1 \cdot \boldsymbol{r} = \boldsymbol{k}_2 \cdot \boldsymbol{r}. \tag{5.160}$$

The vector $\boldsymbol{r}$ restricted to the interface $z = 0$ has only components x and y, $\boldsymbol{r} = x\,\hat{\boldsymbol{x}} + y\,\hat{\boldsymbol{y}}$.

Without loss of generality we can assume that the incident wave vector has no y-component (otherwise, one can rotate Cartesian frame to eliminate the y-component), $k_{0y} := 0$ *i.e.* $\boldsymbol{k}_0 \cdot \hat{\boldsymbol{y}} = 0$. The condition (5.160) must be true for any x and y. It leads to sequence of equalities involving coefficients that multiply the variables x and y

$$\boldsymbol{k}_0 \cdot \hat{\boldsymbol{x}} = \boldsymbol{k}_1 \cdot \hat{\boldsymbol{x}} = \boldsymbol{k}_2 \cdot \hat{\boldsymbol{x}}, \tag{5.161}$$

$$0 = \boldsymbol{k}_1 \cdot \hat{\boldsymbol{y}} = \boldsymbol{k}_2 \cdot \hat{\boldsymbol{y}}. \tag{5.162}$$

The second condition (5.162) gives $k_{1y} = k_{2y} = 0$. It means that the refraction and reflection wave vectors *belong to the plane of incidence*.

Vectors $\boldsymbol{k}_0$, $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ belong to the plane of incidence.

In terms of angles

$$\boldsymbol{k}_a \cdot \hat{\boldsymbol{x}} = k_a \cos(\theta_a + \pi/2) = -k_a \sin\theta_a, \qquad a = 0, 1, 2$$

the condition (5.161) reads

$$k_0 \sin\theta_0 = k_1 \sin\theta_1 = k_2 \sin\theta_2. \tag{5.163}$$

Taking into account the dispersion relation in homogeneous media, one gets the following sequence of equalities

$$\frac{\omega}{c} = \frac{k_0}{n_1} = \frac{k_1}{n_1} = \frac{k_2}{n_2}. \tag{5.164}$$

(5.164) implies $k_1 = k_0$ and $k_2 = \frac{n_2}{n_1} k_0$. The first equality in (5.163) implies equality of the incidence and refraction angle

Law of reflection

$$\sin\theta_0 = \sin\theta_1 \qquad \Rightarrow \qquad \boxed{\theta_0 = \theta_1} \tag{5.165}$$

which is known as *law of reflection*.

The second equality in known as *Snell's law* or *law of refraction*

Law of refraction

$$\boxed{n_1 \sin\theta_0 = n_2 \sin\theta_2.} \tag{5.166}$$

Snell's law determines the angle of refraction $\theta_2 = \arcsin(\frac{n_1}{n_2} \sin\theta_0)$ in dependence on the angle of incidence. Note that the solution θ_2 could not exist. This depends on the ratio of coefficients n_1/n_2. For $n_2 > n_1$ the solution θ_2 exists for any value of the angle of incidence θ_0. On the other hand, for $n_2 < n_1$ it is not true. There exists the critical value of the angle θ_0,

$$\boxed{\theta_c = \arcsin\frac{n_2}{n_1}} \tag{5.167}$$

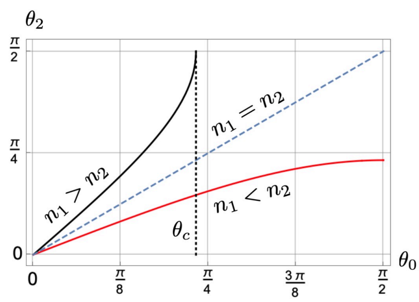


Figure 5.6: The refraction angle $\theta_2 = \arcsin(\frac{n_1}{n_2} \sin\theta_0)$ in dependence on the angle of incidence for $n_1 < n_2$ and $n_1 > n_2$.

such that $\cos\theta_2 = \sqrt{1-\left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_0}$ is not real number for $\theta_0 > \theta_c$. This subject is discussed in the further part. The relation between angles of incidence θ_0 and refraction θ_2 is shown in Fig.5.6.

Conditions for amplitudes

The amplitudes are solutions of the boundary conditions (5.155) and (5.156). Since $\boldsymbol{B} = n\,\hat{\boldsymbol{k}} \times \boldsymbol{E}$, or equivalently, $\boldsymbol{E} = -\frac{1}{n}\,\hat{\boldsymbol{k}} \times \boldsymbol{B}$ one gets that $\boldsymbol{E}$ and $\boldsymbol{H} = \frac{1}{\mu}\boldsymbol{B}$ satisfy relation

$$\boldsymbol{H} = \frac{1}{Z}\,\hat{\boldsymbol{k}} \times \boldsymbol{E}, \qquad \boldsymbol{E} = -Z\,\hat{\boldsymbol{k}} \times \boldsymbol{H}, \tag{5.168}$$

where $Z \equiv \frac{\mu}{n} = \sqrt{\frac{\mu}{\varepsilon}}$ is called *impedance*.[14] The field amplitudes are proportional $E = Z\,H$. It can be seen by taking square of equations (5.168).

[14] For instance, $\sqrt{\frac{\mu_0}{\varepsilon_0}} = 377\,\Omega$. Moreover, in most of dielectrics it is sufficient to approximate $\mu \approx \mu_0$.

The generic monochromatic incident wave can be decomposed into two components – perpendicular and parallel to the plane of incidence

$$\boldsymbol{E}_a = \boldsymbol{E}_{a\perp} + \boldsymbol{E}_{a\|}, \qquad \boldsymbol{H}_a = \boldsymbol{H}_{a\perp} + \boldsymbol{H}_{a\|}, \qquad a = 0, 1, 2. \tag{5.169}$$

In further sections we study incidence waves which has electric field perpendicular to the plane of incidence (s-polarized waves) or parallel to this plane (p-polarized waves). In the last case, the magnetic field is perpendicular to the plane of incidence. Such plane waves with linear polarizations are also called *transverse electric* (TE) and *transverse magnetic* (TM).

The ratios of amplitudes of reflected (and refracted) electric field by the amplitude of the incident field define *amplitude coefficients* of reflection and refraction. These coefficients were obtained by A. J. Fresnel in 1818. Fresnel analyzed oscillations of light in hypothetical luminiferous ether. The result was obtained before the final formulation of Maxwell's equations. Fresnel was the first who understood that light is a transverse wave.

Figure 5.7: Augustin-Jean Fresnel, 1788–1827.

Perpendicular electric field (s–polarization)

We shall consider the case of incident electric field $\boldsymbol{E}_0 = E_0\hat{\boldsymbol{y}}$ which is *perpendicular* to the plane of incidence (transverse electric field TE).[15] This is also true at the interface $z = 0$. The amplitudes $\boldsymbol{E}_1^0$ and $\boldsymbol{E}_2^0$ can be determined from boundary conditions at $z = 0$. Their direction is is not known beforehand. For simplicity, we shall *assume* that they have the same polarization as the incident field *i.e.* they are perpendicular to the plane of incidence: $\boldsymbol{E}_1 = E_1\hat{\boldsymbol{y}}$ and $\boldsymbol{E}_2 = E_2\hat{\boldsymbol{y}}$. The continuity conditions

[15] To simplify formulas, we shall omit the subscript $\perp$.

([5.155](#)) and ([5.156](#)) read

$$\hat{\boldsymbol{n}} \times \hat{\boldsymbol{y}} \, [E_0 + E_1] = \hat{\boldsymbol{n}} \times \hat{\boldsymbol{y}} \, E_2, \tag{5.170}$$

$$\underbrace{\hat{\boldsymbol{n}} \times (\hat{\boldsymbol{k}}_0 \times \hat{\boldsymbol{y}})}_{-(\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{k}}_0)\hat{\boldsymbol{y}}} E_0 + \underbrace{\hat{\boldsymbol{n}} \times (\hat{\boldsymbol{k}}_1 \times \hat{\boldsymbol{y}})}_{-(\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{k}}_1)\hat{\boldsymbol{y}}} E_1 = \frac{Z_1}{Z_2} \underbrace{\hat{\boldsymbol{n}} \times (\hat{\boldsymbol{k}}_2 \times \hat{\boldsymbol{y}})}_{-(\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{k}}_2)\hat{\boldsymbol{y}}} E_2, \tag{5.171}$$

where $\hat{\boldsymbol{n}} \times (\hat{\boldsymbol{k}}_a \times \hat{\boldsymbol{y}}) = (\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{y}})\hat{\boldsymbol{k}}_a - (\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{k}}_a)\hat{\boldsymbol{y}} = -(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{k}}_a)\hat{\boldsymbol{y}}$ because $\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{y}} = 0$. Note that $\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{k}}_1 = -\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{k}}_0$.

The conditions ([5.170](#)) and ([5.171](#)) takes the form

$$-E_1 + E_2 = E_0, \tag{5.172}$$

$$Z_2(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{k}}_0)E_1 + Z_1(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{k}}_2)E_2 = Z_2(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{k}}_0)E_0. \tag{5.173}$$

Solving equations ([5.172](#)) and ([5.173](#)) one gets *amplitude coefficients*

$$\frac{E_1}{E_0} = \frac{Z_2(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{k}}_0) - Z_1(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{k}}_2)}{Z_2(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{k}}_0) + Z_1(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{k}}_2)}, \tag{5.174}$$

$$\frac{E_2}{E_0} = \frac{2Z_2(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{k}}_0)}{Z_2(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{k}}_0) + Z_1(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{k}}_2)}. \tag{5.175}$$

The amplitude coefficients can be parametrized by angles of incidence θ_0 and refraction θ_2. The ratio of impedances reads

$$\frac{Z_1}{Z_2} = \frac{\mu_1}{\mu_2}\frac{n_2}{n_1} = \frac{\mu_1 \sin\theta_0}{\mu_2 \sin\theta_2}, \tag{5.176}$$

where the last equality is true for all angles of incidence θ_0 providing that $n_1 < n_2$. Otherwise, it holds only for θ_0 which is *smaller* than the critical angle θ_c. The value of this angle shall be determined in Section [5.5](#). All three wave vectors are real-valued providing that θ_2 is a solution. In such a case the vectors read

$$\hat{\boldsymbol{k}}_0 = -\sin\theta_0 \, \hat{\boldsymbol{x}} + \cos\theta_0 \, \hat{\boldsymbol{z}},$$

$$\hat{\boldsymbol{k}}_1 = -\sin\theta_0 \, \hat{\boldsymbol{x}} - \cos\theta_0 \, \hat{\boldsymbol{z}},$$

$$\hat{\boldsymbol{k}}_2 = -\sin\theta_2 \, \hat{\boldsymbol{x}} + \cos\theta_2 \, \hat{\boldsymbol{z}}.$$

The amplitude coefficients read

$$\frac{E_1}{E_0} = \frac{Z_2\cos\theta_0 - Z_1\cos\theta_2}{Z_2\cos\theta_0 + Z_1\cos\theta_2}, \tag{5.177}$$

$$\frac{E_2}{E_0} = \frac{2Z_2\cos\theta_0}{Z_2\cos\theta_0 + Z_1\cos\theta_2}, \tag{5.178}$$

or equivalently

$$\frac{E_1}{E_0} = \frac{\mu_2 \tan\theta_2 - \mu_1 \tan\theta_0}{\mu_2 \tan\theta_2 + \mu_1 \tan\theta_0} \overset{\mu_1=\mu_2}{=} \frac{\sin(\theta_2 - \theta_0)}{\sin(\theta_2 + \theta_0)}, \tag{5.179}$$

$$\frac{E_2}{E_0} = \frac{2\mu_2 \tan\theta_2}{\mu_2 \tan\theta_2 + \mu_1 \tan\theta_0} \overset{\mu_1=\mu_2}{=} \frac{2\cos\theta_0 \sin\theta_2}{\sin(\theta_2 + \theta_0)}. \tag{5.180}$$

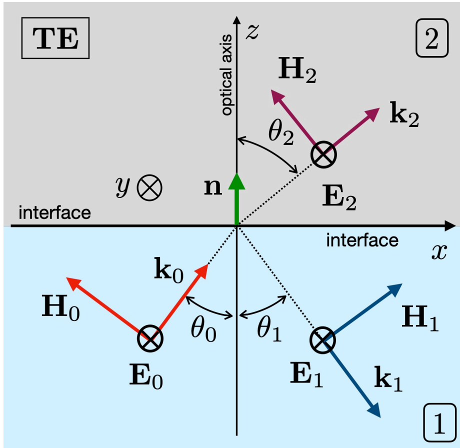


Figure 5.8: The electric field vector perpendicular to the plane of incidence.

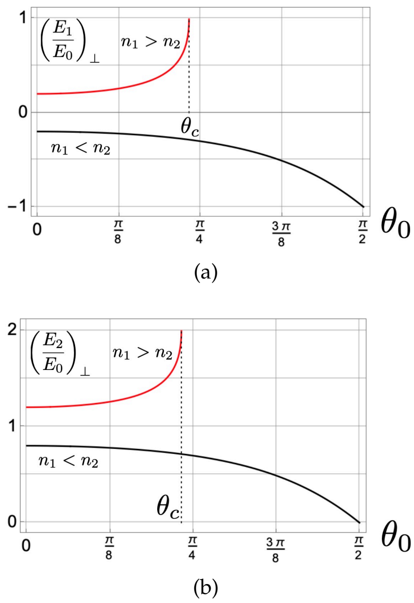


Figure 5.9: The ratios of amplitudes of electric field perpendicular to the plane of incidence for $\mu_1 = \mu_2 \approx \mu_0$.

Last equalities in (5.179) and (5.180) were obtained for $\mu_1 = \mu_2 \approx \mu_0$. Each amplitude coefficent E_1/E_0 and E_2/E_0 is considered for two cases: $n_1 < n_2$ and $n_1 > n_2$. Figure.5.9 shows these ratios in dependence on the angle of incidence for the case $\mu_1 = \mu_2$.

For $n_1 < n_2$ the ratio E_1/E_0 (at $z = 0$) is *negative* independently on the value of angle of incidence. It means that the electric field vector of the wave reflected at the interface separating the material media *changes its phase* by π, $e^{i\pi} = -1$. The absolute electric field amplitude $|E_2|$ is thus less than the amplitude $|E_0|$.

Both ratios satisfy the condition $E_2/E_0 - E_1/E_0 = 1$ which follows from (5.172). For the special case of *normal incidence*, $\theta_0 = 0$, $\theta_2 = 0$, the amplitude coefficients simplify to the form

$$\frac{E_1}{E_0} = \frac{\mu_2 n_1 - \mu_1 n_2}{\mu_2 n_1 + \mu_1 n_2}, \qquad \frac{E_2}{E_0} = \frac{2\mu_2 n_1}{\mu_2 n_1 + \mu_1 n_2}, \tag{5.181}$$

where first equality of (5.176) has been used.

For $n_1 > n_2$ the reflected wave and the incident wave have *equal phases*. However, the refracted wave exists only for $\theta_0 < \theta_c$. This phenomenon, called *total internal reflection*, is discussed in Section 5.5.

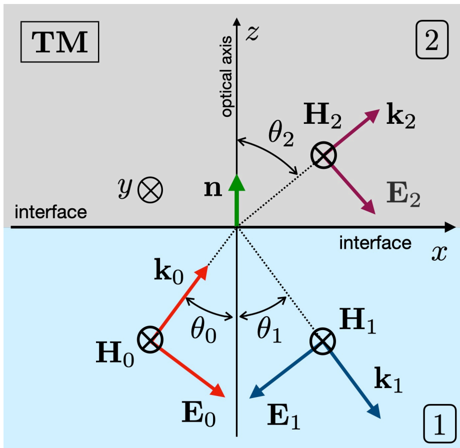


Figure 5.10: The magnetic field vector perpendicular to the plane of incidence.

Parallel electric field (p–polarization)

For the case of electric field parallel to the plane of incidence (p-polarization), the magnetic field is perpendicular to this plane (TM waves). We assume magnetic field in the form

$$\boldsymbol{H}_a = H_a \hat{\boldsymbol{y}}, \qquad a = 0, 1, 2.$$

The electric field is given by expression $\boldsymbol{E} = -Z\hat{\boldsymbol{k}} \times \boldsymbol{H}$. Plugging these expressions into continuity the conditions (5.155) and (5.156) one gets

$$\underbrace{\hat{\boldsymbol{n}} \times (\hat{\boldsymbol{k}}_0 \times \hat{\boldsymbol{y}})}_{-(\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{k}}_0)\hat{\boldsymbol{y}}} H_0 + \underbrace{\hat{\boldsymbol{n}} \times (\hat{\boldsymbol{k}}_1 \times \hat{\boldsymbol{y}})}_{-(\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{k}}_1)\hat{\boldsymbol{y}}} H_1 = \frac{Z_2}{Z_1} \underbrace{\hat{\boldsymbol{n}} \times (\hat{\boldsymbol{k}}_2 \times \hat{\boldsymbol{y}})}_{-(\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{k}}_2)\hat{\boldsymbol{y}}} H_2, \tag{5.182}$$

$$\hat{\boldsymbol{n}} \times \hat{\boldsymbol{y}}\,[H_0 + H_1] = \hat{\boldsymbol{n}} \times \hat{\boldsymbol{y}} H_2. \tag{5.183}$$

Taking into account that $\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{k}}_1 = -\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{k}}_0$ and plugging the amplitude $ZH = E$ one gets from (5.182) and (5.183)

$$(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{k}}_0)E_1 + (\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{k}}_2)E_2 = (\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{k}}_0)E_0. \tag{5.184}$$

$$-Z_2 E_1 + Z_1 E_2 = Z_2 E_0. \tag{5.185}$$

The solution of (5.184) and (5.185) gives amplitude coefficients

$$\frac{E_1}{E_0} = \frac{Z_1(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{k}}_0) - Z_2(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{k}}_2)}{Z_1(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{k}}_0) + Z_2(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{k}}_2)}, \tag{5.186}$$

$$\frac{E_2}{E_0} = \frac{2Z_2(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{k}}_0)}{Z_1(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{k}}_0) + Z_2(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{k}}_2)}. \tag{5.187}$$

The solutions (5.187) and (5.186) can be cast in the form

$$\frac{E_1}{E_0} = \frac{Z_1 \cos\theta_0 - Z_2 \cos\theta_2}{Z_1 \cos\theta_0 + Z_2 \cos\theta_2}, \qquad (5.188)$$

$$\frac{E_2}{E_0} = \frac{2Z_2 \cos\theta_0}{Z_1 \cos\theta_0 + Z_2 \cos\theta_2}. \qquad (5.189)$$

$$\frac{E_1}{E_0} = \frac{\mu_1 \sin(2\theta_0) - \mu_2 \sin(2\theta_2)}{\mu_1 \sin(2\theta_0) + \mu_2 \sin(2\theta_2)} \overset{\mu_1=\mu_2}{=} \frac{\tan(\theta_0 - \theta_2)}{\tan(\theta_0 + \theta_2)}, \qquad (5.190)$$

where

$$\frac{\sin(2\theta_0) - \sin(2\theta_2)}{\sin(2\theta_0) + \sin(2\theta_2)} = \frac{2\sin\left[\frac{2\theta_0 - 2\theta_2}{2}\right]\cos\left[\frac{2\theta_0+2\theta_2}{2}\right]}{2\sin\left[\frac{2\theta_0+2\theta_2}{2}\right]\cos\left[\frac{2\theta_0-2\theta_2}{2}\right]} = \frac{\tan(\theta_0 - \theta_2)}{\tan(\theta_0 + \theta_2)}.$$

Plugging $\frac{Z_1}{Z_2} = \frac{\mu_1}{\mu_2}\frac{\sin\theta_0}{\sin\theta_2}$ into (5.188) one gets

The second coefficient (5.189) takes the form

$$\frac{E_2}{E_0} = \frac{4\mu_2 \cos\theta_0 \sin\theta_2}{\mu_1 \sin(2\theta_0) + \mu_2 \sin(2\theta_2)} \overset{\mu_1=\mu_2}{=} \frac{2\cos\theta_0 \sin\theta_2}{\sin(\theta_0+\theta_2)\cos(\theta_0-\theta_2)}.$$

Figure 5.11 shows the amplitude coefficients for $\mu_1 = \mu_2 \approx \mu_0$ and for both cases of $n_1 < n_2$ and $n_1 > n_2$. For the case $n_1 < n_2$ the solutions exist for all angles of incidence θ_0 and for $n_1 > n_2$ they exist in a limited range of that angle.

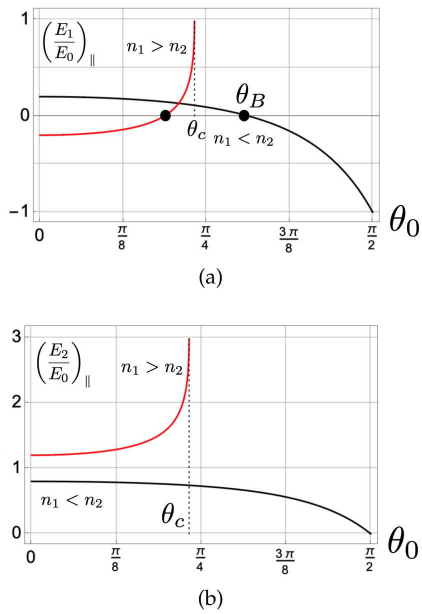


Figure 5.11: The amplitude coefficients for electric field parallel to the plane of incidence (TM waves) for $\mu_1 = \mu_2 \approx \mu_0$. There is no reflected wave for the Brewster angle θ_B.

For $n_1 < n_2$ the ratio E_1^0/E_0^0 is positive for $\theta_0 < \theta_B$ and it is negative for $\theta_0 > \theta_B$. The angle θ_B, called *Brewster's angle*, represents a very special value of the incidence angle which results in *absence of reflected wave*. According to the formula (5.190) (the case of equal magnetic permeabilities) an intensity of the reflected beam vanishes (the denominator tends to infinity) for

$$\theta_B + \theta_2 = \frac{\pi}{2}.$$

Applying Snell's law one gets

$$n_1 \sin\theta_B = n_2 \sin\theta_2 = n_2 \sin\left(\frac{\pi}{2} - \theta_B\right) = n_2 \cos\theta_B$$

which gives expression for Brewster's angle in terms of the ratio of refractive indices

$$\boxed{\tan\theta_B = \frac{n_2}{n_1}.} \qquad (5.191)$$

Getting polarized light

A general electromagnetic wave (for instance natural light beam) is a mixture components with different polarisations. If the incident light beam form Brewster's angle with normal to the interface then reflected beam necessarily does not contain light with p-polarization. Consequently, reflected beam is *polarized* – its electric field vector is perpendicular to the plane of incidence (s-polarization). This method is used to obtain polarized electromagnetic wave.

Total reflection

We shall present the formulas that involve θ_0 and θ_2 in the form such that they are sound for $n_1 > n_2$ and $\theta_0 > \theta_c$. The vectors $\boldsymbol{k}_a$ belong to the plane of incidence and thus they do not contain components in direction of $\hat{\boldsymbol{y}}$, $k_{ay} := 0$. The components normal to the interface are $k_{an} \equiv k_{az}$ whereas the tangent components are $k_{at} \equiv k_{ax}$ where

$$k_{ax} = \boldsymbol{k}_a \cdot \hat{\boldsymbol{x}} = k_a(\hat{\boldsymbol{k}}_a \cdot \hat{\boldsymbol{x}}) = k_a \cos(\pi/2 - \theta_a) = k_a \sin\theta_a$$

with $a = 0, 1, 2$. According to our previous results, the components k_{ax} are equal

$$\underbrace{k_0 \sin\theta_0}_{k_{0t}=k_{0x}} = \underbrace{k_1 \sin\theta_1}_{k_{1t}=k_{1x}} = \underbrace{k_2 \sin\theta_2}_{k_{2t}=k_{2x}}.$$

The reflected and incoming normal components have opposite values, $k_{1n} = -k_{0n}$. Taking into account that $k_{2n}^2 = k_2^2 - k_{2t}^2 = k_2^2 - k_{0t}^2$ one gets

$$k_{2t} = k_0 \sin\theta_0, \tag{5.192}$$

$$k_{2n}^2 = k_2^2 - k_0^2 \sin^2\theta_0 = k_2^2 \left[1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_0\right]. \tag{5.193}$$

These formulas are valid for *any angle* θ_0 and for any value of the ratio n_1/n_2. Plugging $k_{2t} = k_2 \sin\theta_2$ and $k_{2n} = k_2 \cos\theta_2$ into (5.192) and (5.193) and using (5.164) one gets

$$n_2 \sin\theta_2 = n_1 \sin\theta_0, \tag{5.194}$$

$$\cos^2\theta_2 = 1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_0. \tag{5.195}$$

The expression (5.194) is just Snell's law. Now, the point is that the angle θ_2 *may or may not have geometric meaning*. The formula (5.195) is used to determine this variable independently on its interpretation.

The amplitude coefficients are given by ratios of amplitude field components. Going back to to this subject we write down electric fields of the incident and transmitted wave

$$\begin{aligned}
\boldsymbol{E}_0 &= \boldsymbol{E}_0^0\, e^{i[\boldsymbol{k}_0\cdot\boldsymbol{r}-\omega t]} = \boldsymbol{E}_0^0\, e^{i[(k_{0x}x+k_{0z}z)-\omega t]} \\
&= \boldsymbol{E}_0^0\, e^{i[k_0(x\sin\theta_0+z\cos\theta_0)-\omega t]}, \\
\boldsymbol{E}_2 &= \boldsymbol{E}_2^0\, e^{i[\boldsymbol{k}_2\cdot\boldsymbol{r}-\omega t]} = \boldsymbol{E}_2^0\, e^{i[(k_{2x}x+k_{2z}z)-\omega t]} \\
&= \boldsymbol{E}_2^0\, e^{i[(k_0 x\sin\theta_0+k_{2z}z)-\omega t]},
\end{aligned}$$

where $\boldsymbol{r} = x\hat{\boldsymbol{x}} + z\hat{\boldsymbol{z}}$, $k_{2x} \equiv k_{2t} = k_{0t}$ and $k_{2z} \equiv k_{2n}$.

The main question is a problem of angles for $n_1 > n_2$ and $\theta_0 > \theta_c$. In such a case the right hand side of (5.195) is *negative* and consequently

$\cos^2\theta_2 < 0$. It shows that θ_2 has *no geometric meaning* for the considered case. Thus, we define

$$s^2 := -k_{2n}^2 = k_2^2\left[\left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_0 - 1\right] = k_2^2\left[\left(\frac{\sin\theta_0}{\sin\theta_c}\right)^2 - 1\right]. \quad (5.196)$$

We made use of the fact [16] that for $\theta_0 \to \theta_c$ the refraction angle tends to $\pi/2$. The Snell's law gives $n_1 \sin\theta_c = n_2$. The formula (5.196) gives $k_{2z} = \pm is$ where

[16] In the range $\theta_0 \leq \theta_c$

$$s = k_2\sqrt{\left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_0 - 1}. \quad (5.197)$$

Plugging k_{2z} into $\boldsymbol{E}_2$ one gets

$$\boldsymbol{E}_2 = \boldsymbol{E}_2^0 e^{-sz} e^{i[k_0 x \sin\theta_0 - \omega t]}, \quad (5.198)$$

where e^{+sz} is not allowed because it leads to unlimited grow of amplitude for $z \to \infty$ (what is not consistent with energy conservation).

Penetration depth

Penetration depth is defined as

$$\delta := \frac{1}{s} = \frac{1}{2\pi}\frac{\lambda_2}{\sqrt{\left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_0 - 1}} \quad (5.199)$$

where $k = 2\pi/\lambda$ with λ being wavelength. The formula (5.198) shows that the refracted electromagnetic wave penetrate the second medium, however, the penetration depth is very short – of the order of magnitude of wavelength λ_2. The refracted wave has phase $\psi = k_0 x \sin\theta_0 - \omega t$. Hence, its *phase velocity* reads

$$v_p = -\frac{\partial_t \psi}{|\nabla\psi|} = \frac{\omega}{k_0 \sin\theta_0} = \frac{c}{n_1 \sin\theta_0} = \frac{c/n_2}{(n_1/n_2)\sin\theta_0} = \left(\frac{\sin\theta_c}{\sin\theta_0}\right) v_2.$$

The expression $v_2 = c/n_2$ stands for phase velocity of *plane wave* in the medium with refractive index n_2. Since the phase velocity v_p is smaller than v_2, ($\theta_c < \theta_0$), then we can conclude that (5.198) does not represent electric field of plane waves. In other words, the wave is *not plane*. Its amplitude and phase are not constant on planes perpendicular to the wave vector $\boldsymbol{k}_2$. The phase of this wave clearly depends on properties of both media, $\theta_c = \arcsin\frac{n_2}{n_1}$, and the angle of incidence θ_0.

Replacing $\cos\theta_2$ in Fresnel formulas (5.177) and (5.188) by

$$\cos\theta_2 = \sqrt{1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_0} = \frac{is}{k_2}$$

one gets

$$\left(\frac{E_1}{E_0}\right)_\perp = \frac{Z_2\cos\theta_0 - iZ_1(s/k_2)}{Z_2\cos\theta_0 + iZ_1(s/k_2)} = e^{2i\varphi_\perp}, \quad (5.200)$$

$$\left(\frac{E_1}{E_0}\right)_\parallel = \frac{Z_1\cos\theta_0 - iZ_2(s/k_2)}{Z_1\cos\theta_0 + iZ_2(s/k_2)} = e^{2i\varphi_\parallel}. \quad (5.201)$$

Parameters $\varphi_\perp$ and $\varphi_\parallel$ are defined as follows

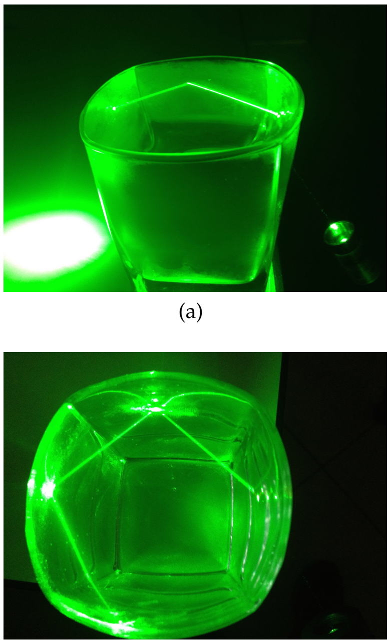
$$\tan\varphi_\perp := -\frac{Z_1(s/k_2)}{Z_2\cos\theta_0}, \qquad \tan\varphi_\parallel := -\frac{Z_2(s/k_2)}{Z_1\cos\theta_0}, \qquad \frac{Z_1}{Z_2} \equiv \frac{\mu_1}{\mu_2}\frac{n_2}{n_1}.$$

In the case of first formula we get

$$\frac{1+i\left(-\frac{Z_1(s/k_2)}{Z_2\cos\theta_0}\right)}{1-i\left(-\frac{Z_1(s/k_2)}{Z_2\cos\theta_0}\right)} = \frac{1+i\tan\varphi_\perp}{1-i\tan\varphi_\perp} = \frac{\cos\varphi_\perp + i\sin\varphi_\perp}{\cos\varphi_\perp - i\sin\varphi_\perp} = \frac{e^{i\varphi_\perp}}{e^{-i\varphi\perp}} = e^{2i\varphi_\perp}$$

and similarly for $\varphi_\parallel$. Since ratios of the amplitudes have absolute values $|e^{2i\varphi_\perp}| = 1$ and $|e^{2i\varphi_\parallel}| = 1$ then magnitudes of reflected and incident wavs are equal (only their phases are different).

It means that the energy flux of reflected wave is equal to energy flux of incident wave. Consequently, there is no energy transfer to the other medium. This is the reason why this phenomenon is called *total reflection*. The existence of penetration depth does not contradict the above result that whole wave is reflected. Experiments show that there exists small region in the second medium such that the refracted wave propagates parallel to the surface of interface. Finally, the wave goes back to the first medium. Total reflection is responsible for formation of *mirages*.



(a)

(b)

Figure 5.12: Total internal reflection; (a) water–air, (b) glass–air.

Energy fluxes of reflected and refracted wave

The energy flux is represented by time average of Poynting vector given by

$$\langle \boldsymbol{S}\rangle = \frac{c}{8\pi\mu} Re\left[\boldsymbol{E}\times\boldsymbol{B}^*\right] = \frac{c}{8\pi\, Z}|\boldsymbol{E}|^2\hat{\boldsymbol{k}}. \tag{5.202}$$

The normal component of this vector is projection of $\langle \boldsymbol{S}\rangle$ onto a versor $\hat{\boldsymbol{n}}$ normal to the interface, $\langle \boldsymbol{S}\rangle_n = \langle \boldsymbol{S}\rangle \cdot \hat{\boldsymbol{n}}$. Its tangent component is given by $\langle \boldsymbol{S}\rangle - \langle \boldsymbol{S}\rangle_n\, \hat{\boldsymbol{n}}$.

Reflection coefficient

Reflection coefficient is given by the ratio

$$R := \frac{|\langle \boldsymbol{S}_1\rangle\cdot\hat{\boldsymbol{n}}|}{|\langle \boldsymbol{S}_0\rangle\cdot\hat{\boldsymbol{n}}|} = \frac{|\boldsymbol{E}_1|^2}{|\boldsymbol{E}_0|^2}, \tag{5.203}$$

where $\hat{\boldsymbol{k}}_0\cdot\hat{\boldsymbol{n}} = \hat{\boldsymbol{k}}_1\cdot\hat{\boldsymbol{n}}$ since $\theta_1 = \theta_0$. The reflection coefficient R represents total flux of energy reflected on the interface. This flux can be split into two parts perpendicular and parallel to the plane of incidence. Squares of the amplitude coefficients (5.177) and (5.188) give reflection coefficients $R_\perp$ (for s-polarized component) and $R_\parallel$ (for p-polarized component)

$$R_\perp = \left(\frac{E_1}{E_0}\right)^2_\perp = \left[\frac{Z_2\cos\theta_0 - Z_1\cos\theta_2}{Z_2\cos\theta_0 + Z_1\cos\theta_2}\right]^2,$$

$$R_\parallel = \left(\frac{E_1}{E_0}\right)^2_\parallel = \left[\frac{Z_1\cos\theta_0 - Z_2\cos\theta_2}{Z_1\cos\theta_0 + Z_2\cos\theta_2}\right]^2.$$

In the case of normal incidence $\theta_0 = 0$ and for $\mu_1 = \mu_2 \approx \mu_0$ the reflection coefficient has the form

Reflection coefficient for normal incidence

$$R_n = \left(\frac{n_1 - n_2}{n_1 + n_2} \right)^2. \tag{5.204}$$

Transmission coefficient is defined as follows

Transmission coefficient

$$T := \frac{|\langle \boldsymbol{S}_2 \rangle \cdot \hat{\boldsymbol{n}}|}{|\langle \boldsymbol{S}_0 \rangle \cdot \hat{\boldsymbol{n}}|} = \frac{Z_1 \cos\theta_2}{Z_2 \cos\theta_0} \frac{|\boldsymbol{E}_2|^2}{|\boldsymbol{E}_0|^2}. \tag{5.205}$$

The perpendicular and parallel transmission coefficients are proportional to squares of the amplitude coefficients (5.178) and (5.189) and they read

$$T_\perp = \frac{Z_1 \cos\theta_2}{Z_2 \cos\theta_0} \left(\frac{E_2}{E_0} \right)_\perp^2 = \frac{4 Z_1 Z_2 \cos\theta_0 \cos\theta_2}{(Z_2 \cos\theta_0 + Z_1 \cos\theta_2)^2},$$
$$T_\parallel = \frac{Z_1 \cos\theta_2}{Z_2 \cos\theta_0} \left(\frac{E_2}{E_0} \right)_\parallel^2 = \frac{4 Z_1 Z_2 \cos\theta_0 \cos\theta_2}{(Z_1 \cos\theta_0 + Z_2 \cos\theta_2)^2}.$$

For normal incidence and for $\mu_1 = \mu_2$ one gets

$$T_n = \frac{4 n_1 n_2}{(n_2 + n_1)^2}. \tag{5.206}$$

Let us observe that $R_\perp + T_\perp = 1$ and similarly $R_\parallel + T_\parallel = 1$. The energy conservation requires that it must hold for total coefficients

$$\boxed{R + T = 1.} \tag{5.207}$$

Reflection and transmission coefficients are shown in Figure 5.13. All coefficient belong to the interval $[0, 1]$. In the case $n_1 > n_2$ the reflection coefficient is equal to unity and the transmission coefficient is equal to zero for $\theta_0 \geq \theta_c$.

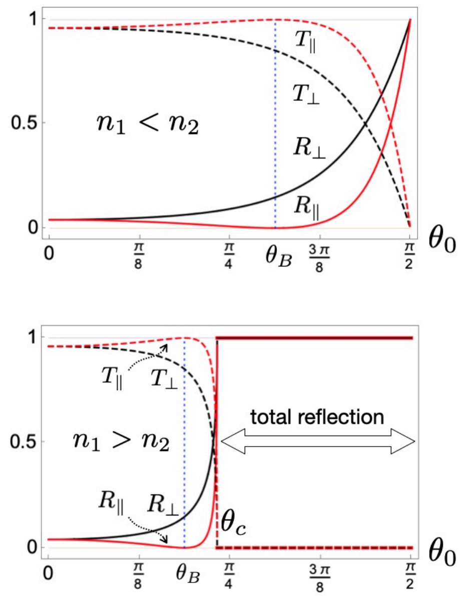


Figure 5.13: Reflection and transmission coefficients.

5.6 *Electromagnetic waves in conductors*

Constitutive relations in conducting media contain Ohm's law which relates the current density $\boldsymbol{J}$ with the electric field $\boldsymbol{E}$ which exists inside the material. Ohm is a local relation between these two quantities. We shall study the case of linear and non-perfect conductors (with dielectric and magnetic properties) characterized by

$$\boldsymbol{D} = \varepsilon \boldsymbol{E}, \qquad \boldsymbol{B} = \mu \boldsymbol{H}, \qquad \boldsymbol{J} = \sigma \boldsymbol{E}, \qquad \rho = 0, \tag{5.208}$$

where σ stands for electric conductivity and ρ for density of free electric charges. A natural extension of a single conductor is a material containing many domains made of different type of conductors separated by interfaces. We shall consider the homogeneous and non-dispersive medium, $\varepsilon = const$, $\mu = const$. Electric and magnetic fields in such a medium satisfy Maxwell's equations

$$\nabla \cdot \boldsymbol{E} = 0, \qquad \nabla \times \boldsymbol{B} - \frac{\varepsilon\mu}{c}\partial_t \boldsymbol{E} - \frac{4\pi}{c}\sigma\mu\boldsymbol{E} = 0 \tag{5.209}$$

$$\nabla \cdot \boldsymbol{B} = 0, \qquad \nabla \times \boldsymbol{E} + \frac{1}{c}\partial_t \boldsymbol{B} = 0 \tag{5.210}$$

where (5.208) have been used.

Relaxation time

We shall study propagation of plane electromagnetic waves in non-perfect conductor. Without loss of generality we can choose the $\hat{\boldsymbol{e}}_3$ Cartesian versor as being aligned with direction in which the wave propagates. It has been shown that electromagnetic fields are transversal in empty space. Now we shall look again at this problem. In order to get a general solution we *shall not assume* the perpendicularity of wave vector and the fields. First of all, we observe that for electric field which depends on t and spatial component x^3 one gets

$$\begin{aligned} \nabla \cdot \boldsymbol{E} &= \hat{\boldsymbol{e}}_i \cdot (\partial_i \boldsymbol{E}) = \hat{\boldsymbol{e}}_3 \cdot (\partial_3 \boldsymbol{E}), \\ \nabla \times \boldsymbol{E} &= \hat{\boldsymbol{e}}_i \times (\partial_i \boldsymbol{E}) = \hat{\boldsymbol{e}}_3 \times (\partial_3 \boldsymbol{E}). \end{aligned}$$

Thus, electric and magnetic Gauss' law reads

$$\hat{\boldsymbol{e}}_3 \cdot (\partial_3 \boldsymbol{E}) = 0, \qquad \hat{\boldsymbol{e}}_3 \cdot (\partial_3 \boldsymbol{B}) = 0, \tag{5.211}$$

whereas Ampère-Maxwell law and Faraday's law have the form

$$\hat{\boldsymbol{e}}_3 \times (\partial_3 \boldsymbol{B}) - \frac{\varepsilon\mu}{c}\partial_t \boldsymbol{E} = \frac{4\pi}{c}\sigma\mu\boldsymbol{E}, \tag{5.212}$$

$$\hat{\boldsymbol{e}}_3 \times (\partial_3 \boldsymbol{E}) + \frac{1}{c}\partial_t \boldsymbol{B} = 0. \tag{5.213}$$

Taking a scalar product of $\hat{\boldsymbol{e}}_3$ and (5.212) one gets

$$\hat{\boldsymbol{e}}_3 \cdot \partial_t \boldsymbol{E} = -\frac{4\pi\sigma}{\varepsilon}\hat{\boldsymbol{e}}_3 \cdot \boldsymbol{E} \tag{5.214}$$

Taking sum of (5.214) multiplied by dt and (5.211) multiplied by dx^3 one gets equation

$$\hat{\boldsymbol{e}}_3 \cdot \underbrace{\left[\partial_t \boldsymbol{E} dt + \partial_3 \boldsymbol{E} dx^3\right]}_{d\boldsymbol{E}} = -\frac{4\pi\sigma}{\varepsilon}\hat{\boldsymbol{e}}_3 \cdot \boldsymbol{E} dt$$

which can be cast in the form

$$\hat{\boldsymbol{e}}_3 \cdot \left[\frac{d\boldsymbol{E}}{dt} + \frac{4\pi\sigma}{\varepsilon}\boldsymbol{E}\right] = 0. \tag{5.215}$$

In fact, solution of(5.215) represents *longitudinal component* of the electric field,$E_\parallel := \hat{\boldsymbol{e}}_3 \cdot \boldsymbol{E}$. In a similar way, we get from other two Maxwell's equations $\hat{\boldsymbol{e}}_3 \cdot \frac{d\boldsymbol{B}}{dt} = 0$. Thus, one gets

$$\frac{dE_\parallel}{dt} + \frac{1}{\tau}E_\parallel = 0, \qquad \frac{dB_\parallel}{dt} = 0, \qquad \tau := \frac{\varepsilon}{4\pi\sigma} \tag{5.216}$$

where *relaxation time* is denoted by τ. First of equations (5.216) has solution

$$E_\parallel(t, x^3) = E_\parallel(0, x^3)e^{-\frac{t}{\tau}}. \tag{5.217}$$

This solution shows that there could exist time dependent longitudinal component of the electric field in *non-perfect conductors*. This component decreases with time in exponential manner, $e^{-\frac{t}{\tau}}$. On the other hand, magnetic longitudinal component being solution of Maxwell's equations is *static*, $B_\parallel = const$. Any electromagnetic wave has both, electric and magnetic fields which depend on time. Thus the longitudinal component of electric field cannot be related with any wave and therefore all electromagnetic waves in conductors are *transversal*.

Dispersion relation in conductors

In similarity to the case of non-conductors one can derive the second-order equations for electric and magnetic fields. Taking rotational of Ampere's-Maxwell equation and using remaining equations one can get equation for magnetic field. Similarly, acting with rotational on Faraday's law one gets equation for electric field. The resulting equations have the form

$$\mathcal{L}\boldsymbol{E} = 0, \qquad \mathcal{L}\boldsymbol{B} = 0, \tag{5.218}$$

where linear differential operator $\mathcal{L}$ reads

$$\mathcal{L} := \frac{\varepsilon\mu}{c^2}\partial_t^2 + 4\pi\frac{\sigma\mu}{c^2}\partial_t - \nabla^2. \tag{5.219}$$

The first order differential operator $4\pi\frac{\sigma\mu}{c^2}\partial_t$ plays the role of dissipative term (like in diffusion equation). For $\sigma \to 0$ the operator $\mathcal{L}$ is reduced to the d'Alembert (wave) operator. We assume the following general form of solutions describing electromagnetic plane wave in conducting media

$$\boldsymbol{E} = \boldsymbol{E}_0 e^{i(\boldsymbol{\kappa}\cdot\boldsymbol{r} - \omega t)}, \qquad \boldsymbol{B} = \boldsymbol{B}_0 e^{i(\boldsymbol{\kappa}\cdot\boldsymbol{r} - \omega t)}. \tag{5.220}$$

The equation (5.218) gives condition for $\boldsymbol{\kappa}$. There is only a single condition for both the electric and magnetic parts, namely $\mathcal{L}e^{i(\boldsymbol{\kappa}\cdot\boldsymbol{r} - \omega t)} = 0$. It gives the following algebraic equation

$$\left[-\frac{\varepsilon\mu}{c^2}\omega^2 - i\frac{4\pi\sigma\mu}{c^2}\omega + \kappa^2\right] e^{i(\boldsymbol{\kappa}\cdot\boldsymbol{r} - \omega t)} = 0$$

which must be satisfied for any t and $\boldsymbol{r}$. It leads to *dispersion relation*

$$\boxed{\kappa^2 = \varepsilon\mu\frac{\omega^2}{c^2}\left[1 + i\frac{4\pi\sigma}{\varepsilon\omega}\right].} \tag{5.221}$$

Note, that right hand side of (5.221) is complex. It means that $\boldsymbol{\kappa}$ must be some complex-valued vector, *i.e.* $\boldsymbol{\kappa}^2 = \kappa^2 \in \mathbb{C}$. The complex number κ can be represented by its real and imaginary parts

$$\kappa = k + is, \qquad k, s \in \mathbb{R}.$$

Plugging this expression into (5.221) and comparing real and imaginary parts of both sides one gets

$$k^2 - s^2 = \varepsilon\mu\frac{\omega^2}{c^2}, \qquad ks = 2\pi\sigma\mu\frac{\omega}{c^2}. \tag{5.222}$$

Combining equations (5.222) one eliminates variables s or k

$$k^4 - \frac{\varepsilon\mu\,\omega^2}{c^2}k^2 - \left(\frac{2\pi\sigma\mu\,\omega}{c^2}\right)^2 = 0, \tag{5.223}$$

$$s^4 + \frac{\varepsilon\mu\,\omega^2}{c^2}s^2 - \left(\frac{2\pi\sigma\mu\,\omega}{c^2}\right)^2 = 0. \tag{5.224}$$

Both these equations are quadratic in variables k^2, s^2 and they have equal discriminants

$$\Delta = \left(\frac{\varepsilon\mu\omega^2}{c^2}\right)^2\left[1 + \left(\frac{4\pi\sigma}{\varepsilon\omega}\right)^2\right]. \tag{5.225}$$

Physically sound solutions are of the form

$$k^2 = \frac{\varepsilon\mu\,\omega^2}{2c^2}\left[\sqrt{1 + \left(\frac{4\pi\sigma}{\varepsilon\omega}\right)^2} + 1\right], \tag{5.226}$$

$$s^2 = \frac{\varepsilon\mu\,\omega^2}{2c^2}\left[\sqrt{1 + \left(\frac{4\pi\sigma}{\varepsilon\omega}\right)^2} - 1\right].$$

Terms with $-\sqrt{\Delta}$ were discarded because they would give $k^2 < 0$, $s^2 < 0$ and consequently the variables k and s would be imaginary. Thus, the final form of the solution reads

$$k = \frac{n}{\sqrt{2}}\left(\frac{\omega}{c}\right)\left[\sqrt{1 + \left(\frac{4\pi\sigma}{\varepsilon\omega}\right)^2} + 1\right]^{\frac{1}{2}} \tag{5.227}$$

$$s = \frac{n}{\sqrt{2}}\left(\frac{\omega}{c}\right)\left[\sqrt{1 + \left(\frac{4\pi\sigma}{\varepsilon\omega}\right)^2} - 1\right]^{\frac{1}{2}}. \tag{5.228}$$

Relation between amplitudes

The field (5.220) where $\boldsymbol{\kappa}$ is a solution of (5.221) solve equations (5.218). However, the electromagnetic wave must satisfy not only the wave-like equation but a complete set of Maxwell's equations. Plugging the ansatz (5.220) into Maxwell's equations we get algebraic conditions for amplitudes $\boldsymbol{E}_0$ and $\boldsymbol{B}_0$. Thus, we consider an electromagnetic wave which propagates along the axis x^3. Taking the wave vector in the form $\boldsymbol{\kappa} = (k + is)\hat{\boldsymbol{e}}_3$ one gets

$$\boldsymbol{E} = \boldsymbol{E}_0 e^{-s\,x^3} e^{i(k\,x^3 - \omega t)} \tag{5.229}$$

$$\boldsymbol{B} = \boldsymbol{B}_0 e^{-s\,x^3} e^{i(k\,x^3 - \omega t)}. \tag{5.230}$$

Electric and magnetic fields depend on spatial coordinate x^3 and time t. Faraday's law $\hat{\boldsymbol{e}}_3 \times \partial_3 \boldsymbol{E} + \frac{1}{c}\partial_t \boldsymbol{B} = 0$, where $\partial_3 \boldsymbol{E} = i(k + is)\boldsymbol{E}$ and $\partial_t \boldsymbol{B} = -i\omega \boldsymbol{B}$ results in

$$\boldsymbol{B}_0 = \frac{c}{\omega}(k + is)\,\hat{\boldsymbol{e}}_3 \times \boldsymbol{E}_0. \tag{5.231}$$

where complex number κ is parametrized by its amplitude $|\kappa|$ and phase ϕ,

$$\kappa = |\kappa| e^{i\phi}, \qquad |\kappa| = \sqrt{k^2 + s^2}, \qquad \phi = \arctan\left(\frac{s}{k}\right). \tag{5.232}$$

Plugging (5.227) and (5.228) into these formulas one gets

$$|\kappa| = \frac{n\,\omega}{c}\left[1 + \left(\frac{4\pi\sigma}{\varepsilon\,\omega}\right)^2\right]^{\frac{1}{4}}, \tag{5.233}$$

$$\phi = \frac{1}{2}\arctan\left(\frac{4\pi\sigma}{\varepsilon\,\omega}\right). \tag{5.234}$$

Proof. We consider expression

$$\tan\phi = \frac{s}{k} = \left[\frac{\sqrt{u} - 1}{\sqrt{u} + 1}\right]^{\frac{1}{2}}, \qquad u \equiv 1 + \left(\frac{4\pi\sigma}{\varepsilon\,\omega}\right)^2. \tag{5.235}$$

Then, applying some trigonometric identities we get

$$\begin{aligned}
\tan(2\phi) &= \frac{\sin(2\phi)}{\cos(2\phi)} = \frac{2\sin\phi\cos\phi}{\cos^2\phi - \sin^2\phi} = \frac{2\tan\phi}{1 - \tan^2\phi} \\
&= \frac{2\frac{s}{k}}{1 - \left(\frac{s}{k}\right)^2} = 2\frac{\left[\frac{\sqrt{u}-1}{\sqrt{u}+1}\right]^{\frac{1}{2}}}{1 - \frac{\sqrt{u}-1}{\sqrt{u}+1}} = 2\frac{\left[\frac{\sqrt{u}-1}{\sqrt{u}+1}\right]^{\frac{1}{2}}}{\frac{\sqrt{u}+1-\sqrt{u}+1}{\sqrt{u}+1}} \\
&= (\sqrt{u} + 1)\left[\frac{\sqrt{u} - 1}{\sqrt{u} + 1}\right]^{\frac{1}{2}} \\
&= [(\sqrt{u})^2 - 1]^{1/2} = \sqrt{u - 1} = \frac{4\pi\sigma}{\varepsilon\,\omega}.
\end{aligned}$$

Finally, we obtain relation between complex electric and magnetic field amplitudes

$$\boldsymbol{B}_0 = n \left[1 + \left(\frac{4\pi\sigma}{\varepsilon\,\omega}\right)^2\right]^{\frac{1}{4}} e^{i\phi}\, \hat{\boldsymbol{e}}_3 \times \boldsymbol{E}_0. \tag{5.236}$$

The factor $e^{i\phi}$ is responsible for *time lag* associated with relative phase shift for fields $\boldsymbol{B}$ and $\boldsymbol{E}$. Since $\hat{\boldsymbol{e}}_3 \cdot \boldsymbol{E}_0 = 0 = \hat{\boldsymbol{e}}_3 \cdot \boldsymbol{B}_0$ then taking square root of a square of (5.236) one gets

Time lag

$$|\boldsymbol{B}_0| = n \left[1 + \left(\frac{4\pi\sigma}{\varepsilon\,\omega}\right)^2\right]^{\frac{1}{4}} |\boldsymbol{E}_0|. \tag{5.237}$$

Limit cases

In this section we study the equation

$$\frac{\varepsilon\mu}{c^2}\partial_t^2 \boldsymbol{E} + 4\pi\frac{\sigma\mu}{c^2}\partial_t \boldsymbol{E} - \nabla^2 \boldsymbol{E} = 0$$

looking at some characteristic limit cases. The term $\frac{\varepsilon\mu}{c^2}\partial_t^2 \boldsymbol{E}$ which is second order in time derivatives originates in *displacement current* whereas the term $4\pi\frac{\sigma\mu}{c^2}\partial_t \boldsymbol{E}$ originates in *conduction current*. Bothe these terms contribute to the dispersion relation through expressions

$$\frac{\varepsilon\mu}{c^2}\partial_t^2 \boldsymbol{E} \to -\frac{\varepsilon\,\mu}{c^2}\omega^2, \qquad 4\pi\frac{\sigma\mu}{c^2}\partial_t \boldsymbol{E} \to -i\frac{4\pi\sigma\,\mu}{c^2}\omega.$$

The ratio of absolute values of these two terms determines the phase shift and it reads

$$\frac{\frac{4\pi\sigma\,\mu}{c^2}\omega}{\frac{\varepsilon\,\mu}{c^2}\omega^2} = \frac{4\pi\sigma}{\varepsilon\,\omega} = \tan(2\phi). \tag{5.238}$$

- **Case** $\frac{4\pi\sigma}{\varepsilon\,\omega} \ll 1$

  In this case the effect of conduction current is small comparing with the effect of displacement current. Expanding k in powers of $\frac{4\pi\sigma}{\varepsilon\,\omega}$ one gets

$$\begin{aligned} k =& \frac{n}{\sqrt{2}}\left(\frac{\omega}{c}\right)\left[\sqrt{1 + \left(\frac{4\pi\sigma}{\varepsilon\omega}\right)^2} + 1\right]^{\frac{1}{2}} \\ =& \frac{n}{\sqrt{2}}\left(\frac{\omega}{c}\right)\left[1 + \frac{1}{2}\left(\frac{4\pi\sigma}{\varepsilon\omega}\right)^2 + \ldots + 1\right]^{\frac{1}{2}} \\ =& n\frac{\omega}{c}\left[1 + \frac{1}{4}\left(\frac{4\pi\sigma}{\varepsilon\omega}\right)^2 + \ldots\right]^{\frac{1}{2}} \\ =& n\frac{\omega}{c}\left[1 + \frac{1}{2}\left(\frac{2\pi\sigma}{\varepsilon\omega}\right)^2\right] + \ldots \end{aligned} \tag{5.239}$$

and similarly for s,

$$\begin{aligned}
s =& \frac{n}{\sqrt{2}}\left(\frac{\omega}{c}\right)\left[\sqrt{1+\left(\frac{4\pi\sigma}{\varepsilon\omega}\right)^2}-1\right]^{\frac{1}{2}} \\
=& \frac{n}{\sqrt{2}}\frac{\omega}{c}\left[1+\frac{1}{2}\left(\frac{4\pi\sigma}{\varepsilon\omega}\right)^2+\ldots+(-1)\right]^{\frac{1}{2}} \\
=& \sqrt{\frac{\varepsilon\mu}{2}}\frac{\omega}{c}\frac{1}{\sqrt{2}}\frac{4\pi\sigma}{\varepsilon\,\omega}+\ldots \\
=& \frac{2\pi\sigma}{c}\sqrt{\frac{\mu}{\varepsilon}}+\ldots, \qquad (5.240)
\end{aligned}$$

where $Z := \sqrt{\frac{\mu}{\varepsilon}}$. Notice that the imaginary part $s = Im(\kappa)$ does not depend on the frequency ω. For $\sigma \approx 0$ the phase shift is very small, $\phi \approx 0$. Consequently, the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ have approximately the same phase.

- **Case** $\frac{4\pi\sigma}{\varepsilon\omega} \gg 1$

In this limit the conduction current dominates. This is the case of metals where $\sigma/\varepsilon \approx 10^{18}$. The conduction current dominates for frequencies lower than 10^{17} Hz (microwaves, radio-frequency, light, some range of X-ray). It means that the real and imaginary parts are almost equal,

$$k \approx s \approx \frac{1}{c}\sqrt{2\pi\sigma\,\mu\,\omega}.$$

It gives

$$|\kappa| = \sqrt{k^2+s^2} \approx \frac{1}{c}\sqrt{4\pi\sigma\,\mu\,\omega} \qquad (5.241)$$

$$\phi = \arctan\left(\frac{s}{k}\right) \approx \arctan(1) = \frac{\pi}{4}. \qquad (5.242)$$

The relative phase shift of fields takes value $\phi = \pi/4$ whereas absolute values of their amplitudes satisfy the proportionality relation

$$|\boldsymbol{B}_0| \approx n\sqrt{\frac{4\pi\sigma}{\varepsilon\,\omega}}|\boldsymbol{E}_0|, \qquad (5.243)$$

where n takes values of the order of unity. Since $\sqrt{\frac{4\pi\sigma}{\varepsilon\,\omega}} \gg 1$ then $|\boldsymbol{B}_0| \gg |\boldsymbol{E}_0|$.

Distribution of electric current in conductors

The term $\partial_t\boldsymbol{D}$, that represents the displacement term, is irrelevant for good conductors. In such a case the equation

$$\nabla^2\boldsymbol{E} - \frac{4\pi\sigma\mu}{c^2}\partial_t\boldsymbol{E} = 0$$

is equivalent to the following one

$$\nabla^2 \boldsymbol{J} - \frac{4\pi\sigma\mu}{c^2}\partial_t \boldsymbol{J} = 0, \tag{5.244}$$

where $\boldsymbol{J} = \sigma\boldsymbol{E}$. (5.244) is a *diffusion equation*. Since the electric field depends on time through $e^{-i\omega t}$, then the current $\boldsymbol{J}$ one can expect that the current density is proportional to this exponential function, namely $\boldsymbol{J} = \boldsymbol{J}_0(\boldsymbol{r})e^{-i\omega t}$. Plugging this expression into (5.244) we get[17]

[17] In SI units $\delta = \sqrt{\frac{2}{\sigma\mu\omega}}$.

$$\nabla^2 \boldsymbol{J}_0 + \tau^2 \boldsymbol{J}_0 = 0, \qquad \tau^2 := i\frac{4\pi\sigma\mu\omega}{c^2}. \tag{5.245}$$

Considering that

$$\sqrt{i} = \frac{1}{\sqrt{2}}(1+i) \qquad \text{and} \qquad \frac{4\pi\sigma}{\varepsilon\,\omega} \gg 1$$

we get

$$\tau = \frac{1+i}{\delta}, \qquad \delta = \frac{c}{\sqrt{2\pi\sigma\mu\,\omega}} \approx \frac{1}{s}.$$

We shall look at the example of a conductor which occupies the region $x^1 \geq 0$. The border $x^1 \equiv x = 0$ is the interface that separates conducting material and empty space. The electromagnetic wave propagates along the axis $x^1 \equiv x$. This case corresponds with normal incidence of the wave. We assume that the wave is linearly polarized in direction of the x^3 axes and then $\boldsymbol{J}_0(\boldsymbol{r}) = J_0^3(x)\hat{\boldsymbol{e}}_3$. The resulting equation

$$\left[\frac{d^2}{d(x)^2} + \tau^2\right] J_0^3(x) = 0 \tag{5.246}$$

has the solution

$$J_0^3(x) = J_0^3(0)e^{i\tau\,x} = J_0^3(0)e^{-\frac{x}{\delta}}e^{i\frac{x}{\delta}}. \tag{5.247}$$

We have to reject the second solution

$$e^{-i\tau\,x} = e^{\frac{x}{\delta}}e^{-i\frac{x}{\delta}}$$

because it leads to non-physical behaviour in the conductor $x \geq 0$. The electric current density vanishes exponentially inside the conductor. The characteristic length δ is called *skin depth*.

In the case of silver, $\sigma^{-1} = 1.58 * 10^{-8}\Omega$m, and wave frequency $\nu = 4 * 10^9$ Hz which skin depth has value $\delta = \frac{1}{\sqrt{\pi\mu\sigma\nu}} = 10^{-6}$m whereas for copper, $\sigma^{-1} = 1.68 * 10^{-8}\Omega$m, and UV light with $\nu = 10^{15}$ Hz it reads $\delta = 10^{-9}$m.

5.7 Electromagnetic waves in dispersive dielectric media

Dispersive media

The dependence of dielectric permittivity ε and magnetic permeability μ on the electromagnetic wave frequency ω is called *dispersion*. Many dielectric media, like for instance water, are characterized by permittivity ε which is constant for slowly varying electromagnetic field and varies (differently for different substances) for higher electromagnetic field frequencies.

The study of high-frequency electromagnetic field in *polarizable* media is a very interesting subject. The electromagnetic field which is periodic in time is also periodic in space. Spatial oscillations are characterized by wavelength $\lambda \sim \frac{c}{\omega}$, where ω is the frequency of the field. The wavelength is getting shorter as frequency ω is increasing. For sufficiently high frequencies the corresponding wavelength is comparable with typical size of atoms represented by *atomic scale*.[18] The macroscopic description of the electromagnetic field is meaningless for $\lambda \sim a$. Thus, the macroscopic regime for electromagnetic waves is charcterized by wavelengths $\lambda \gg a$. For this reason we are interested in the range of frequencies for which

- there is a meaningful macroscopic description,
- there are present some new effects caused by dispersion.

The existence of dispersion for most of materials is expected within the scenario of electronic mechanism. This mechanism is the most rapid manner of establishment of the electric or magnetic polarization in matter. For v being a typical velocity of electrons in atoms, the ratio a/v is a characteristic time which is of the order of relaxation time. The relaxation time is of the order of characteristic time $v \ll c$. The wavelength corresponding to these times $\lambda \sim ac/v$ is much larger than a because $v \ll c$. In what follows we shall consider electromagnetic waves with wavelength $\lambda \gg a$. This condition, applicable dielectrics, may not be sufficient in generality. For instance, meta ls in low temperatures possess region of frequencies in which the macroscopic theory has not application, although the condition $c/\omega \gg a$ is satisfied.

Macroscopic Maxwell's equations in material media have formally the same form as in empty space

$$\nabla \cdot \boldsymbol{D} = 0, \qquad \nabla \times \boldsymbol{H} = \frac{1}{c}\partial_t \boldsymbol{D},$$

and

$$\nabla \cdot \boldsymbol{B} = 0, \qquad \nabla \times \boldsymbol{E} = -\frac{1}{c}\partial_t \boldsymbol{B}.$$

Differential equations in such a form have no solution until they are complemented by relations between fields $\boldsymbol{E}$, $\boldsymbol{D}$ and $\boldsymbol{B}$, $\boldsymbol{H}$ *i.e. constitutive*

[18] A characteristic magnitude of a is 1 Angstrom equals to 10^{-10}m.

relations. For static and quase-static fields in homogeneous and isotropic media these relations are just proportionality relations $\boldsymbol{D} = \varepsilon \boldsymbol{E}$, $\boldsymbol{B} = \mu \boldsymbol{H}$, where parameters ϵ and μ are material constants. The constitutive relations in such a case are *local in time*. In other words, fields $\boldsymbol{D}$ at instant of time t is determined bt $\boldsymbol{E}$ taken at the same instant of time. The same is true for the pair $\boldsymbol{B}$, $\boldsymbol{H}$.

Temporal locality of equations is a consequence of the fact that the electric and magnetic dipoles have enough time for changing their orientation responding for slowly varying electromagnetic field. The situation is different for quickly varying electromagnetic field. Since dipoles cannot immediately change their orientation then there exists certain time delay in response of the material. In such a case, the electric permittivity and magnetic permeability are functions of the frequency ω. The dependence on ω is different for functions describing electric and magnetic properties of matter.

Dielectrics

The polarization vector $\boldsymbol{P}$ is defined taking into account neutrality of dielectrics, $\int_V d^3x \, \langle \rho_b \rangle = 0$, where V is the region occupied by dielectric body and $\langle \rho_b \rangle$ is volume density of bounded charges. Electrical neutrality is assured by the assumption that the polarization divergence is proportional to the electric charge density, $\langle \rho_b \rangle = -\nabla \cdot \boldsymbol{P}$ where $\boldsymbol{P} \equiv 0$. The electric dipole moment of the body is given by expression

$$\boldsymbol{p} = \int_V d^3x \, \boldsymbol{r} \, \langle \rho_b \rangle = \int_V d^3x \boldsymbol{P}. \tag{5.248}$$

It means that the polarisation vector $\boldsymbol{P}$ has interpretation of *electric dipole moment density.* The derivation and interpretation of the polarization vector holds for time-varying fields. Thus the vector $\boldsymbol{P} = \frac{1}{4\pi}(\boldsymbol{D} - \boldsymbol{E})$ represents the electric polarization independently on dispersion in the medium.

Fields which oscillates with high frequencies have usually small amplitudes. For fields which are not too strong the fields $\boldsymbol{D}(t, \boldsymbol{x})$ and $\boldsymbol{E}(t, \boldsymbol{x})$ are related by linear transformation. The most general such transformation is given by integral

$$\boldsymbol{D}(t, \boldsymbol{x}) = \boldsymbol{E}(t, \boldsymbol{x}) + \int_{-\infty}^{t} dt' f(t - t') \boldsymbol{E}(t', \boldsymbol{x}),$$

where the function $f(t)$ is determined by properties of the medium. The upper limit $t' = t$ of the integral represents *causal cut-off*. For further convenience the term $\boldsymbol{E}(t, \boldsymbol{x})$ has been separated from the integral expression. [19] Changing the variable of integration

$$\tau := t - t', \qquad \int_{-\infty}^{t} dt' \to \int_{\infty}^{0} (-d\tau),$$

[19] Note that absence of the integral is physically interpret as absence of dielectric medium. The adopted way of representing the electric displacement field allows us to remove the Dirac delta from the function $f(t)$.

one gets

$$\boxed{\boldsymbol{D}(t,\boldsymbol{x}) = \boldsymbol{E}(t,\boldsymbol{x}) + \int_0^\infty d\tau f(\tau)\boldsymbol{E}(t-\tau,\boldsymbol{x}).} \tag{5.249}$$

One can put the constitutive relation for dispersive media (5.249) in the similar form to the relation $\boldsymbol{D} = \varepsilon \boldsymbol{E}$, namely

$$\boldsymbol{D} = \widehat{\varepsilon}\,\boldsymbol{E} \tag{5.250}$$

where $\widehat{\varepsilon}$ is some *linear integral operator*. The presented formalism has also application in metals.

In what follows we shall apply the method of Fourier decomposition of time-dependent field. For a *single frequency* field

$$\boldsymbol{E}(t,\boldsymbol{x}) = e^{-i\omega t}\widetilde{\boldsymbol{E}}(\omega,\boldsymbol{x})$$

the expression (5.249) takes the form

$$\boldsymbol{D}(t,\boldsymbol{x}) = \boldsymbol{E}(t,\boldsymbol{x}) + \left(\int_0^\infty d\tau f(\tau)e^{i\omega\tau}\right)\underbrace{e^{-i\omega t}\widetilde{\boldsymbol{E}}(\omega,\boldsymbol{x})}_{\boldsymbol{E}(t,\boldsymbol{x})} = \varepsilon(\omega)\boldsymbol{E}(t,\boldsymbol{x}) \tag{5.251}$$

where

$$\boxed{\varepsilon(\omega) = 1 + \int_0^\infty d\tau f(\tau)e^{i\omega\tau}.} \tag{5.252}$$

Hence, for dingle frequency fields the integral operator $\widehat{\varepsilon}$ takes the form of proportionality coefficient between fields $\boldsymbol{D}$ and $\boldsymbol{E}$. This coefficient depends on properties of the material medium and the field frequency. The *dispersion law* $\varepsilon(\omega)$ is a complex-valued function and thus it can be represented in the form

$$\varepsilon(\omega) = \varepsilon'(\omega) + i\varepsilon''(\omega). \tag{5.253}$$

where $\varepsilon'(\omega), \varepsilon''(\omega) \in \mathbb{R}$ are, respectively, its real and imaginary parts. The expression (5.252) implies the relation

$$\varepsilon(-\omega) = \varepsilon^*(\omega) \tag{5.254}$$

which gives

$$\varepsilon'(-\omega) + i\varepsilon''(-\omega) = \varepsilon'(\omega) - i\varepsilon''(\omega).$$

It follows from the above expression that

$$\varepsilon'(-\omega) = \varepsilon'(\omega), \qquad \varepsilon''(-\omega) = -\varepsilon''(\omega), \tag{5.255}$$

i.e. the real part of $\epsilon(\omega)$ is even function of ω whereas the imaginary part is odd. Expanding the real and imaginary part of the function $\varepsilon(\omega)$ in Taylor series at $\omega = 0$ one gets that

$$\varepsilon'(\omega) = \varepsilon_0 + \sum_{n=1}^{\infty} \varepsilon_{2n}\omega^{2n}, \qquad \varepsilon''(\omega) = \varepsilon_1\omega + \sum_{n=1}^{\infty} \varepsilon_{2n+1}\omega^{2n+1},$$

i.e. they contain either even or odd powers of ω.

Only the lowest order expansion terms are significant in the limit $\omega \to 0$. Thus, the first term of approximation for the real part ε_0[20] corresponds with non-dispersive *dielectrics*. For the imaginary part the lower order term id linear in frequency ω.

[20] Note that ε_0 is not an electric constant in vacuum because in Gaussian units this constant is equal unit.

The function $\epsilon(\omega)$ is physically sound for **metals** in the low range of frequencies. Ampere's law in dielectric media (without free current) with this law in non perfect conductors have the form

$$\nabla \times \boldsymbol{H} = \frac{4\pi}{c} \underbrace{\left(\frac{1}{4\pi}\partial_t \boldsymbol{D}\right)}_{\boldsymbol{J}_d}, \qquad \nabla \times \boldsymbol{H} = \frac{4\pi}{c} \underbrace{(\sigma \boldsymbol{E})}_{\boldsymbol{J}_c} \tag{5.256}$$

where $\boldsymbol{J}_d$ and $\boldsymbol{J}_c$ are, respectively, displacement and conduction currents. Thus the expression (5.249) can be formally applied for metals with substitution $\partial_t \boldsymbol{D} \to 4\pi\sigma \boldsymbol{E}$. It means that

$$\partial_t \boldsymbol{D}(t, \boldsymbol{x}) = \partial_t \left(\varepsilon(\omega) e^{-i\omega t} \widetilde{\boldsymbol{E}}(\omega, \boldsymbol{x})\right) = \underbrace{-i\omega\varepsilon(\omega)}_{4\pi\sigma} \boldsymbol{E}(t, \boldsymbol{x})$$

and thus the function $\omega(\varepsilon)$ is of the form

$$\boxed{\varepsilon(\omega) = i\frac{4\pi\sigma}{\omega}.} \tag{5.257}$$

Hence, the function $\varepsilon(\omega)$ for conductors contains the singular imaginary term, proportional to ω^{-1}. The subsequent expansion term is a real constant, however, in the case of metals this constant has no similar interpretation as ε_0 in dielectrics.

The Lorentz model

The Lorentz model is a *classical* model of polarizability. Typical velocities v of electrons in atoms are small when comparing with the speed of light c. It mens that dislocations of electrons in atoms are of order v/ω and thus they are much smaller than the electromagnetic wavelength c/ω. Consequently, the electric field experienced by bounded electron is approximately *uniform* in space and can be approximated by $\boldsymbol{E}(t) = \boldsymbol{E}_0 e^{-i\omega t}$ at some point $\boldsymbol{r}_0$ in vicinity of the electron.

We shall consider a *classical model* in which the electron-nucleus interaction is modeled by harmonic force $\boldsymbol{F} = -k\boldsymbol{x}$, where $\boldsymbol{x}$ denotes dislocation of the electron from the position of the equilibrium. By assumption the electric field is uniform in whole region where the electron moves. The loses of energy are modeled by the force $\boldsymbol{F}_d = -\eta \frac{d\boldsymbol{x}}{dt}$. Thus, the dynamics of the electron is governed by classical equation of motion

$$m\frac{d^2\boldsymbol{x}}{dt^2} = -k\boldsymbol{x} - \eta\frac{d\boldsymbol{x}}{dt} - e\boldsymbol{E}(t), \tag{5.258}$$

or equivalently

$$\frac{d^2\boldsymbol{x}}{dt^2} + \gamma\frac{d\boldsymbol{x}}{dt} + \omega_0^2\boldsymbol{x} = -\frac{e}{m}\boldsymbol{E}_0 e^{-i\omega t}. \quad (5.259)$$

We shall consider a complex-valued form of the solution, namely $\boldsymbol{x}(t) = \boldsymbol{x}_0 e^{-i\omega t}$. It gives[21]

$$\boldsymbol{x}_0 = -\frac{e}{m}\frac{\boldsymbol{E}_0}{(\omega_0^2 - \omega^2) - i\gamma\omega}. \quad (5.260)$$

[21] This is so-called particular solution of non homogeneous differential equation.

Note that electric field $\boldsymbol{E}(t)$ which depend on multiple frequencies can be represented by the Fourier integral $\boldsymbol{E}(t) = \frac{1}{2\pi}\int d\omega e^{-i\omega t}\widetilde{\boldsymbol{E}}(\omega)$, where $\widetilde{\boldsymbol{E}}(\omega)$ plays the role of $\boldsymbol{E}_0$. In such a case the solution $\boldsymbol{x}(t)$ is assumed to have the form of Fourier integral $\boldsymbol{x}(t) = \frac{1}{2\pi}\int d\omega e^{-i\omega t}\widetilde{\boldsymbol{x}}(\omega)$. The coefficients $\widetilde{\boldsymbol{x}}(\omega)$ are of the form

$$\tilde{\boldsymbol{x}}(\omega) = -\frac{e}{m}\frac{\tilde{\boldsymbol{E}}(\omega)}{(\omega_0^2 - \omega^2) - i\gamma\omega}.$$

The electric dipole moment of a single electron reads

$$\boldsymbol{p}(t) = -e\,\boldsymbol{x}(t).$$

If the material has N such electrons per unit volume than the polarization vector reads $\boldsymbol{P}(t) = N\boldsymbol{p}(t)$ and thus

$$\boldsymbol{P}(t) = \frac{Ne^2}{m}\frac{\boldsymbol{E}(t)}{(\omega_0^2 - \omega^2) - i\gamma\omega} = \chi_e\boldsymbol{E}(t). \quad (5.261)$$

More realistic models assume that different electrons interact differently with the electric field. For instance, a dielectric material with N_m molecules in the unit of volume and Z electrons in each molecule ($N = N_m Z$) possesses f_j electrons (of total Z electrons) that are characterized by frequencies ω_j and damping constants γ_j. Thus the electric susceptibility is generalized to the form

$$\chi_e = \frac{N_m e^2}{m}\sum_j \frac{f_j}{(\omega_j^2 - \omega^2) - i\gamma_j}, \qquad \sum_j f_j = Z. \quad (5.262)$$

The function $\varepsilon(\omega)$ reads

$$\varepsilon(\omega) = 1 + \frac{\omega_p^2}{(\omega_0^2 - \omega^2) - i\gamma\omega}, \qquad \omega_p^2 = \frac{4\pi Ne^2}{m}. \quad (5.263)$$

wgere ω_p is *plasma frequency*. The real and imaginary parts of $\varepsilon(\omega)$ have the form

$$\varepsilon(\omega) = \underbrace{1 + \frac{\omega_p^2(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2) + \gamma^2\omega^2}}_{\varepsilon'(\omega)} + i\underbrace{\frac{\omega_p^2\omega}{(\omega_0^2 - \omega^2) + \gamma^2\omega^2}}_{\varepsilon''(\omega)}. \quad (5.264)$$

In the limit of high frequencies ω the function $\varepsilon(\omega)$ is approximated by expression

$$\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2} \tag{5.265}$$

which tends to unity for $\omega \to \infty$. This approximation is valid for light atoms in deep UV region and for heavier ones in the region of X-ray frequencies.

Dispersion of magnetic permeability

In contrast to $\varepsilon(\omega)$, the magnetic permeability $\mu(\omega)$ loses its physical meaning for high frequencies. The expression

$$\boldsymbol{M} = \frac{1}{4\pi}(\boldsymbol{B} - \boldsymbol{H})$$

can be interpreted as magnetic moment density providing that $\partial_t \boldsymbol{P}$ can be neglected. This can be seen from magnetic moment definition

$$\boldsymbol{m} = \frac{1}{2c}\int_V d^3x\, \boldsymbol{r} \times \langle \boldsymbol{j}_b \rangle \tag{5.266}$$

Ampere's law in absence of macroscopic conduction currents can be put in two different ways, namely

$$\nabla \times \boldsymbol{B} = \frac{1}{c}\partial_t \boldsymbol{E} + \frac{4\pi}{c}\langle \boldsymbol{j}_b \rangle, \qquad \nabla \times \boldsymbol{H} = \frac{1}{c}\partial_t \boldsymbol{D}.$$

Subtraction of these equations gives

$$\nabla \times \underbrace{(\boldsymbol{B} - \boldsymbol{H})}_{4\pi \boldsymbol{M}} = \frac{1}{c}\partial_t \underbrace{(\boldsymbol{E} - \boldsymbol{D})}_{-4\pi \boldsymbol{P}} + \frac{4\pi}{c}\langle \boldsymbol{j}_b \rangle .$$

Thus the current density of bounded charges reads

$$\langle \boldsymbol{j}_b \rangle = c\nabla \times \boldsymbol{M} + \partial_t \boldsymbol{P}. \tag{5.267}$$

Clearly, this current depends on time derivative of a polarization vector. Plugging (5.267) into (5.266) and using the fact that $\boldsymbol{M} = 0$ in empty space we get

$$\boldsymbol{m} = \int_V d^3x\, \boldsymbol{M} + \frac{1}{2c}\int_V d^3x\, \boldsymbol{r} \times \partial_t \boldsymbol{P}.$$

This expression shows that $\boldsymbol{M}$ has interpretation of magnetic moment density only if the term $\partial_t \boldsymbol{P}$ is very small comparing with the other term

$$\partial_t \boldsymbol{P}(t) \ll c\nabla \times \boldsymbol{M}. \tag{5.268}$$

For instance, we can take a small body with characteristic size l (*e.g. a sphere* of radius l) in oscillating and approximately uniform magnetic field $\boldsymbol{B} \approx \boldsymbol{B}_0 e^{-i\omega t}$.

Integration of (5.268) over small disc localized *inside* the body and having radius approximately equal to l gives

$$\int_S d\boldsymbol{a} \cdot \partial_t \boldsymbol{P}(t) \sim l^2 \partial_t P(t)$$

and

$$\int_S d\boldsymbol{a} \cdot (\nabla \times \boldsymbol{M}) = \oint_C d\boldsymbol{l} \cdot \boldsymbol{M} \sim l M(t) = l \chi_m H(t).$$

Thus the condition (5.268) results in the following one

$$\partial_t P(t) \ll \frac{\chi_m c}{l} H(t). \tag{5.269}$$

The polarization vector is given by $\boldsymbol{P} = \frac{1}{4\pi}(\boldsymbol{D} - \boldsymbol{E}) = \frac{\varepsilon - 1}{4\pi}\boldsymbol{E}$. Taking $\varepsilon - 1 \sim 1$ (*e.g.* for teflon $\varepsilon = 2.1$) we get

$$\partial_t P(t) \sim \partial_t E(t).$$

The electric field magnitude can be estimated from Faraday's law

$$\int_C d\boldsymbol{l} \cdot \boldsymbol{E} = -\frac{1}{c} \int_S d\boldsymbol{a} \cdot \boldsymbol{B}.$$

Choosing the integration region S as a disc of radius l we get integrals

$$\int_C d\boldsymbol{l} \cdot \boldsymbol{E} \sim l E(t), \qquad -\frac{1}{c}\frac{d}{dt} \int_S d\boldsymbol{a} \cdot \boldsymbol{B} \sim -\frac{l^2}{c} \partial_t B(t) \approx -\frac{l^2}{c} \partial_t H(t)$$

where the approximation of the field B by H follows from the fact that χ_m is usually vary small for paramagnetics and diamagnetics. Thus we have from Faraday's laws

$$E(t) \sim -\frac{l}{c} \partial_t H(t).$$

Since $\boldsymbol{H} \approx \boldsymbol{B}_0 e^{-i\omega t}$, then

$$\partial_t P(t) \sim -\frac{l}{c} \partial_t^2 H(t) = \frac{l\omega^2}{c} H(t).$$

Plugging this expression into (5.268) we get

$$\boxed{l^2 \ll \frac{\chi_m c^2}{\omega^2}.} \tag{5.270}$$

However, l cannot be arbitrary small. The macroscopic character of the body is given by the condition $l \gg a$ where a is the atomic size. The condition (5.270) does not hold for optical frequencies because $\chi_m \sim \frac{v^2}{c^2}$ and $\omega \sim \frac{v}{a}$. Thus the right hand side of (5.270) is proportional to a^2. Consequently, the concept of magnetic permeability is *meaningless for optical frequencies*. In such a case $\mu = 1$.

The energy of the electromagnetic field in dispersive media

The electromagnetic energy flux is given by *Poynting vector*

$$\boldsymbol{S} = \frac{c}{4\pi}\boldsymbol{E} \times \boldsymbol{H}. \tag{5.271}$$

This formula holds for any time-varying electromagnetic field including the case of dispersive media. The rate of change of energy density $\partial_t u$ is given by divergence of Poynting vector $\nabla \cdot \boldsymbol{S}$, namely

$$\begin{aligned}
\partial_t u = -\nabla \cdot \boldsymbol{S} &= -\frac{c}{4\pi}\epsilon_{ijk}\partial_i(E^j H^k) \\
&= -\frac{c}{4\pi}\left(\epsilon_{ijk}\partial_i E^j H^k - \epsilon_{jik}E^j\partial^i H^k\right) \\
&= -\frac{c}{4\pi}\underbrace{(\boldsymbol{H}\cdot(\nabla \times \boldsymbol{E})}_{-\frac{1}{c}\partial_t \mathbf{B}} - \boldsymbol{E}\cdot\underbrace{(\nabla \times \boldsymbol{H})}_{\frac{1}{c}\partial_t \mathbf{D}}) \\
&= \frac{1}{4\pi}(\boldsymbol{E}\cdot\partial_t\boldsymbol{D} + \boldsymbol{H}\cdot\partial_t\boldsymbol{B}).
\end{aligned} \tag{5.272}$$

In non-dispersive dielectric media the constitutive relations $\boldsymbol{D} = \varepsilon\boldsymbol{E}$ and $\boldsymbol{B} = \varepsilon\boldsymbol{H}$ allows us to put energy density in the form

$$u = \frac{1}{8\pi}(\varepsilon\boldsymbol{E}^2 + \mu\boldsymbol{H}^2). \tag{5.273}$$

Thermodynamic interpretation of energy density (5.273) is the following one: it is equal to the difference of total energy inside unit volume with and without electromagnetic field. Such interpretation of the electromagnetic energy is not possible in presence of dispersion due to existence of absorption process (energy dissipation).

Dispersion for monochromatic wave

We shall consider the simplest case of monochromatic electromagnetic field. We assume constancy of the field amplitude which means that whole energy dissipated inside unit volume in unit of time is compensate by transfer of energy from external source. The average value of the rate of change of energy density $\langle \partial_t u \rangle_t$ where

$$\langle f(t) \rangle_t := \frac{1}{T}\int_0^T f(t)dt$$

represents averaging process on a period represents the amount of heat per unit of time transferred into unit volume. This quantity can be cast in the form

$$Q = \langle -\nabla \cdot \boldsymbol{S} \rangle_t = \frac{1}{4\pi}\Big(\langle \boldsymbol{E}\cdot\partial_t\boldsymbol{D}\rangle_t + \langle \boldsymbol{H}\cdot\partial_t\boldsymbol{B}\rangle_t \Big), \tag{5.274}$$

where all the fields are real-valued. This expression is quadratic in fields. For auxiliary complex fields this expression must be replaced by the following one

$$Q = \frac{1}{4\pi}\Big(\langle Re(\boldsymbol{E}) \cdot \partial_t Re(\boldsymbol{D})\rangle_t + \langle Re(\boldsymbol{H}) \cdot \partial_t Re(\boldsymbol{B})\rangle_t \Big) \tag{5.275}$$

where physical fields are given by real parts of auxiliary fields

$$Re(\boldsymbol{E}) = \frac{1}{2}(\boldsymbol{E} + \boldsymbol{E}^*), \qquad Re(\boldsymbol{D}) = \frac{1}{2}(\boldsymbol{D} + \boldsymbol{D}^*).$$

A similar procedure is applied to magnetic fields $\boldsymbol{H}$ and $\boldsymbol{B}$. The fields $\boldsymbol{D}(t, \boldsymbol{x}) = \boldsymbol{D}_0(\boldsymbol{x})e^{-i\omega t}$ and $\boldsymbol{B}(t, \boldsymbol{x}) = \boldsymbol{B}_0(\boldsymbol{x})e^{-i\omega t}$ are monochromatic. According to (5.251) the constitutive relations read

$$\boldsymbol{D}(t, \boldsymbol{x}) = \varepsilon(\omega)\boldsymbol{E}(t, \boldsymbol{x}),$$
$$\boldsymbol{B}(t, \boldsymbol{x}) = \mu(\omega)\boldsymbol{H}(t, \boldsymbol{x}).$$

Their derivatives with respect to time are of the form

$$\partial_t Re(\boldsymbol{D}) = -\frac{i\omega}{2}\left(\varepsilon \boldsymbol{E} - \varepsilon^* \boldsymbol{E}^*\right),$$
$$\partial_t Re(\boldsymbol{B}) = -\frac{i\omega}{2}\left(\varepsilon \boldsymbol{B} - \varepsilon^* \boldsymbol{B}^*\right).$$

The scalar products in (5.275) take the form

$$\begin{aligned}
\langle Re(\boldsymbol{E}) \cdot \partial_t Re(\boldsymbol{D})\rangle_t &= -\frac{i\omega}{4}\left\langle (\boldsymbol{E} + \boldsymbol{E}^*) \cdot (\varepsilon \boldsymbol{E} - \varepsilon^* \boldsymbol{E}^*)\right\rangle_t \\
&= -\frac{i\omega}{4}\Big[(\varepsilon - \varepsilon^*)\underbrace{\left\langle |\boldsymbol{E}|^2\right\rangle_t}_{|\boldsymbol{E}|^2} + \varepsilon \underbrace{\left\langle \boldsymbol{E}^2\right\rangle_t}_{0} - \varepsilon^* \underbrace{\left\langle \boldsymbol{E}^{*2}\right\rangle_t}_{0}\Big] \\
&= -\frac{i\omega}{4}\underbrace{(\varepsilon - \varepsilon^*)}_{2i\varepsilon''}|\boldsymbol{E}|^2 = \frac{\omega}{2}\varepsilon''|\boldsymbol{E}|^2
\end{aligned}$$

and

$$\langle Re(\boldsymbol{H}) \cdot \partial_t Re(\boldsymbol{B})\rangle_t = -\frac{i\omega}{4}\underbrace{(\mu - \mu^*)}_{2i\mu''}|\boldsymbol{H}|^2 = \frac{\omega}{2}\mu''|\boldsymbol{H}|^2.$$

Thus (5.275) reads

$$\boxed{Q = \frac{\omega}{8\pi}\left(\varepsilon''|\boldsymbol{E}|^2 + \mu''|\boldsymbol{H}|^2\right).} \tag{5.276}$$

The obtained result shows significance of the imaginary parts of $\varepsilon(\omega)$ and $\mu(\omega)$. They express the absorption properties of material media. Since quantity Q is strictly positive, $Q > 0$, absorption coefficients must be also positive, $\varepsilon'' > 0$ and $\mu'' > 0$. Thus ε'' describes *electric loses* whereas μ'' describes *magnetic loses*.

Dispersion for non-monochromatic wave

Non-monochromatic fields are represented in the form of Fourier transform

$$\boldsymbol{E}(t,\boldsymbol{x}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{-i\omega t}\widetilde{\boldsymbol{E}}(\omega,\boldsymbol{x}), \tag{5.277}$$

$$\boldsymbol{D}(t,\boldsymbol{x}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{-i\omega t}\widetilde{\boldsymbol{D}}(\omega,\boldsymbol{x}). \tag{5.278}$$

Plugging expression (5.277) into (5.249) one gets

$$\begin{aligned}\boldsymbol{D}(t,\boldsymbol{x}) &= \boldsymbol{E}(t,\boldsymbol{x}) + \int_0^{\infty} d\tau f(\tau)\boldsymbol{E}(t-\tau,\boldsymbol{x}) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{-i\omega t}\left(1+\int_0^{\infty} d\tau f(\tau)e^{i\omega\tau}\right)\widetilde{\boldsymbol{E}}(\omega,\boldsymbol{x}) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{-i\omega t}\underbrace{\varepsilon(\omega)\widetilde{\boldsymbol{E}}(\omega,\boldsymbol{x})}_{\widetilde{\boldsymbol{D}}(\omega,\boldsymbol{x})}\end{aligned} \tag{5.279}$$

and thus

$$\partial_t\boldsymbol{D}(t,\boldsymbol{x}) = -\frac{i}{2\pi}\int_{-\infty}^{\infty} d\omega e^{-i\omega t}\omega\varepsilon(\omega)\widetilde{\boldsymbol{E}}(\omega,\boldsymbol{x}). \tag{5.280}$$

The *real-valued* electric field has the form $\boldsymbol{E}(t,\boldsymbol{x}) = \boldsymbol{E}^*(t,\boldsymbol{x})$ where

$$\boldsymbol{E}^*(t,\boldsymbol{x}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{i\omega t}\widetilde{\boldsymbol{E}}^*(\omega,\boldsymbol{x}).$$

Changing the variable $\omega \to -\omega$ in (5.277) one gets

$$\boldsymbol{E}(t,\boldsymbol{x}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{i\omega t}\widetilde{\boldsymbol{E}}(-\omega,\boldsymbol{x}).$$

Comparing both formulas one concludes that the Fourier amplitudes must satisfy condition

$$\widetilde{\boldsymbol{E}}(-\omega,\boldsymbol{x}) = \widetilde{\boldsymbol{E}}^*(\omega,\boldsymbol{x}).$$

We assume that non-monochromatic fields vanish sufficiently quickly for $t \to \pm\infty$. Total energy dissipated in unit volume is given by the integral

$$\begin{aligned}\int_{-\infty}^{\infty} dt\, Q &= \int_{-\infty}^{\infty} dt(-\nabla\cdot\boldsymbol{S}) \\ &= \frac{1}{4\pi}\int_{-\infty}^{\infty} dt(\boldsymbol{E}\cdot\partial_t\boldsymbol{D} + \boldsymbol{H}\cdot\partial_t\boldsymbol{B}).\end{aligned} \tag{5.281}$$

The quantity Q in (5.281) is local in time, $Q = \partial_t u$, which means that unlike (5.274) it is not average over the period (the field is not necessarily periodic). The first integral in (5.281) can be cast in the form

$$\begin{aligned}
&\frac{1}{4\pi}\int_{-\infty}^{\infty} dt \boldsymbol{E}\cdot\partial_t \boldsymbol{D} = \\
&= -\frac{i}{4\pi(2\pi)^2}\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' e^{-i(\omega+\omega')t}\omega\varepsilon(\omega)\widetilde{\boldsymbol{E}}(\omega,\boldsymbol{x})\cdot\widetilde{\boldsymbol{E}}(\omega',\boldsymbol{x}) \\
&= -\frac{i}{4\pi(2\pi)^2}\int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' \underbrace{\left(\int_{-\infty}^{\infty} dt e^{-i(\omega+\omega')t}\right)}_{2\pi\delta(\omega+\omega')}\omega\varepsilon(\omega)\widetilde{\boldsymbol{E}}(\omega,\boldsymbol{x})\cdot\widetilde{\boldsymbol{E}}(\omega',\boldsymbol{x}) \\
&= -\frac{i}{8\pi^2}\int_{-\infty}^{\infty} d\omega\, \omega\varepsilon(\omega)\widetilde{\boldsymbol{E}}(\omega,\boldsymbol{x})\cdot\underbrace{\widetilde{\boldsymbol{E}}(-\omega,\boldsymbol{x})}_{\widetilde{\boldsymbol{E}}^*(\omega,\boldsymbol{x})} \\
&= -\frac{i}{8\pi^2}\int_{-\infty}^{\infty} d\omega\, \omega\varepsilon(\omega)|\widetilde{\boldsymbol{E}}(\omega,\boldsymbol{x})|^2 \\
&= -\frac{i}{8\pi^2}\underbrace{\int_{-\infty}^{\infty} d\omega\, \omega\varepsilon'(\omega)|\widetilde{\boldsymbol{E}}(\omega,\boldsymbol{x})|^2}_{0} + \frac{1}{8\pi^2}\int_{-\infty}^{\infty} d\omega\, \omega\varepsilon''(\omega)|\widetilde{\boldsymbol{E}}(\omega,\boldsymbol{x})|^2.
\end{aligned}$$

First integral vanishes because its integrand is odd. Not that the expression ω is odd whereas $\varepsilon'(-\omega) = \varepsilon'(\omega)$ is even and

$$|\widetilde{\boldsymbol{E}}(-\omega,\boldsymbol{x})|^2 = \underbrace{\widetilde{\boldsymbol{E}}(-\omega,\boldsymbol{x})}_{\widetilde{\boldsymbol{E}}^*(\omega,\boldsymbol{x})}\underbrace{\widetilde{\boldsymbol{E}}^*(-\omega,\boldsymbol{x})}_{\widetilde{\boldsymbol{E}}(\omega,\boldsymbol{x})} = |\widetilde{\boldsymbol{E}}(\omega,\boldsymbol{x})|^2$$

is even as well. The magnetic contribution to the integral is obtained in a similar way. It gives

$$\begin{aligned}
\int_{-\infty}^{\infty} dt\, Q &= \frac{1}{8\pi^2}\int_{-\infty}^{\infty} d\omega\, \omega\left[\varepsilon''(\omega)|\widetilde{\boldsymbol{E}}(\omega,\boldsymbol{x})|^2 + \mu''(\omega)|\widetilde{\boldsymbol{H}}(\omega,\boldsymbol{x})|^2\right] \\
&= \frac{1}{4\pi^2}\int_{0}^{\infty} d\omega\, \omega\left[\varepsilon''(\omega)|\widetilde{\boldsymbol{E}}(\omega,\boldsymbol{x})|^2 + \mu''(\omega)|\widetilde{\boldsymbol{H}}(\omega,\boldsymbol{x})|^2\right].
\end{aligned} \tag{5.282}$$

where we made use of the fact that the integrand is even function of ω, namely $g(-\omega) = g(\omega)$, and thus the integral over a negative range of ω can be represented the integral over a positive range,

$$\int_{-\infty}^{0} d\omega g(\omega) = \int_{+\infty}^{0}(-d\omega)g(-\omega) = \int_{0}^{\infty} d\omega g(\omega).$$

(5.282) shows that coefficients ε'' and μ'' are responsible for energy dissipation in dispersive media. The dissipated energy is converted into heat which means that $Q > 0$. Since all frequencies in (5.282) are positive, then it must hold

$$\varepsilon'' > 0, \qquad \mu'' > 0 \tag{5.283}$$

for any material. There are no such restriction on the sign of real parts ε' and μ'.

Although loses are always present in real materials there are ranges of frequencies for which they are not significant. They are called *transparency regions*. In such regions there is possible to introduce the concept of internal energy of the body in the same sense as in a static field.

Dispersion for quase-monochromatic wave

For single monochromaic wave there is no steady accumulation of electromagnetic energy. For this reason we consider a quase-monochromatic wave *i.e.* a wave whose components have frequencies in narrow range about some mean value ω_0. The auxiliary complex-valued electromagnetic field has the form

$$\boldsymbol{E}(t,\boldsymbol{x}) = \boldsymbol{E}_0(t,\boldsymbol{x})e^{-i\omega_0 t}, \qquad \boldsymbol{H}(t,\boldsymbol{x}) = \boldsymbol{H}_0(t,\boldsymbol{x})e^{-i\omega_0 t} \tag{5.284}$$

where $\boldsymbol{E}_0(t,\boldsymbol{x})$ and $\boldsymbol{H}_0(t,\boldsymbol{x})$ are slowly-varying functions of time. Plugging

$$\begin{aligned} Re(\boldsymbol{E}) &= \frac{1}{2}(\boldsymbol{E}+\boldsymbol{E}^*), & Re(\boldsymbol{D}) &= \frac{1}{2}(\boldsymbol{D}+\boldsymbol{D}^*), \\ Re(\boldsymbol{H}) &= \frac{1}{2}(\boldsymbol{H}+\boldsymbol{H}^*), & Re(\boldsymbol{B}) &= \frac{1}{2}(\boldsymbol{B}+\boldsymbol{B}^*), \end{aligned} \tag{5.285}$$

into expression

$$\partial_t u = \frac{1}{4\pi}\Big(Re(\boldsymbol{E})\cdot\partial_t Re(\boldsymbol{D}) + Re(\boldsymbol{H})\cdot\partial_t Re(\boldsymbol{B})\Big)$$

and skipping quickly oscillating terms $\boldsymbol{E}\cdot\boldsymbol{D}\sim e^{-2i\omega_0 t}$, $\boldsymbol{E}^*\cdot\boldsymbol{D}^*\sim e^{2i\omega_0 t}$ *e.t.c.* we get

$$\partial_t u = \frac{1}{16\pi}\left(\boldsymbol{E}\cdot\partial_t\boldsymbol{D}^* + \boldsymbol{E}^*\cdot\partial_t\boldsymbol{D} + \boldsymbol{H}\cdot\partial_t\boldsymbol{B}^* + \boldsymbol{H}^*\cdot\partial_t\boldsymbol{B}\right). \tag{5.286}$$

Quickly oscillating terms $\sim e^{\pm 2i\omega_0 t}$ can be formally eliminated by time averaging over the interval $T = \frac{2\pi}{\omega_0}$. Such averaging does not change slowly-varying functions $\langle \boldsymbol{E}_0(t,\boldsymbol{x})\rangle_t \approx \boldsymbol{E}_0(t,\boldsymbol{x})$ *e.t.c.* To obtain expressions $\partial_t\boldsymbol{D}$ and $\partial_t\boldsymbol{B}$ we expand the electric field in Fourier series

$$\boldsymbol{E}(t,\boldsymbol{x}) = \sum_{\omega'}\widetilde{\boldsymbol{E}}_0(\omega',\boldsymbol{x})e^{-i(\omega_0+\omega')t}.$$

Since $\widetilde{\boldsymbol{E}}_0(t,\boldsymbol{x})$ is a *slowly* varying function of time, the frequencies ω' of the expansion $\boldsymbol{E}_0(t,\boldsymbol{x}) = \sum_{\omega'}\widetilde{\boldsymbol{E}}_0(\omega',\boldsymbol{x})e^{-i\omega' t}$ are *small* comparing with ω_0, thus $\omega' \ll \omega_0$. It leads to the expression

$$\begin{aligned} \boldsymbol{D}(t,\boldsymbol{x}) &= \boldsymbol{E}(t,\boldsymbol{x}) + \int_0^\infty d\tau\, f(\tau)\boldsymbol{E}(t-\tau,\boldsymbol{x}) \\ &= \sum_{\omega'}\underbrace{\left(1+\int_0^\infty d\tau\, f(\tau)e^{i(\omega_0+\omega')\tau}\right)}_{\varepsilon(\omega_0+\omega')}\widetilde{\boldsymbol{E}}_0(\omega',\boldsymbol{x})e^{-i(\omega_0+\omega')t}. \end{aligned} \tag{5.287}$$

Thus temporal derivative is of the form

$$\begin{aligned}
\partial_t \boldsymbol{D}(t,\boldsymbol{x}) &= \sum_{\omega'} -i \overbrace{[(\omega_0+\omega')\varepsilon(\omega_0+\omega')]}^{\omega\varepsilon(\omega)} \widetilde{\boldsymbol{E}}_0(\omega',\boldsymbol{x}) e^{-i(\omega_0+\omega')t} \\
&= \sum_{\omega'} -i \left[\omega_0\varepsilon(\omega_0) + \left.\frac{d(\omega\varepsilon(\omega))}{d\omega}\right|_{\omega_0} \omega' + \ldots \right] \widetilde{\boldsymbol{E}}_0(\omega',\boldsymbol{x}) e^{-i(\omega_0+\omega')t} \\
&= -i\omega_0\varepsilon(\omega_0) e^{-i\omega_0 t} \sum_{\omega'} \widetilde{\boldsymbol{E}}_0(\omega',\boldsymbol{x}) e^{-i\omega' t} + \\
&+ e^{-i\omega_0 t} \left.\frac{d(\omega\varepsilon(\omega))}{d\omega}\right|_{\omega_0} \sum_{\omega'} \widetilde{\boldsymbol{E}}_0(\omega',\boldsymbol{x}) \underbrace{(-i\omega') e^{-i\omega' t}}_{\frac{d}{dt} e^{-i\omega t}} + \ldots \\
&= \left(-i\omega_0\varepsilon(\omega_0) \boldsymbol{E}_0(t,x) + \left.\frac{d(\omega\varepsilon)}{d\omega}\right|_{\omega_0} \partial_t \boldsymbol{E}_0(t,x) + \ldots \right) e^{-i\omega_0 t}. \quad (5.288)
\end{aligned}$$

The quadratic expression is of the form

$$\boldsymbol{E}^*(t,\boldsymbol{x}) \cdot \partial_t \boldsymbol{D}(t,\boldsymbol{x}) = -i\omega^0 \varepsilon(\omega_0) \boldsymbol{E}_0^* \cdot \boldsymbol{E}_0 + \left.\frac{d(\omega\varepsilon)}{d\omega}\right|_{\omega_0} \underbrace{\boldsymbol{E}_0^* \cdot \partial_t \boldsymbol{E}_0}_{\frac{1}{2}\partial_t(\boldsymbol{E}_0^* \cdot \boldsymbol{E}_0)} + \ldots$$

Hence, only real-valued terms do not cancel in the sum

$$\boldsymbol{E}^*(t,\boldsymbol{x}) \cdot \partial_t \boldsymbol{D}(t,\boldsymbol{x}) + \boldsymbol{E}(t,\boldsymbol{x}) \cdot \partial_t \boldsymbol{D}^*(t,\boldsymbol{x}) = \left.\frac{d(\omega\varepsilon)}{d\omega}\right|_{\omega_0} \partial_t (\boldsymbol{E}^* \cdot \boldsymbol{E}) + \ldots$$

where $\boldsymbol{E}_0^* \cdot \boldsymbol{E}_0 = \boldsymbol{E}^* \cdot \boldsymbol{E}$. A similar considerations for magnetic field leads to expression

$$\boldsymbol{H}^*(t,\boldsymbol{x}) \cdot \partial_t \boldsymbol{B}(t,\boldsymbol{x}) + \boldsymbol{H}(t,\boldsymbol{x}) \cdot \partial_t \boldsymbol{B}^*(t,\boldsymbol{x}) = \left.\frac{d(\omega\mu)}{d\omega}\right|_{\omega_0} \partial_t (\boldsymbol{H}^* \cdot \boldsymbol{H}) + \ldots$$

Taking the lowest terms of approximation we get

$$\langle \partial_t u \rangle_t = \frac{1}{16\pi} \partial_t \left(\left.\frac{d(\omega\varepsilon)}{d\omega}\right|_{\omega_0} (\boldsymbol{E}^* \cdot \boldsymbol{E}) + \left.\frac{d(\omega\mu)}{d\omega}\right|_{\omega_0} (\boldsymbol{H}^* \cdot \boldsymbol{H}) \right). \quad (5.289)$$

The rate of change of energy density is slow in timescale T describing the averaging interval. Thus $\langle \partial_t u \rangle_t = \partial_t \langle u \rangle_t$ and it follows from from (5.289) the expression

$$\langle u \rangle_t = \frac{1}{16\pi} \left(\left.\frac{d(\omega\varepsilon)}{d\omega}\right|_{\omega_0} (\boldsymbol{E}^* \cdot \boldsymbol{E}) + \left.\frac{d(\omega\mu)}{d\omega}\right|_{\omega_0} (\boldsymbol{H}^* \cdot \boldsymbol{H}) \right). \quad (5.290)$$

The scalar product $\boldsymbol{E}^* \cdot \boldsymbol{E}$ can be put in the form containing *real-valued* fields

$$\boldsymbol{E}^* \cdot \boldsymbol{E} = 2 \left\langle \boldsymbol{E}^2 \right\rangle_t.$$

Dropping symbols ω_0 and t one gets expression

$$\boxed{\langle u \rangle = \frac{1}{18\pi} \left(\frac{d(\omega\varepsilon)}{d\omega} \left\langle \boldsymbol{E}^2 \right\rangle + \frac{d(\omega\mu)}{d\omega} \left\langle \boldsymbol{H}^2 \right\rangle \right).} \quad (5.291)$$

The formula (5.291) is valid only for electromagnetic fields characterized by slowly-varying amplitude $\boldsymbol{E}_0(t, \boldsymbol{x})$. Interruption of electromagnetic energy supply results in conversion of energy stored in the body (volume integral of $\langle u \rangle$) into heat. It means that $\langle u \rangle > 0$, and thus

$$\frac{d(\omega\varepsilon)}{d\omega} > 0, \qquad \frac{d(\omega\mu)}{d\omega} > 0. \tag{5.292}$$

Analytic properties of $\varepsilon(\omega)$

Properties of $f(\tau)$

The function $f(\tau)$ is represents dispersive properties of material media. Once given it determines the form of permittivity $\varepsilon(\omega)$. Below we list main properties of $f(\tau)$.

- $f(\tau) \in \mathbb{R}$ is *finite* for all values of τ, including $\tau = 0$.

  The isolation of the term $\boldsymbol{E}(t, \boldsymbol{x})$ in expression

  $$\begin{aligned} \boldsymbol{D}(t, \boldsymbol{x}) &= \boldsymbol{E}(t, \boldsymbol{x}) + \int_0^\infty d\tau\, f(\tau) \boldsymbol{E}(t - \tau, \boldsymbol{x}) \\ &= \int_{-\infty}^\infty d\tau \left[\delta(\tau) + \theta(\tau) f(\tau)\right] \boldsymbol{E}(t - \tau, \boldsymbol{x}) \end{aligned} \tag{5.293}$$

  avoids the presence of Dirac delta $\delta(\tau)$.

- For *dielectrics* $f(\tau) \to 0$ for $\tau \to \infty$.

  According to expression

  $$\boldsymbol{D}(t, \boldsymbol{x}) = \boldsymbol{E}(t, \boldsymbol{x}) + \int_{-\infty}^t dt'\, f(t - t') \boldsymbol{E}(t', \boldsymbol{x}),$$

  which is just (5.293) for $\tau = t - t'$, the remote past $t' \to -\infty$ corresponds with $\tau \to \infty$. For this reason $f(\tau)$ for $\tau \to \infty$ describes dielectric media in their very early stage. Actual value of $\mathbf{D}$ at t cannot be strongly influenced by values of the electric field $\boldsymbol{E}$ in the remote past. This statement is motivated by the fact that physical mechanism behind (5.293) is a polarisation of media. The function $f(\tau)$ is significantly different from zero for τ being of order of relaxation time.

- For *conductors* $f(\tau) - 4\pi\sigma \to 0$ for $\tau \to \infty$.

  The mathematical formalism developed for dielectrics possesses applications for conductors despite the physical mechanism behind (5.293) is different. Although the stationary current does not change the physical state of conductors, it is *formally* responsible for the appearance of the displacement function $\boldsymbol{D}$. Indeed, comparing the displacement current with the conduction current

  $$\frac{1}{c}\frac{\partial \boldsymbol{D}}{\partial t} = \frac{4\pi}{c}\sigma \boldsymbol{E}$$

and integrating this expression one gets

$$\boldsymbol{D}(t) = 4\pi\sigma \int_{-\infty}^{t} dt' \boldsymbol{E}(t', \boldsymbol{x}) = 4\pi\sigma \int_{0}^{\infty} d\tau \boldsymbol{E}(t - \tau, \boldsymbol{x}),$$

where the condition that there is no external electric field at the remote past, $\boldsymbol{E}(t', \boldsymbol{x}) \to 0$ for $t' \to \infty$ is used.

In further part we shall discuss properties of function $\varepsilon(\omega)$ which was defined as

$$\varepsilon(\omega) = 1 + \int_{0}^{\infty} d\tau f(\tau) e^{i\omega\tau}.$$

The parameter ω was real-valued in our previous analysis. However, some important properties of the function $\varepsilon(\omega)$ can be obtained using method of analysis of complex functions. For this reason we shall *assume* that argument of the function $\varepsilon(z)$ is a complex variable, $z \in \mathbb{C}$ *i.e.*

$$z = z' + iz'', \qquad z', z'' \in \mathbb{R}.$$

Properties of $\varepsilon(\omega)$ in the upper complex half-plane

Function $\varepsilon(\omega)$ has the following properties in the upper complex half-plane.

1. *Function $\varepsilon(z)$ is a single-valued regular function in the upper half-plane*. It means that the function has no singular points in this region. Indeed, for $z'' > 0$ (upper half-plane) the expression $e^{iz\tau}$ is proportional to $e^{-z''\tau}$. Finiteness of $f(\tau)$ in whole region of integration leads to convergence of the integral

$$\varepsilon(z) = 1 + \int_{0}^{\infty} d\tau f(\tau) e^{iz\tau} = 1 + \int_{0}^{\infty} d\tau f(\tau) e^{-z''\tau} e^{iz'\tau}. \qquad (5.294)$$

It means that the function is single-valued and finite in this region. Function $\varepsilon(z)$ has no singularities at the real axis ($z'' = 0$), except the case of metals, where it has *simple pole* at the origin. A regular character of $\varepsilon(z)$ in the upper half-plane is a consequence of *causality principle*, implemented by the presence of $\theta(t' - t)$ in the integral

$$\int_{-\infty}^{\infty} dt'\, \theta(t' - t) f(t' - t) \boldsymbol{E}(t', \boldsymbol{x}) = \int_{0}^{\infty} d\tau f(\tau) \boldsymbol{E}(\tau, \boldsymbol{x}).$$

The expression (5.294) is meaningless in the lower half-plane because the integral diverges. For this reason the integral in the region $z'' < 0$ can be defined as analytic continuation of the expression $\varepsilon(z)$ given in the upper half plane. However, in general such defined function has some singularities. Only in the upper half-plane the function $\varepsilon(z)$ well-defined from mathematical point of view and has physical interpretation.

2. *Function $\varepsilon(z)$ is odd at the real $z''=0$ axis and changes its sign at $z'=0$.* Since $f(\tau)$ is real-valued, then

$$\varepsilon(-z^*) = 1+\int_0^\infty d\tau\, f(\tau)e^{-iz^*\tau} = \left(1+\int_0^\infty d\tau\, f(\tau)e^{iz\tau}\right)^* \\ = \varepsilon^*(z).$$

It implies that the real and imaginary parts of $\varepsilon(z)$ satisfy

$$\varepsilon'(-z^*) = \varepsilon'(z), \\ \varepsilon''(-z^*) = -\varepsilon''(z). \tag{5.295}$$

In particular, $\varepsilon(z)$ is *real-valued*, $\varepsilon''(iz'')=0$, at the imaginary $z'=0$ axis

$$\varepsilon(iz'') = \varepsilon^*(iz'').$$

According to our previous considerations, the imaginary part of $\varepsilon=\varepsilon'+i\varepsilon''$ is *positive*, $\varepsilon''>0$, for positive frequencies $\omega>0$. It means that imaginary part of permittivity is positive, $\varepsilon''(z')>0$, at real $z'>0$ semi-axis. The equality (5.295) takes the form $\varepsilon''(-z') = -\varepsilon''(z')<0$. We conclude that $\varepsilon(z')$ is *negative* on the negative part of the real z' axis. The imaginary part of $\varepsilon(z)$ changes its sign at $z'=0$ passing through zero for dielectrics or through infinity for metals. The point $z'=0$ is the only point at the real axis in which the function $\varepsilon(z)$ can vanish.

3. *Function $\varepsilon(z)$ tends to unity for $z\to\infty$ in the upper-half plane.*

It has been shown in the particular case that $\varepsilon(\omega)=1-\frac{\omega_p^2}{\omega^2}\to 1$ for $\omega\to\infty$ which corresponds with the case of the real z' axis ($z''=0$). In generality, $\varepsilon(z)\to 1$ for $z\to\infty$ in any manner in the upper half plane. It follows directly from the integral

$$\varepsilon(z) = 1+\int_0^\infty d\tau\, f(\tau)e^{-z''\tau}e^{iz'\tau}$$

which vanishes in the limit $z''\to\infty$ for $z'=0$. For finite z'' and $z'\neq 0$ the integral vanishes because of the oscillating factor $e^{iz'\tau}$.

4. **Theorem:** *Function $\varepsilon(z)$ has no real values at any finite point localized in the upper half-plane, except the points at the imaginary axis where it decreases monotonically*

$$\begin{aligned} &\text{dielectrics}: && \varepsilon_0 \searrow 1 \quad \text{for} \quad z'': 0\to\infty, \\ &\text{metals}: && \infty \searrow 1 \quad \text{for} \quad z'': 0\to\infty. \end{aligned}$$

This theorem implies that $\varepsilon(z)$ has no zeros in the upper half-plane.

Proof: Let us consider the integral over closed curve C in the complex plane z

$$\frac{1}{2\pi i}\oint_C \frac{dg(z)}{dz}\frac{dz}{g(z)-a} = N_0 - N_\infty, \tag{5.296}$$

where N_0 and N_∞ stand, respectively, for number of zeros and number of poles of the function $g(z) - a$ in the region enclosed by C.

Assumptions:

- $g(z)$ is a function which has no poles in the upper half plane (the pole at $z = 0$ is allowed),
- a is a real number,
- $g_0 \in \mathbb{R}$ is represents the value of the function $g(z)$ at the origin $z = 0$ *i.e.* $g_0 = \lim_{z\to 0} g(z)$,
- the contour of integration C consists on the real axis $z'' = 0$ and semi circle of infinite radius in the upper half-plane,
- the function $g(z)$ vanishes at the infinite semicircle.

The contour C in the complex pane z is mapped on the closed contour C' in the complex plane g such that the point $z = 0$ is mapped on $g_0 \in \mathbb{R}$, infinite semicircle is mapped on $g = 0$ and the real positive/negative semi-axes are mapped on two curves (with self-crossing points) lying, respectively, in the upper/lower half planes. The contours are plotted in Figure 5.14. These curves have no crossing points with the real axis ($z'' = 0$) except two points $g = 0$ and $g = g_0$.

Since $g(z)$ has no poles in the upper half-plane (so is $g(z) - a$) then the integral (5.296) equals to the number of zeros

$$\frac{1}{2\pi i}\oint_{C'} \frac{dg}{g-a} = N_0. \tag{5.297}$$

There are two possibilities:

- For $g_0 < \infty$ it follows from the form of the contour C' that the total increase of the argument of $g(z) - a$ is equal to 2π providing that a is located inside the contour. Otherwise, it is equal to zero. It gives $N_0 = 1$ for $0 < a < g_0$ and $N_0 = 0$ for $a > g_0$. It means that $g(z)$ in the upper half-plane takes the value $a \in \mathbb{R}$ *only once* and it does not take real values $\mathbb{R}\backslash[0, g_0]$. We conclude that $g(z)$ *cannot have maximum neither minimum at the imaginary upper semi-axis* $z' = 0$ (otherwise it would take some values twice). Hence, this function changes monotonically at the upper imaginary semi-axis of the complex plane z.
- For $g_0 = \infty$ the function $g(z)$ has a pole at $z = 0$. The proof is similar. The only difference is presence of additional small semicircle around $z = 0$ in the upper half-plane. The function at the upper imaginary semi-axis decreases from $g = \infty$ to $g = 0$.

Properties of the function $\varepsilon(z)$ follows from properties of $g(z)$ for

$$g(z) = \varepsilon(z) - 1.$$

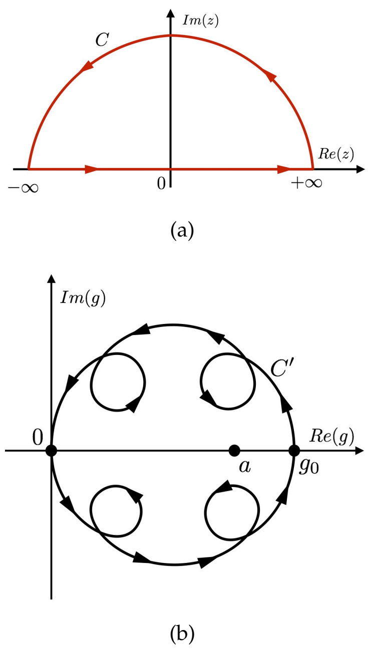


Figure 5.14: (a) The integration contour Cat the complex plane z and (b) its image C' on the complex plane g.

The Kramers-Kronig relations

Analytical properties of the function $\varepsilon(z)$ in the upper half-plane allows for obtaining some important relations between their real and imaginary parts – calculated for points lying on the real axis.

Let $g(z)$ be a complex function that has continuous complex derivative (Cauchy-Riemann equations are satisfied *i.e.* $\partial_x Re(g) - \partial_y Im(g) = 0$ and $\partial_x Im(g) + \partial_y Re(g) = 0$, where $z = x + iy$). According to the integral Cauchy theorem the integral over the border ∂D of an open set D vanishes,

$$\oint_{\partial D} g(z)dz = 0.$$

It implies the integral Cauchy formula

$$\oint_{\partial D} dz \frac{g(z)}{z - z_0} = \begin{cases} 2\pi i g(z_0) & \text{for } z_0 \in D, \\ 0 & \text{for } z_0 \notin D. \end{cases} \tag{5.298}$$

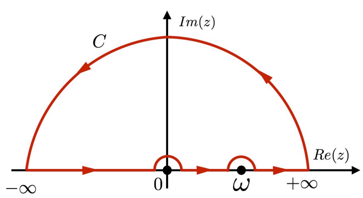


Figure 5.15: The integration contour C.

In the following part we set the function $g(z) := \varepsilon(z) - 1$ (regular in the upper half-plane) and denote the pole $z_0 := \omega \in \mathbb{R}_+$.

The integration contour C consists on the real axis ($z'' = 0$) with small semicircle(s) in the upper half plane: C_ω and C_0 (only for metals) with centers at $z = \omega$ and $z = 0$ and the semicircle with infinite radius C_∞ in the upper half-plane, see Figure 5.15. The function $\varepsilon(z) - 1$ vanishes at the infinite semicircle so the function $(\varepsilon - 1)/(z - \omega)$ tends to zero for $z \to \infty$ quicker than the function $1/z$. Hence, the integral $\oint_C dz \frac{\varepsilon(z)-1}{z-\omega}$ *converges*. Moreover, since $\varepsilon(z)$ has no poles in the upper half-plane and the point $z = \omega$ does not belong to region D then $(\varepsilon(z) - 1)/(z - \omega)$ is *analytic function*. In such a case the integral along C must vanish

$$\oint_C dz \frac{\varepsilon(z) - 1}{z - \omega} = 0.$$

The integral at the large semicircle C_∞ vanishes. In the case of *dielectrics* there is only one small semicircle is C_ω. We choose the parametrization $z = \omega + \rho e^{i\varphi}$ at C_ω and take the limit $\rho \to 0$,

$$\underbrace{\lim_{\rho \to 0} \left(\int_{-\infty}^{-\rho} + \int_{\rho}^{\infty} \right) \frac{\varepsilon(z) - 1}{z - \omega} dz}_{\mathcal{P} \int_{-\infty}^{\infty} \frac{\varepsilon(z)-1}{z-\omega} dz} + \lim_{\rho \to 0} \int_{C_\omega(\rho)} \frac{\varepsilon(z) - 1}{z - \omega} dz = 0. \tag{5.299}$$

The second integral can be split into two integrals

$$\int_{C_\omega(\rho)} \frac{\varepsilon(z) - 1}{z - \omega} dz = \int_{C_\omega(\rho)} \frac{\varepsilon(z) - \varepsilon(\omega)}{z - \omega} dz + \int_{C_\omega(\rho)} \frac{\varepsilon(\omega) - 1}{z - \omega} dz$$

where

$$\int_{C_\omega(\rho)} \frac{\varepsilon(z)-\varepsilon(\omega)}{z-\omega} dz \le \left| \int_{C_\omega(\rho)} \frac{\varepsilon(z)-\varepsilon(\omega)}{z-\omega} dz \right|$$
$$\le \int_{C_\omega(\rho)} \frac{|\varepsilon(z)-\varepsilon(\omega)|}{|z-\omega|} |dz| = \int_{C_\omega(\rho)} \frac{|\varepsilon(z)-\varepsilon(\omega)|}{|\rho e^{i\varphi}|} |\rho e^{i(\varphi+\frac{\pi}{2})} d\varphi|$$
$$= \int_{C_\omega(\rho)} |\varepsilon(z)-\varepsilon(\omega)| d\varphi \le \sup_{x\in C_\omega(\rho)} |\varepsilon(z)-\varepsilon(\omega)| \int_\pi^0 d\varphi.$$

The function $\varepsilon(z)$ is continuous in $z=\omega$ so

$$\lim_{\rho\to 0} \left(-\pi \sup_{z\in C_\omega(\rho)} |\varepsilon(z)-\varepsilon(\omega)| \right) = 0.$$

It leads to vanishing of the integral

$$\lim_{\rho\to 0} \int_{C_\omega(\rho)} \frac{\varepsilon(z)-\varepsilon(\omega)}{z-\omega} dz = 0.$$

The second integral reads

$$\lim_{\rho\to 0} \int_{C_\omega(\rho)} \frac{\varepsilon(\omega)-1}{z-\omega} dz = (\varepsilon(\omega)-1) \lim_{\rho\to 0} \int_{C_\omega(\rho)} \frac{i\rho e^{i\varphi} d\varphi}{\rho e^{i\varphi}} = -i\pi(\varepsilon(\omega)-1).$$

Then the equality (5.299) takes the form

$$\mathcal{P} \int_{-\infty}^{\infty} \frac{\varepsilon(z)-1}{z-\omega} dz - i\pi(\varepsilon(\omega)-1) = 0$$

and thus

$$\boxed{\varepsilon(\omega) - 1 = \frac{1}{i\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\varepsilon(x)-1}{x-\omega} dx} \tag{5.300}$$

where the symbol $x \in \mathbb{R}$ emphasizes that the integration is taken along the real axis. Plugging $\varepsilon = \varepsilon' + i\varepsilon''$ we get

$$(\varepsilon'(\omega)-1) + i\varepsilon''(\omega) = -\frac{i}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{(\varepsilon'(x)-1) + i\varepsilon''(x)}{x-\omega} dx.$$

It leads to two equalities

$$\varepsilon'(\omega) - 1 = \frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\varepsilon''(x)}{x-\omega} dx, \tag{5.301}$$
$$\varepsilon''(\omega) = -\frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\varepsilon'(x)-1}{x-\omega} dx. \tag{5.302}$$

where $\omega, x \in \mathbb{R}$. Functions ε' and ε'' have defined parity. The relation $\varepsilon(-z^*) = \varepsilon^*(z)$ gives

$$\varepsilon'(-x) = \varepsilon'(x), \qquad \varepsilon''(-x) = -\varepsilon''(x).$$

The integral in (5.301) can be cast in the form of integral over positive frequencies

$$\begin{aligned}\mathcal{P}\int_{-\infty}^{\infty}\frac{\varepsilon''(x)}{x-\omega}dx &= \mathcal{P}\int_{-\infty}^{0}\frac{\varepsilon''(x)}{x-\omega}dx+\mathcal{P}\int_{0}^{\infty}\frac{\varepsilon''(x)}{x-\omega}dx\\ &= \mathcal{P}\int_{\infty}^{0}\frac{-\varepsilon''(x)}{-x-\omega}(-dx)+\mathcal{P}\int_{0}^{\infty}\frac{\varepsilon''(x)}{x-\omega}dx\\ &= \mathcal{P}\int_{0}^{\infty}\left(\frac{x-\omega}{x^2-\omega^2}+\frac{x+\omega}{x^2-\omega^2}\right)\varepsilon''(x)dx\\ &= 2\mathcal{P}\int_{0}^{\infty}\frac{x\varepsilon''(x)}{x^2-\omega^2}dx.\end{aligned}$$

In a similar way we transform the integral in (5.302)

$$\begin{aligned}\mathcal{P}\int_{-\infty}^{\infty}\frac{\varepsilon'(x)-1}{x-\omega}dx &= \mathcal{P}\int_{-\infty}^{0}\frac{\varepsilon'(x)-1}{x-\omega}dx+\mathcal{P}\int_{0}^{\infty}\frac{\varepsilon'(x)-1}{x-\omega}dx\\ &= \mathcal{P}\int_{\infty}^{0}\frac{\varepsilon'(x)-1}{-x-\omega}(-dx)+\mathcal{P}\int_{0}^{\infty}\frac{\varepsilon'(x)-1}{x-\omega}dx\\ &= \mathcal{P}\int_{0}^{\infty}\left(\frac{x-\omega}{x^2-\omega^2}-\frac{x+\omega}{x^2-\omega^2}\right)(\varepsilon'(x)-1)dx\\ &= 2\omega\mathcal{P}\int_{0}^{\infty}\frac{\varepsilon'(x)-1}{x^2-\omega^2}.\end{aligned}$$

It leads to *Kramers-Kronig relations*

Kramers-Kronig relations

$$\boxed{\varepsilon'(\omega)=1+\frac{2}{\pi}\mathcal{P}\int_{0}^{\infty}\frac{x\varepsilon''(x)}{x^2-\omega^2}dx,}\tag{5.303}$$

$$\boxed{\varepsilon''(\omega)=-\frac{2\omega}{\pi}\mathcal{P}\int_{0}^{\infty}\frac{\varepsilon'(x)-1}{x^2-\omega^2}dx.}\tag{5.304}$$

The above relations have been derived for dielectrics. In order to get relations that are valid for metals we observe that $\varepsilon(z)$ have singular part $i\frac{4\pi\sigma}{\omega}$ at $\omega\to 0$

$$\varepsilon(\omega)=i\frac{4\pi\sigma}{\omega}+\tilde{\varepsilon}(\omega)\tag{5.305}$$

where $\tilde{\varepsilon}(\omega)$ is some *regular term*. The singular term affects only the imaginary part ε'. Thus

$$\tilde{\varepsilon}'(\omega)=\varepsilon(\omega),\qquad \tilde{\varepsilon}''(\omega)=\varepsilon''(\omega)-i\frac{4\pi\sigma}{\omega}.$$

Substituting regular expressions

$$\varepsilon'\to\tilde{\varepsilon}'=\varepsilon',\qquad \varepsilon''\to\tilde{\varepsilon}''=\varepsilon''-\frac{4\pi\sigma}{\omega}$$

into (5.301) and (5.302) one gets

$$\boxed{\varepsilon'(\omega)=1+\frac{2}{\pi}\mathcal{P}\int_{0}^{\infty}\frac{x\varepsilon''(x)}{x^2-\omega^2}dx,}\tag{5.306}$$

and

$$\boxed{\varepsilon''(\omega) = \frac{4\pi\sigma}{\omega} - \frac{2\omega}{\pi}\mathcal{P}\int_0^\infty \frac{\varepsilon'(x)-1}{x^2-\omega^2}dx.} \tag{5.307}$$

The second relation (5.307) follows immediately from (5.302). On the other hand, the first relation (5.306) requires more careful treatment. The integral in (5.301) reads

$$\mathcal{P}\int_0^\infty \frac{x\left(\varepsilon''(x) - \frac{4\pi\sigma}{x}\right)}{x^2-\omega^2}dx = \mathcal{P}\int_0^\infty \frac{x\varepsilon''(x)}{x^2-\omega^2}dx - 4\pi\sigma\mathcal{P}\int_0^\infty \frac{dx}{x^2-\omega^2},$$

where

$$\begin{aligned}
\mathcal{P}\int_0^\infty \frac{dx}{x^2-\omega^2} &= \frac{1}{2\omega}\mathcal{P}\int_0^\infty \left(\frac{1}{x-\omega} - \frac{1}{x+\omega}\right)dx \\
&= \frac{1}{2\omega}\lim_{R\to\infty}\mathcal{P}\int_0^R \left(\frac{1}{x-\omega} - \frac{1}{x+\omega}\right)dx \\
&= \frac{1}{2\omega}\lim_{R\to\infty}\lim_{\eta\to 0}\left(\int_0^{\omega-\eta}\frac{1}{x-\omega} + \int_{\omega+\eta}^R \frac{1}{x-\omega}\right)dx \\
&- \frac{1}{2\omega}\lim_{R\to\infty}\int_0^R \frac{dx}{x+\omega} \\
&= \frac{1}{2\omega}\lim_{R\to\infty}\lim_{\eta\to 0}\Big(\ln(\omega-\eta-\omega) - \ln(-\omega) + \ln(R-\omega) - \\
&- \ln(\omega+\eta-\omega)\Big) - \frac{1}{2\omega}\lim_{R\to\infty}[\ln(R_\omega) - \ln(\omega)] \\
&= \frac{1}{2\omega}\lim_{R\to\infty}\lim_{\eta\to 0}\ln\left(\frac{-\eta(R-\omega)}{\eta(-\omega)}\right) - \frac{1}{2\omega}\lim_{R\to\infty}\ln\left(\frac{R+\omega}{\omega}\right) \\
&= \frac{1}{2\omega}\lim_{R\to\infty}\ln\left(\frac{R-\omega}{R+\omega}\right) = 0.
\end{aligned} \tag{5.308}$$

Formulas (5.301) and (5.306) are especially important because they allows to calculate function $\varepsilon'(\omega)$ if $\varepsilon''(\omega)$ is known (at least empirically). The function $\varepsilon''(\omega)$ can be determined experimentally by the measure of absorption.

5.8 *Waveguides*

Waveguides are formed by conducting tubes filled with dielectric materials. In general such can have magnetic properties. The perpendicular cross section of the waveguide is not necessary a symmetric figure.

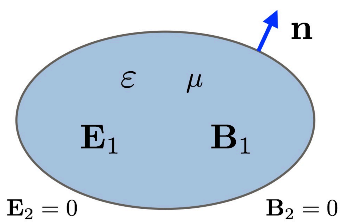


Figure 5.16: A waveguide cross section.

The electric and magnetic field exist only inside the waveguide

$$\boldsymbol{E}_1 \equiv \boldsymbol{E}, \qquad \boldsymbol{B}_1 \equiv \boldsymbol{B}, \qquad \boldsymbol{D}_1 \equiv \boldsymbol{D}, \qquad \boldsymbol{H}_1 \equiv \boldsymbol{H}$$

and

$$\boldsymbol{E}_2 = 0, \qquad \boldsymbol{B}_2 = 0, \qquad \boldsymbol{D}_2 = 0, \qquad \boldsymbol{H}_2 = 0.$$

The field intensities and inductions are related by constitutive relations $\boldsymbol{D} = \varepsilon \boldsymbol{E}$ and $\boldsymbol{B} = \mu \boldsymbol{H}$.

Let $\hat{\boldsymbol{n}}$ be a vector normal to the waveguide surface pointing outside the waveguide. The fields satisfy the boundary conditions at the waveguide surface S, namely

$$\boldsymbol{D} \cdot \hat{\boldsymbol{n}} = -4\pi\sigma, \qquad \boldsymbol{B} \cdot \hat{\boldsymbol{n}} = 0, \tag{5.309}$$

$$\hat{\boldsymbol{n}} \times \boldsymbol{E} = 0, \qquad \hat{\boldsymbol{n}} \times \boldsymbol{H} = -\frac{4\pi}{c}\boldsymbol{K}. \tag{5.310}$$

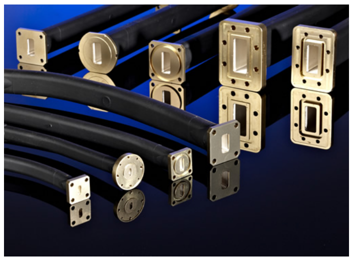


Figure 5.17: Waveguides

Maxwell's equations imply that electric and magnetic field obey wave equation $\partial^2 \boldsymbol{E} = 0$ and $\partial^2 \boldsymbol{B} = 0$ where $\partial^2 := \frac{\varepsilon\mu}{c^2}\partial_t^2 - \nabla^2$. We choose the axis x^3 as being parallel to the longitudinal axis of the waveguide. The electric and magnetic field can be considered in the following form

$$\boldsymbol{E}(t, \boldsymbol{r}) = \boldsymbol{E}(x^1, x^2) e^{i(\pm kx^3 - \omega t)}, \tag{5.311}$$

$$\boldsymbol{B}(t, \boldsymbol{r}) = \boldsymbol{B}(x^1, x^2) e^{i(\pm kx^3 - \omega t)}. \tag{5.312}$$

We shall adopt convention that, if not written explicitly, vectors $\boldsymbol{E}(x^1, x^2) \equiv \boldsymbol{E}$ and $\boldsymbol{B}(x^1, x^2) \equiv \boldsymbol{B}$ are denoted by symbols without arguments.

Plugging this expression into wave equation we get equations for $\boldsymbol{E}(x^1, x^2)$ and $\boldsymbol{B}(x^1, x^2)$. The wave operator acting on electric field gives

Figure 5.18: Coaxial cable

$$\begin{aligned} -\partial^2 \boldsymbol{E}(t, \boldsymbol{r}) &= \left[\nabla_\perp^2 + \partial_3^3 - \frac{\varepsilon\mu}{c^2}\partial_t^2\right] \left(\boldsymbol{E}(x^1, x^2) e^{i(\pm kx^3 - \omega t)}\right) \\ &= \left[\nabla_\perp^2 + \gamma^2\right] \boldsymbol{E}(x^1, x^2) e^{i(\pm kx^3 - \omega t)}, \end{aligned} \tag{5.313}$$

and similarly for the magnetic field, where

$$\boxed{\gamma^2 := \varepsilon\mu \frac{\omega^2}{c^2} - k^2.} \tag{5.314}$$

It gives

$$\begin{aligned} \left[\nabla_\perp^2 + \gamma^2\right] \boldsymbol{E}(x^1, x^2) &= 0, \\ \left[\nabla_\perp^2 + \gamma^2\right] \boldsymbol{B}(x^1, x^2) &= 0. \end{aligned} \tag{5.315}$$

Parallel and transversal components

The electric and magnetic field amplitudes $\boldsymbol{E}(x^1, x^2)$, $\boldsymbol{B}(x^1, x^2)$ can be decomposed on two components which are perpendicular and parallel to the x^3 axis,

$$\begin{aligned} \boldsymbol{E}(t, \boldsymbol{r}) &= \underbrace{E^1 \hat{\boldsymbol{e}}_1 + E^2 \hat{\boldsymbol{e}}_2}_{\boldsymbol{E}_\perp} + \underbrace{E^3 \hat{\boldsymbol{e}}_3}_{\boldsymbol{E}_\parallel}, \\ \boldsymbol{B}(t, \boldsymbol{r}) &= \underbrace{B^1 \hat{\boldsymbol{e}}_1 + B^2 \hat{\boldsymbol{e}}_2}_{\boldsymbol{B}_\perp} + \underbrace{B^3 \hat{\boldsymbol{e}}_3}_{\boldsymbol{B}_\parallel}. \end{aligned} \tag{5.316}$$

Such decomposition is very useful. In some cases (transverse electric TE and transverse magnetic TM waves) the perpendicular $\boldsymbol{E}_\perp$, $\boldsymbol{B}_\perp$ and parallel components satisfy some algebraic relations. It follows that the perpendicular components can be *determined from the parallel components*. The parallel components E^3 and B^3 are determined by equations (5.315). We shall present Maxwell's equations in terms of perpendicular and parallel components. The gradient operator is split as follows $\nabla = \nabla_\perp + \hat{\boldsymbol{e}}_3 \partial_3$.

- **Faraday's law** takes the form

$$\nabla_\perp \times \boldsymbol{E}(t,\boldsymbol{r}) + \underbrace{\hat{\boldsymbol{e}}_3 \times \partial_3 \boldsymbol{E}(t,\boldsymbol{r})}_{\pm ik\hat{\boldsymbol{e}}_3 \times \boldsymbol{E}(t,\boldsymbol{r})} = -\frac{1}{c} \underbrace{\partial_t \boldsymbol{B}(t,\boldsymbol{r})}_{-i\omega \boldsymbol{B}(t,\boldsymbol{r})}. \tag{5.317}$$

  It follows that expression $e^{i(\pm kx^3 - \omega t)}$ became just multiplicative factor, then the amplitude vectors $\boldsymbol{E}(x^1, x^2)$ and $\boldsymbol{B}(x^1, x^2)$ satisfy equation

$$\nabla_\perp \times \boldsymbol{E} \pm ik\, \hat{\boldsymbol{e}}_3 \times \boldsymbol{E} = i\frac{\omega}{c}\boldsymbol{B}.$$

  Plugging decomposition (5.316) into the above equation one gets

$$\nabla_\perp \times \boldsymbol{E}_\parallel + \nabla_\perp \times \boldsymbol{E}_\perp \pm ik\, \hat{\boldsymbol{e}}_3 \times \boldsymbol{E}_\perp = i\frac{\omega}{c}(\boldsymbol{B}_\parallel + \boldsymbol{B}_\perp). \tag{5.318}$$

  Taking the scalar product of (5.318) and $\hat{\boldsymbol{e}}_3$ and considering that $\hat{\boldsymbol{e}}_3 \cdot (\nabla_\perp \times \boldsymbol{E}_\parallel) = 0$ and $\hat{\boldsymbol{e}}_3 \cdot (\hat{\boldsymbol{e}}_3 \times \boldsymbol{E}_\perp) = 0$ we obtain

$$\boxed{\hat{\boldsymbol{e}}_3 \cdot (\nabla_\perp \times \boldsymbol{E}_\perp) = i\frac{\omega}{c} B^3.} \tag{5.319}$$

  Next, taking the cross product of (5.318) with $\hat{\boldsymbol{e}}_3$ and taking into account relations $\hat{\boldsymbol{e}}_3 \times (\nabla_\perp \times \boldsymbol{E}_\perp) = 0$ and $\hat{\boldsymbol{e}}_3 \times (\hat{\boldsymbol{e}}_3 \times \boldsymbol{E}_\perp) = -\boldsymbol{E}_\perp$ and

$$\begin{aligned} \hat{\boldsymbol{e}}_3 \times (\nabla_\perp \times \boldsymbol{E}_\parallel) &= \hat{\boldsymbol{e}}_3 \times (\hat{\boldsymbol{e}}_1 \times \hat{\boldsymbol{e}}_3 \partial_1 E^3 + \hat{\boldsymbol{e}}_2 \times \hat{\boldsymbol{e}}_3 \partial_2 E^3) \\ &= \hat{\boldsymbol{e}}_3 \times (-\hat{\boldsymbol{e}}_2 \partial_1 E^3 + \hat{\boldsymbol{e}}_1 \partial_2 E^3) = \hat{\boldsymbol{e}}_1 \partial_1 E^3 + \hat{\boldsymbol{e}}_2 \partial_2 E^3 \\ &= \nabla_\perp E^3 \end{aligned}$$

  we get equation

$$\boxed{\nabla_\perp E^3 \mp ik\, \boldsymbol{E}_\perp = i\frac{\omega}{c} \hat{\boldsymbol{e}}_3 \times \boldsymbol{B}_\perp.} \tag{5.320}$$

- For **Ampere-Maxwell's** law we obtain

$$\nabla_\perp \times \boldsymbol{B}(t,\boldsymbol{r}) + \underbrace{\hat{\boldsymbol{e}}_3 \times \partial_3 \boldsymbol{B}(t,\boldsymbol{r})}_{\pm ik\hat{\boldsymbol{e}}_3 \times \boldsymbol{B}(t,\boldsymbol{r})} = \frac{\varepsilon\mu}{c} \underbrace{\partial_t \boldsymbol{E}(t,\boldsymbol{r})}_{-i\omega \boldsymbol{E}(t,\boldsymbol{r})}. \tag{5.321}$$

  Then

$$\nabla_\perp \times \boldsymbol{B}_\parallel + \nabla_\perp \times \boldsymbol{B}_\perp \pm ik\, \hat{\boldsymbol{e}}_3 \times \boldsymbol{B}_\perp = -i\varepsilon\mu\frac{\omega}{c}(\boldsymbol{E}_\parallel + \boldsymbol{E}_\perp). \tag{5.322}$$

Taking the scalar product of (5.322) and $\hat{e}_3$ one gets equation

$$\boxed{\hat{e}_3 \cdot (\nabla_\perp \times \boldsymbol{B}_\perp) = -i\varepsilon\mu \frac{\omega}{c} E^3.} \tag{5.323}$$

The cross product yields

$$\boxed{\nabla_\perp B^3 \mp ik\, \boldsymbol{B}_\perp = -i\varepsilon\mu \frac{\omega}{c} \hat{e}_3 \times \boldsymbol{E}_\perp.} \tag{5.324}$$

- Finally for electric and magnetic **Gauss law** one gets

$$\boxed{\nabla_\perp \cdot \boldsymbol{E}_\perp = \mp ik\, E^3,} \qquad \boxed{\nabla_\perp \cdot \boldsymbol{B}_\perp = \mp ik\, B^3.} \tag{5.325}$$

Relations between amplitudes for $\gamma^2 \neq 0$

In order to obtain $\boldsymbol{E}_\perp$ we take left multiplication of (5.324) by $\hat{e}_3$ which gives

$$\hat{e}_3 \times (\nabla_\perp B^3) \mp ik\, \hat{e}_3 \times \boldsymbol{B}_\perp = -i\varepsilon\mu \frac{\omega}{c} \underbrace{\hat{e}_3 \times (\hat{e}_3 \times \boldsymbol{E}_\perp)}_{-\boldsymbol{E}_\perp}. \tag{5.326}$$

Substituting $\hat{e}_3 \times \boldsymbol{B}_\perp = \frac{c}{i\omega}(\nabla_\perp E^3 \mp ik\, \boldsymbol{E}_\perp)$, obtained from (5.320), into (5.326) and multiplying by $-i\frac{\omega}{c}$ we obtain

$$\boxed{\boldsymbol{E}_\perp = \frac{i}{\gamma^2}\left[\pm k\, \nabla_\perp E^3 - \frac{\omega}{c} \hat{e}_3 \times \nabla_\perp B^3\right],} \tag{5.327}$$

where in the last step divide by γ^2. This relation is true unless $\gamma^2 = 0$ *i.e.* when $\varepsilon\mu\frac{\omega^2}{c^2} = k^2$. Similar result can be obtained for $\boldsymbol{B}_\perp$. In this case we take left cross product of (5.320) with $\hat{e}_3$

$$\hat{e}_3 \times (\nabla_\perp E^3) \mp ik\, \hat{e}_3 \times \boldsymbol{E}_\perp = i\frac{\omega}{c} \underbrace{\hat{e}_3 \times (\hat{e}_3 \times \boldsymbol{B}_\perp)}_{-\boldsymbol{B}_\perp}. \tag{5.328}$$

and substitute $\hat{e}_3 \times \boldsymbol{E}_\perp = -\frac{c}{i\omega\varepsilon\mu}(\nabla_\perp B^3 \mp ik\, \boldsymbol{B}_\perp)$ obtained from (5.324). Multiplying the result by $i\varepsilon\mu\frac{\omega}{c}$ and and dividing by γ^2 we obtain

$$\boxed{\boldsymbol{B}_\perp = \frac{i}{\gamma^2}\left[\pm k\, \nabla_\perp B^3 + \varepsilon\mu \frac{\omega}{c} \hat{e}_3 \times \nabla_\perp E^3\right].} \tag{5.329}$$

Note that in this case the amplitudes $\boldsymbol{E}_\perp$ and $\boldsymbol{B}_\perp$ are uniquely determined by components E^3 and B^3.

TE and TM waves

The electromagnetic field takes non-zero values only inside the waveguide. The electric field $\boldsymbol{E}$ satisfy the equation $(\nabla_\perp^2 + \gamma^2)\boldsymbol{E} = 0$ (and similarly for $\boldsymbol{B}$). Moreover, since the waveguide surface S is a conductor then tangential component of the electric field *must vanish at this surface*

$E^3|_S = 0$. Taking the scalar product of the equation (5.324) with the vector $\hat{\boldsymbol{n}}$ normal to the waveguide surface we obtain

$$\underbrace{\hat{\boldsymbol{n}} \cdot (\nabla_\perp B^3)}_{\frac{\partial B^3}{\partial n}} \mp ik \underbrace{\hat{\boldsymbol{n}} \cdot \boldsymbol{B}_\perp}_{\hat{\boldsymbol{n}} \cdot \boldsymbol{B}} = -i\varepsilon\mu\frac{\omega}{c} \underbrace{\hat{\boldsymbol{n}} \cdot (\hat{\boldsymbol{e}}_3 \times \boldsymbol{E}_\perp)}_{-\hat{\boldsymbol{e}}_3 \cdot (\hat{\boldsymbol{n}} \times \boldsymbol{E})}. \tag{5.330}$$

At the surface S where $\hat{\boldsymbol{n}} \cdot \boldsymbol{B}|_S = 0$ and $\hat{\boldsymbol{n}} \times \boldsymbol{E}|_S = 0$ the equation (5.330) implies that

$$\left.\frac{\partial B^3}{\partial n}\right|_S = 0.$$

Any solution representing electromagnetic wave that propagates in the waveguide must satisfy the following boundary conditions

$$\boxed{\hat{\boldsymbol{n}} \cdot \boldsymbol{B}|_S = 0, \qquad \hat{\boldsymbol{n}} \times \boldsymbol{E}|_S = 0, \qquad E^3|_S = 0, \qquad \left.\frac{\partial B^3}{\partial n}\right|_S = 0.} \tag{5.331}$$

- *Transverse electric waves TE* is a group of electromagnetic waves such that $E^3 = 0$ *everywhere*. The relation between $\boldsymbol{E}_\perp$ and $\boldsymbol{B}_\perp$ for TE waves follows directly from (5.320) and it reads

$$\boldsymbol{E}_\perp = \mp\frac{\omega}{kc}\hat{\boldsymbol{e}}_3 \times \boldsymbol{B}_\perp \qquad \Leftrightarrow \qquad \boldsymbol{B}_\perp = \pm\frac{kc}{\omega}\hat{\boldsymbol{e}}_3 \times \boldsymbol{E}_\perp. \tag{5.332}$$

  The equation (5.329) gives

$$\boldsymbol{B}_\perp = \pm\frac{ik}{\gamma^2}\nabla_\perp B^3, \tag{5.333}$$

  where B^3 is the solution of $(\nabla_\perp^2 + \gamma^2)B^3 = 0$. This solution must satisfy the Neumann boundary condition $\frac{\partial B^3}{\partial n}|_S = 0$. The particular form of the solution depends on the waveguide geometry.

- *Transverse magnetic waves TM* are solutions such that $B^3 = 0$ *everywhere*. The equation (5.324) gives

$$\boldsymbol{B}_\perp = \pm\varepsilon\mu\frac{\omega}{kc}\hat{\boldsymbol{e}}_3 \times \boldsymbol{E}_\perp \qquad \Leftrightarrow \qquad \boldsymbol{E}_\perp = \mp\frac{kc}{\varepsilon\mu\,\omega}\hat{\boldsymbol{e}}_3 \times \boldsymbol{B}_\perp, \tag{5.334}$$

  where $\boldsymbol{E}_\perp$ follows from (5.327) and it reads

$$\boldsymbol{E}_\perp = \pm\frac{ik}{\gamma^2}\nabla_\perp E^3. \tag{5.335}$$

  The transverse component of the electric field is obtained from $(\nabla_\perp^2 + \gamma^2)E^3 = 0$ with the Dirichlet boundary condition $E^3|_S = 0$.

Transverse electromagnetic waves TEM

Transverse electromagnetic waves TEM are solutions such that $E^3 = 0$ and $B^3 = 0$ **everywhere**. Equations (5.319) and (5.323) take the form

$$\hat{\boldsymbol{e}}_3 \cdot (\nabla_\perp \times \boldsymbol{E}_\perp) = 0, \qquad \hat{\boldsymbol{e}}_3 \cdot (\nabla_\perp \times \boldsymbol{B}_\perp) = 0. \tag{5.336}$$

From (5.325) we have

$$\nabla_\perp \cdot \boldsymbol{E}_\perp = 0, \qquad \nabla_\perp \cdot \boldsymbol{B}_\perp = 0. \tag{5.337}$$

The electric field in Cartesian coordinates satisfies equations

$$\partial_1 E^2 - \partial_2 E^1 = 0, \qquad \partial_1 E^1 + \partial_2 E^2 = 0. \tag{5.338}$$

Taking derivative of the first equation with respect to x^1 and using the second equation we obtain $(\partial_1^2 + \partial_2^2)E^2 = 0$. Similarly, taking derivative with respect to x^2 one gets $(\partial_1^2 + \partial_2^2)E^1 = 0$. Repeating the same steps with respect to the magnetic field one gets identical equations, thus

$$\nabla_\perp^2 \boldsymbol{E}_\perp = 0, \qquad \nabla_\perp^2 \boldsymbol{B}_\perp = 0. \tag{5.339}$$

Comparing (5.339) and (5.315) one can conclude that

$$\gamma^2 = \varepsilon\mu\frac{\omega^2}{c^2} - k^2 = 0.$$

It means that dispersion relation for electromagnetic waves TEM has the same form as for waves in infinite space

$$k = n\frac{\omega}{c}. \tag{5.340}$$

The algebraic relations between electric and magnetic fields follow directly from (5.320) and (5.324). They have the form

$$\boldsymbol{B} = \pm n\, \hat{\boldsymbol{e}}_3 \times \boldsymbol{E}_\perp, \qquad \boldsymbol{E} = \mp\frac{1}{n}\, \hat{\boldsymbol{e}}_3 \times \boldsymbol{B}_\perp. \tag{5.341}$$

Another important property of TEM waves is fact that such waves *cannot exist in hollow waveguides*. The expression $\nabla_\perp \times \boldsymbol{E}_\perp$ has only component parallel to x^3, thus the equation $\hat{\boldsymbol{e}}_3 \cdot (\nabla_\perp \times \boldsymbol{E}_\perp) = 0$ is equivalent to $\nabla_\perp \times \boldsymbol{E}_\perp = 0$. However, since $\boldsymbol{E}_\perp$ is a function of two spatial variables then

$$\nabla_\perp \times \boldsymbol{E}_\perp = 0 \qquad \Rightarrow \qquad \boldsymbol{E}_\perp = -\nabla\varphi(x^1, x^2). \tag{5.342}$$

Plugging this expression into $\nabla_\perp \cdot \boldsymbol{E}_\perp = 0$ we obtain Laplace equation in two dimensions φ

$$\nabla_\perp^2 \varphi = 0. \tag{5.343}$$

Since the electric field vanishes on S, then the potential $\varphi = const$ on S. The Laplace equation has only trivial *constant solution* in such a case. Then $\boldsymbol{E}_\perp = 0$ inside the waveguide. For this reason there are no electromagnetic TEM waves in hollow waveguides.

Chapter 6
Electromagnetic radiation

6.1 Fundamental solutions of the wave equation

Maxwell's equations with sources

The electromagnetic field in the presence of external (prescribed) four-currents $J^\mu(x)$ satisfies Maxwell's equations

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu. \tag{6.1}$$

Imposing the Lorenz[1] gauge condition $\partial_\mu A^\mu(x) = 0$ one gets (6.1) in the form of the inhomogeneous wave equation

[1] Ludvig Lorenz.

$$\partial^2 A^\mu(x) = \frac{4\pi}{c} J^\mu(x). \tag{6.2}$$

Since equation (6.2) is *linear* then its general solution is a sum of *general* solution of the homogeneous wave equation $\partial^2 A^\mu(x) = 0$ and any *particular* solution of the inhomogeneous equation (6.2)

$$A^\mu(x) = A^\mu_{hom}(x) + A^\mu_{part}(x).$$

Particular solutions of (6.2) can be cast in the form[2]

[2] This is the case of continuous superposition.

$$A^\mu_{part}(x) = \frac{4\pi}{c} \int d^4x'\, D(x - x') J^\mu(x'). \tag{6.3}$$

Particular solution of the inhomogeneous equation

The function $D(x - x')$ shall be determined in the next section. The expression (6.3) must satisfy equation (6.2), which is possible only if

$$\partial^2 D(x) = \delta^4(x). \tag{6.4}$$

Indeed, plugging (6.3) into (6.2) one gets

$$\partial^2 A^\mu_{part}(x) = \frac{4\pi}{c} \int d^4x' \underbrace{\partial^2 D(x - x')}_{\delta^4(x-x')} J^\mu(x') = \frac{4\pi}{c} J^\mu(x) \tag{6.5}$$

The function $D(x)$ is termed *fundamental solution* of the wave equation. Its precise mathematical meaning (as well as the meaning of $\delta^4(x)$) is given in the context of *generalized functions (distributions)*. [3] In language of distributions the fundamental solution for the d'Alembert operator satisfies the equation

[3] Distributions act on test functions as linear functionals $(f, \varphi) := \int d^n x f(x)\varphi(x)$. Test functions are $\mathcal{C}^\infty$. They vanish at infinity together with all their derivatives.

$$\boxed{(\partial^2 D(x), \varphi(x)) = (\delta^4(x), \varphi(x))} \tag{6.6}$$

which means that the equality of $\partial^2 D(x), \varphi(x))$ and $(\delta^4(x), \varphi(x))$ is required but not the equality of $\partial^2 D(x)$ and $\delta^4(x)$. Thus, the solution of the inhomogeneous wave equation has been reduced to the problem which does not depend on a particular form of four-current density J^μ. Note that the four-vector x' can be omitted in the equation (6.6) because the differential operator ∂^2 act on x.[4]

[4] Alternatively, one can define new variable $z := x - x'$.

Fundamental solution of the wave equation in 3+1 dimensions

Problem with poles in the inverse Fourier transform

The equation (6.6) can be solved by means of the Fourier transform. The transform with respect to all four space-time coordinates yields an algebraic equation. The solution of this equation has simple poles and thus it cannot be plugged directly into the Fourier integral. This problem is usually solved by replacing the real integration variable by complex variable and prescribing the integration in means of contour integral on a complex plane.

Fourier transform with respect to spatial variables x^1, x^2 and x^3

We shall derive fundamental solution using an alternative approach *i.e.* taking the Fourier transform with respect to spatial coordinates x^1, x^2 and x^3 (omitting x^0). Plugging the Fourier integral

$$D(x^0, \boldsymbol{x}) = F_x^{-1}[\widetilde{D}(x^0, \boldsymbol{k})] = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} d^3k \, \widetilde{D}(x^0, \boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

and

$$\delta^4(x) = \delta(x^0)\delta^3(\boldsymbol{x}) = \delta(x^0)\frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} d^3k \, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} d^3k \, \delta(x^0) e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

into the equation (6.6) one gets

$$\frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} d^3k \, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \left(\partial_0^2 \widetilde{D}(x^0, \boldsymbol{k}) + \boldsymbol{k}^2 \widetilde{D}(x^0, \boldsymbol{k}) - \delta(x^0) \right) = 0 \tag{6.7}$$

i.e. the equation (6.4) takes the form of the harmonic oscillator equation with the Dirac delta force

$$\partial_0^2 \widetilde{D}(x^0, \boldsymbol{k}) + \boldsymbol{k}^2 \widetilde{D}(x^0, \boldsymbol{k}) = \delta(x^0). \tag{6.8}$$

The equations (6.7) and (6.8) are considered as *distributional equations*. The Fourier transform[5] of distributions gives distributions which act on test functions φ according to

[5] More precisely: the generalized Fourier transform

Distributional version of (6.7) and (6.8)

$$\left(F_x^{-1}\left[(\partial_0^2 + \boldsymbol{k}^2) \widetilde{D}(x^0, \boldsymbol{k}) \right], \varphi(x^0, \boldsymbol{x}) \right) = \left(F_x^{-1}\left[\delta(x^0) \right], \varphi(x^0, \boldsymbol{x}) \right).$$

This equation can be cast in the form

$$\Big((\partial_0^2 + \boldsymbol{k}^2)\widetilde{D}(x^0,\boldsymbol{k}), \widetilde{\varphi}(x^0,\boldsymbol{k})\Big) = \Big(\delta(x^0), \widetilde{\varphi}(x^0,\boldsymbol{k})\Big), \tag{6.9}$$

where $\widetilde{\varphi}(x^0,\boldsymbol{k}) \equiv F_x^{-1}[\varphi](x^0,\boldsymbol{k}))$ is a test function. The equation (6.9) is a distributional version of (6.8).

The solution of (6.8) is called *fundamental solution of the harmonic oscillator* and it reads

The fundamental solution of harmonic oscillator equation

$$\boxed{\widetilde{D}_{ret/adv}(x^0,\boldsymbol{k}) = \pm\theta(\pm x^0)\frac{\sin(kx^0)}{k},} \tag{6.10}$$

where $k = |\boldsymbol{k}|$. $\widetilde{D}_{ret}$ and $\widetilde{D}_{adv}$ are, respectively, *retarded* (upper sign) and *advanced* (bottom sign) solutions.

We shall show that $\widetilde{D}_{ret}$ is the solution in distributional sense. Denoting for convenience $\xi \equiv x^0$ and $Z(\xi) \equiv \frac{\sin(k\xi)}{k}$ and using definition of generalized derivative of distributions

Generalized derivative of a distribution

$$(f^{(n)}(\xi), \varphi(\xi)) := (-1)^n (f(\xi), \varphi^{(n)}(\xi))$$

one gets

Proof for the retarded function

$$\begin{aligned}
(\widetilde{D}'' + k^2\widetilde{D}, \varphi) &= (\widetilde{D}, \varphi'') + (k^2\widetilde{D}, \varphi) = \\
&= \int_{-\infty}^{\infty} d\xi\, \theta(\xi)Z(\xi)\varphi''(\xi) + \int_{-\infty}^{\infty} d\xi\, \theta(\xi)k^2 Z(\xi)\varphi(\xi) \\
&= \int_0^{\infty} d\xi\, Z(\xi)\varphi''(\xi) + \int_0^{\infty} d\xi\, k^2 Z(\xi)\varphi(\xi) = \cdots
\end{aligned}$$

Integration by parts gives

$$\begin{aligned}
\cdots &= Z(\xi)\varphi'(\xi)\Big|_0^{\infty} - \int_0^{\infty} d\xi Z'(\xi)\varphi'(\xi) + \int_0^{\infty} d\xi\, k^2 Z(\xi)\varphi(\xi) = \\
&= Z(\xi)\varphi'(\xi)\Big|_0^{\infty} - Z'(\xi)\varphi(\xi)\Big|_0^{\infty} + \int_0^{\infty} d\xi\, \underbrace{(Z''(\xi) + k^2 Z(\xi))}_{0}\varphi(\xi).
\end{aligned}$$

Since test functions and all their derivatives vanish at infinity $\varphi(\infty) \equiv 0$, $\varphi^{(n)}(\infty) \equiv 0$ and $Z(0) = 0$ and $Z'(0) = 1$ then

$$(\widetilde{D}'' + k^2\widetilde{D}, \varphi) = \varphi(0) = (\delta(\xi), \varphi(\xi)), \tag{6.11}$$

where the second equality reflects the definition of the $\delta-$distribution. The proof for the advanced function is almost identical. In such a case we take $\widetilde{D}(\xi) = -\theta(-\xi)Z(\xi)$ which gives

$(\delta(x), \varphi(x)) := \varphi(0)$

Proof for the advanced function

$$\begin{aligned}
(\widetilde{D}'' + k^2\widetilde{D}, \varphi) &= -\int_{-\infty}^{0} d\xi\, Z(\xi)\varphi''(\xi) - \int_{-\infty}^{0} d\xi\, k^2 Z(\xi)\varphi(\xi) \\
&= -Z(\xi)\varphi'(\xi)\Big|_{-\infty}^{0} + Z'(\xi)\varphi(\xi)\Big|_{-\infty}^{0} - \\
&\quad - \int_{-\infty}^{0} d\xi\, \underbrace{(Z''(\xi) + k^2 Z(\xi))}_{0}\varphi(\xi) \\
&= \varphi(0) = (\delta(\xi), \varphi(\xi)).
\end{aligned}$$

For explicitly given the function $\widetilde{D}_{ret/adv}(x^0, \boldsymbol{k})$ one gets solutions of (6.6) as the inverse Fourier transform

The inverse Fourier transform of $\widetilde{D}_{ret/adv}(x^0, \boldsymbol{k})$

$$\begin{aligned} D_{ret/adv}(x^0, \boldsymbol{x}) &= \frac{1}{(2\pi)^3} \int d^3k \, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \widetilde{D}_{ret/adv}(x^0, \boldsymbol{k}) \\ &= \pm \frac{\theta(\pm x^0)}{(2\pi)^3} \int d^3k \, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \frac{\sin(kx^0)}{k}. \end{aligned}$$

We choose spherical coordinates in such a way that the vector $\boldsymbol{k}$ is given by (k, ϑ, φ), where $\vartheta = 0$ stands for $\boldsymbol{k}$ colinear with $\boldsymbol{x}$. It gives

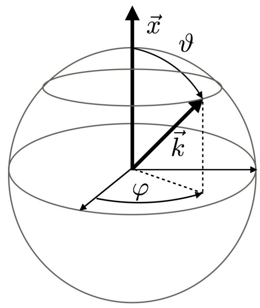


Figure 6.1: Spherical coordinates in the space k^1, k^2, k^3.

$$d^3k = k^2 \sin\vartheta \, dk d\vartheta d\varphi, \qquad \boldsymbol{k}\cdot\boldsymbol{x} = kr\cos\vartheta, \qquad r \equiv |\boldsymbol{x}|.$$

In terms of the variable $u := \cos\vartheta$ one gets

$$\begin{aligned} D_{ret/adv}(x^0, \boldsymbol{x}) &= \pm \frac{\theta(\pm x^0)}{(2\pi)^3} \int_0^{2\pi} d\varphi \int_0^\infty k^2 dk \, \frac{\sin(kx^0)}{k} \int_{-1}^1 du \, e^{ikru} \\ &= \pm \frac{\theta(\pm x^0)}{(2\pi)^2} \int_0^\infty k^2 \, dk \left(\frac{e^{ikx^0} - e^{-ikx^0}}{2ik} \right) \left(\frac{e^{ikr} - e^{-ikr}}{ikr} \right) \\ &= \pm \frac{\theta(\pm x^0)}{2r(2\pi)^2} \Bigg(\int_0^\infty dk \big[e^{ik(x^0-r)} - e^{ik(x^0+r)} \big] + \\ &\quad + \underbrace{\int_0^\infty dk \left[e^{-ik(x^0-r)} - e^{-ik(x^0+r)} \right]}_{\int_{-\infty}^0 dk [e^{ik(x^0-r)} - e^{ik(x^0+r)}]} \Bigg) \\ &= \pm \frac{\theta(\pm x^0)}{2r(2\pi)^2} \int_{-\infty}^\infty dk \left[e^{ik(x^0-r)} - e^{ik(x^0+r)} \right] \\ &= \pm \frac{\theta(\pm x^0)}{4\pi r} \Big[\delta(x^0 - r) - \delta(x^0 + r) \Big]. \end{aligned} \tag{6.12}$$

Taking into account identities

$$\theta(\pm x^0)\delta(x^0 \mp r) = \delta(x^0 \mp r), \qquad \theta(\pm x^0)\delta(x^0 \pm r) = 0$$

and the fact that $r > 0$ one can put (6.12) in slightly different form

Fundamental solutions of the wave equation in 3+1 dimensions

$$\begin{aligned} D_{ret/adv}(x^0, \boldsymbol{x}) &= \frac{\theta(\pm x^0)}{4\pi r} [\delta(x^0 - r) + \delta(x^0 + r)] \\ &= \frac{\theta(\pm x^0)}{2\pi} \delta(x^\mu x_\mu) \end{aligned} \tag{6.13}$$

where $x^\mu x_\mu = (x^0)^2 - r^2$. The last equality follows from the formula which gives the Dirac delta of a function with zeros at z_k

$$\delta(f(z)) = \sum_k \frac{\delta(z - z_k)}{|f'(z_k)|}, \qquad \text{where} \qquad f(z_k) = 0.$$

Note that the fundamental solution (6.13) is a Lorentz invariant function.

Fundamental solution of the wave equation without explicit dependence on one or two variables

When a physical system possesses certain *symmetries i.e.* their solutions do not depend on one or more coordinates, then a part of the d'Alembert operator associated with these variables became irrelevant. For instance, $\partial^2\psi(x^0, x^1, x^2)$ in 3+1 dimensions reads

Physical systems without dependence on x^3 coordinate

$$(\partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2)\psi(x^0, x^1, x^2) = (\partial_0^2 - \partial_1^2 - \partial_2^2)\psi(x^0, x^1, x^2).$$

Such systems behave effectively as some lower-dimensional systems.

The independence of physical fields on certain number of coordinates can be expressed in the formalism of generalized functions assuming that test functions do not depend on these coordinates. Effectively, one has to replace

$$\varphi(x^0, x^1, x^2, x^3) \to \varphi(x^0, x^1, x^2).$$

This assumption allows us to obtain low-dimensional fundamental solutions integrating over irrelevant coordinates.

We shall consider a class of test functions that are constant in direction given by coordinate x^3. The generalized function $(\partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2)D_{ret}^{(3)}(x)$ acts on any test function $\varphi(x)$ as a linear functional in 3+1 dimensions *i.e.* it associates a real number with φ . This number is obtained by integration over all (four) spacetime variables

Reduction $D^{(3)} \to D^{(2)}$

$$\Big(\overset{(3+1)}{\partial^2} D_{ret}^{(3)}(x), \varphi(x)\Big) = \Big(D_{ret}^{(3)}(x^0, \boldsymbol{x}), \overset{(2+1)}{\partial^2}\varphi - \overbrace{\partial_3^2\varphi}^{0}\Big)$$
$$= \int_{\mathbb{R}^3} dx^0 dx^1 dx^2 \Big[\int_{-\infty}^{\infty} dx^3 D_{ret}^{(3)}(x^0, \boldsymbol{x})\Big] \overset{(2+1)}{\partial^2}\varphi \tag{6.14}$$

where the independence of φ on x^3 allows us to integrate out $D_{ret}^{(3)}(x)$ over x^3

$\overset{(2+1)}{\partial^2} := \partial_0^2 - \partial_1^2 - \partial_2^2$

$$\boxed{D_{ret}^{(2)}(x^0, x^1, x^2) := \int_{-\infty}^{\infty} dx^3 D_{ret}^{(3)}(x^0, x^1, x^2, x^3).} \tag{6.15}$$

The right hand side of (6.14) can be cast in the form

$$\Big(D_{ret}^{(2)}, \overset{(2+1)}{\partial^2}\varphi(x^0, x^1, x^2)\Big) = \Big(\overset{(2+1)}{\partial^2} D_{ret}^{(2)}, \varphi(x^0, x^1, x^2)\Big), \tag{6.16}$$

where (f, φ) is a three-dimensional integral

$$(f, \varphi) = \int_{\mathbb{R}^3} dx^0 dx^1 dx^2 f\varphi.$$

Since the test function φ does not depend on the x^3 variable then the $\delta-$distribution acts on it in the following way

$$\Big(\delta^4(x), \varphi(x^0, x^1, x^2)\Big) = \varphi(0, 0, 0) = \Big(\delta(x^0)\delta(x^1)\delta(x^2), \varphi(x^0, x^1, x^2)\Big).$$

One can conclude that $D^{(2)}_{ret}$ obeys the following distributional equation

$$\boxed{\Big(\overset{(2+1)}{\partial^2}\, D^{(2)}_{ret}(x), \varphi(x)\Big) = (\delta^3(x), \varphi(x)),} \qquad (6.17)$$

where $x = (x^0,\, x^1,\, x^2)$. Thus $D^{(2)}_{ret}$ has meaning of fundamental solution of the wave equation in $2+1$ dimensions.

Reduction $D^{(2)} \to D^{(1)}$

In order to get a fundamental solution for the d'Alembert operator in $1+1$ dimensions one has to take a class of test functions which only depends on x^0 and x^1. Expression $(D^{(2)}_{ret}, \varphi)$ contains integral over x^2, however, φ does not depend on this variable. Hence $D^{(2)}_{ret}$ can be explicitly integrated out on the variable x^2

$$\boxed{D^{(1)}_{ret}(x^0, x^1) := \int_{-\infty}^{\infty} dx^2 D^{(2)}(x^0, x^1, x^2).} \qquad (6.18)$$

where $x = (x^0,\, x^1)$. In the next two section we shall derive an explicit form of the solutions $D^{(2)}_{ret}$ and $D^{(1)}_{ret}$.

Fundamental solution in $2+1$ *dimensions*

In this section we shall obtain the explicit form of the generalized function $D^{(2)}_{ret}$. In order to simplify the notation we define $\boldsymbol{y} := (x^1, x^2)$ and $y := |\boldsymbol{y}|$. We consider the integral over $\mathbb{R}^4$ of the product of functions $D^{(3)}_{ret}(x)$ and $\varphi(x^0, x^1, x^2)$ [6]

[6] It is more convenient to study a four-dimensional integral than the integral of $D^{(3)}_{ret}$ over the variable x^3 alone.

$$\Big(D^{(3)}_{ret}(x), \varphi(x^0, \boldsymbol{y})\Big)_{(3+1)} =$$
$$= \int_{-\infty}^{\infty} dx^0 \int_{\mathbb{R}^2} d^2y\, \varphi(x^0, \boldsymbol{y}) \int_{-\infty}^{\infty} dx^3\, \frac{\theta(x^0)}{4\pi|\boldsymbol{x}|}\delta(x^0 - |\boldsymbol{x}|)$$
$$= \int_{0}^{\infty} dx^0 \int_{\mathbb{R}^2} d^2y\, \varphi(x^0, \boldsymbol{y}) \int_{-\infty}^{\infty} dx^3 \frac{1}{4\pi}\, \frac{\delta(x^0 - \sqrt{y^2 + (x^3)^2})}{\sqrt{y^2 + (x^3)^2}} = \dots$$

The $\delta-$function depends on the x^3 variable through the function

$$f(x^3) := x^0 - \sqrt{y^2 + (x^3)^2}$$

and thus it can be cast in the form

$$\delta(f(x^3)) = \sum_k \frac{\delta(x^3 - x^3_k)}{|f'(x^3)|_{x^3 = x^3_k}},$$

where solutions of the equation $f(x^3_k) = 0$ are denoted by x^3_k. There are two such solutions

$$x^3_1 = -\sqrt{(x^0)^2 - y^2}, \qquad x^3_2 = +\sqrt{(x^0)^2 - y^2}.$$

They are real-valued expressions for $|\boldsymbol{y}| \le x^0$, then

Restriction $|\boldsymbol{y}| \le x^0$

$$\delta(f(x^3)) = \frac{\delta(x^3 - x_1^3)}{\left|-\frac{x_1^3}{\sqrt{y^2+(x_1^3)^2}}\right|} + \frac{\delta(x^3 - x_2^3)}{\left|-\frac{x_2^3}{\sqrt{y^2+(x_2^3)^2}}\right|}$$
$$= \frac{x^0}{\sqrt{(x^0)^2 - y^2}}\left[\delta(x^3 - x_1^3) + \delta(x^3 - x_2^3)\right].$$

The condition $|\boldsymbol{y}| \le x^0$ restricts the integration region to the interior of a disc with radius $y = x^0$. One gets

$$\ldots = \int_0^\infty dx^0 \int_{|\boldsymbol{y}|\le x^0} d^2y\, \varphi(x^0, \boldsymbol{y}) \times$$
$$\times \int_{-\infty}^\infty dx^3 \frac{1}{4\pi} \frac{\frac{x^0}{\sqrt{(x^0)^2-y^2}}\left[\delta(x^3 - x_1^3) + \delta(x^3 - x_2^3)\right]}{\sqrt{y^2 + (x^3)^2}} =$$
$$= \int_0^\infty dx^0 \int_{|\boldsymbol{y}|\le x^0} d^2y\, \varphi(x^0, \boldsymbol{y}) \frac{1}{4\pi} \frac{x^0}{\sqrt{(x^0)^2 - y^2}} \frac{2}{x^0} = \ldots$$

Including the Heaviside's function $\theta(x^0 - |\boldsymbol{y}|)$ one can extend the integration area to the space $\mathbb{R} \times \mathbb{R}^2$, thus

$$\ldots = \int_{-\infty}^\infty dx^0 \int_{\mathbb{R}^2} d^2y\, \varphi(x^0, \boldsymbol{y}) \frac{1}{2\pi} \frac{\theta(x^0 - |\boldsymbol{y}|)}{\sqrt{(x^0)^2 - |\boldsymbol{y}|^2}}.$$

The (2+1) dimensional fundamental solution $D_{ret}^{(2)}$ is of the form

The fundamental (retarded) solution for the d'Alembert operator in 2+1 dimensions

$$\boxed{D_{ret}^{(2)}(x) = \frac{1}{2\pi} \frac{\theta(x^0 - |\boldsymbol{x}|)}{\sqrt{(x^0)^2 - |\boldsymbol{x}|^2}},} \tag{6.19}$$

where we the argument x in $D^{(2)}(x)$ means $x = (x^0, \boldsymbol{x})$, where $\boldsymbol{x} \equiv (x^1, x^2)$.

Fundamental solution in $1+1$ dimensions

Let us take the expression

$$(D_{ret}^{(2)}, \varphi)_{(2+1)} = \int_{-\infty}^\infty dx^0 \int_{-\infty}^\infty dx^1 \left[\int_{-\infty}^\infty dx^2 D^{(2)}(x^0, x^1, x^2)\right] \varphi(x^0, x^1)$$
$$= \int_{-\infty}^\infty dx^0 \int_{-\infty}^\infty dx^1 \left[\int_{-\infty}^\infty dx^2 \frac{1}{2\pi} \frac{\theta(x^0 - |\boldsymbol{x}|)}{\sqrt{(x^0)^2 - |\boldsymbol{x}|^2}}\right] \varphi(x^0, x^1)$$
$$= \ldots$$

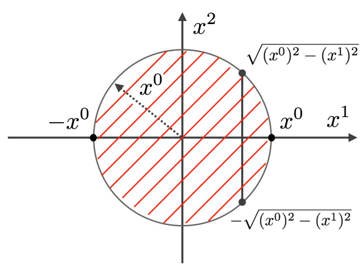


Figure 6.2: The region of integration.

where the condition $\theta(x^0 - |\boldsymbol{x}|)$ restricts the integration to the interior of a circle $|\boldsymbol{x}| \le x^0$ which is shown in Figure 6.2. The presence of a term $\theta(x^0 - |\boldsymbol{x}|)$ establishes the limits of integration over variables x^0, x^1 and x^2:

$$x^0 \ge 0, \qquad -x^0 \le x^1 \le x^0, \qquad x_-^2 \le x^2 \le x_+^2,$$
$$x_\pm^2 := \pm\sqrt{(x^0)^2 - (x^1)^2}. \tag{6.20}$$

It gives

$$\begin{aligned}
\ldots &= \int_0^\infty dx^0 \int_{-x^0}^{x^0} dx^1 \left[\int_{x_-^2}^{x_+^2} dx^2 \frac{1}{2\pi}\frac{1}{\sqrt{(x^0)^2-(x^1)^2-(x^2)^2}}\right] \varphi(x^0,x^1) \\
&= \frac{1}{2\pi}\int_0^\infty dx^0 \int_{-x^0}^{x^0} dx^1 \left[2\int_0^{x_+^2} \frac{dx^2}{\sqrt{(x^0)^2-(x^1)^2}} \frac{1}{\sqrt{1-\frac{(x^2)^2}{(x^0)^2-(x^1)^2}}}\right] \times \\
&\times \varphi(x^0,x^1) = \ldots
\end{aligned}$$

In terms of new variable

$$u := \frac{x^2}{\sqrt{(x^0)^2-(x^1)^2}}, \qquad du = \frac{dx^2}{\sqrt{(x^0)^2-(x^1)^2}}$$

the last expression takes the form

$$\begin{aligned}
\ldots &= \frac{1}{\pi}\int_0^\infty dx^0 \int_{-x^0}^{x^0} dx^1 \Big[\underbrace{\int_0^1 \frac{du}{\sqrt{1-u^2}}}_{\arcsin(1)-\arcsin(0)=\frac{\pi}{2}} \Big] \varphi(x^0,x^1) \\
&= \frac{1}{2}\int_0^\infty dx^0 \int_{-x^0}^{x^0} dx^1 \varphi(x^0,x^1) \\
&= \int_{-\infty}^\infty dx^0 \int_{-\infty}^\infty dx^1 \left[\frac{1}{2}\theta(x^0-|x^1|)\right] \varphi(x^0,x^1).
\end{aligned}$$

Thus, the fundamental solution of the wave equation in $1+1$ dimensions is of the form

Fundamental solution of the wave equation in $1+1$ dimensions

$$\boxed{D_{ret}^{(1)}(x) = \frac{1}{2}\theta\Big(x^0-|x^1|\Big).} \tag{6.21}$$

Huygen's principle

Comparing the fundamental solutions (6.13), (6.19) and (6.21), respectively, in 3+1, 2+1 and 1+1 dimensions one can conclude that in lower dimensions the solutions depend on functions of variables $|\boldsymbol{x}| \leq x^0$. The point is that these functions are different from the $\delta-$ like generalize function (6.13). For this reason the solution of wave equation at any point $(x^0, \boldsymbol{x})$ depends on initial data in certain region of spacetime. Due to causality, such regions are localized *inside* the past light cone of the event $(x^0, \boldsymbol{x})$ considered as the light-cone apex.

On the contrary, the solutions in 3+1 dimensions depend exclusively on events that are causally related with $(x^0, \boldsymbol{x})$ *i.e.* they belong to the conical surface of the light-cone. It means that solutions of the wave equation in 3+1 dimensions propagate *with the speed of light*. This fact is known as *Huygen's principle*. In fact, the Huygen's principle is also true in higher $n+1$ dimensions where $n = 3, 5, 7, \ldots$.

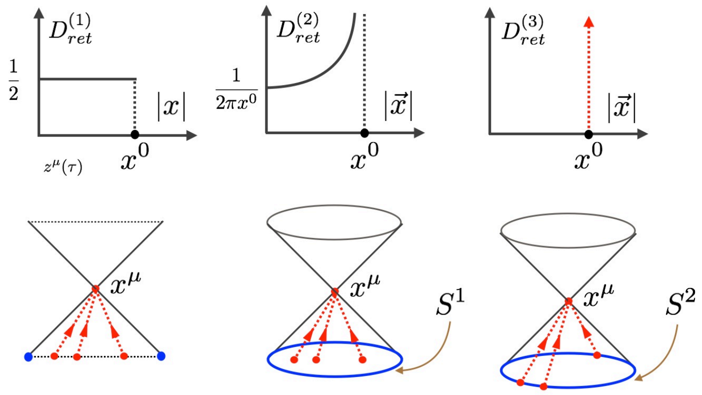


Figure 6.3: Fundamental solutions and their supports in 1+1, 2+1 and 3+1 dimensions.

Note, that wave pulses in $2+1$ and $1+1$ dimensions propagate with subluminal speeds which is caused by dimensional reduction (projection). For instance, two distinct events which differ by the value of coordinate x^3 are considered as a single event in $2+1$ dimensions. The signal propagation trajectories in $2+1$ dimensions are projections of the actual $3+1$ dimensional trajectories. Thus the observed speed of signals in 2+1 dimensions is an effective projection.

Solution $D_{ret} - D_{adv}$

A very special solution of the homogeneous wave equation is discussed in this section. This solution consists on two fundamental which satisfy the non-homogeneous equation $(\partial^2 D_{ret/adv}, \varphi) = (\delta^4, \varphi)$. Their difference $D(x) := D^{(3)}_{ret}(x) - D^{(3)}_{adv}(x)$ is a distribution which solves the *homogeneous* wave equation, $(\partial^2 D, \varphi) = (0, \varphi)$. The solution is of the form

Special solution of homogeneous the wave equation

$$\begin{aligned} D(x) :=& D^{(3)}_{ret}(x) - D^{(3)}_{adv}(x) \\ =& \frac{1}{2\pi}\theta(x^0)\delta(x^\mu x_\mu) - \frac{1}{2\pi}\theta(-x^0)\delta(x^\mu x_\mu) \\ =& \frac{1}{2\pi}\mathrm{sgn}(x^0)\delta(x^\mu x_\mu). \end{aligned} \tag{6.22}$$

Taking $D^{(3)}_{ret/adv}(x)$ as a Fourier transform of $\widetilde{D}^{(3)}_{ret/adv}(x^0, \boldsymbol{k})$ with respect to coordinates $\boldsymbol{k} = (k^1, k^2, k^3)$, one gets $D(x)$ in the form

Integral form of $D(x)$

$$\begin{aligned} D(x) &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} d^3k\, e^{ik\cdot x}[\widetilde{D}^{(3)}_{ret}(x^0, \boldsymbol{k}) - \widetilde{D}^{(3)}_{adv}(x^0, \boldsymbol{k})] \\ &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} d^3k\, e^{ik\cdot x}\underbrace{\left[\theta(x^0) + \theta(-x^0)\right]}_{1}\frac{\sin(kx^0)}{k} \end{aligned}$$

and thus

$$D(x) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} d^3k\, e^{ik\cdot x}\frac{\sin(kx^0)}{k}. \tag{6.23}$$

Other solution of the homogeneous wave equation can be obtained taking the derivative of $D(x)$ with respect to spatial coordinate x^0. It gives

$$D_1(x) := \frac{\partial D(x)}{\partial x^0} = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} d^3k\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \cos(kx^0). \tag{6.24}$$

Solutions (6.23) and (6.24) have the following properties

$$D(0,\boldsymbol{x}) = 0, \qquad D_1(0,\boldsymbol{x}) = \delta^3(\boldsymbol{x}), \qquad \frac{\partial D_1}{\partial x^0}(x^0,\boldsymbol{x})\bigg|_{x^0=0} = 0. \tag{6.25}$$

Functions $D(x^0,\boldsymbol{x})$ and $D_1(x^0,\boldsymbol{x})$ form *two linearly independent solutions of the wave equation*. Moreover, taking into account properties (6.25) one can get the formula for a solution of the wave equation which satisfies the initial data. We shall discuss this subject in the next section.

6.2 *Boundary problems*

Any general solution A^μ of the equation $\partial^2 A^\mu = \frac{4\pi}{c} J^\mu$ can be altered by adding any solution of the homogeneous wave equation $\partial^2 A^\mu = 0$. This freedom allows to choose the general solution in order that it satisfies some condition at certain *space-like* surface Σ. The surface of simultaneity of certain event in a given reference frame is example of such a surface. In such a case the conditions are called *initial conditions*.

It is possible to choose the initial field configuration before (or after) switching on (off) the currents J^μ. In such a case, the electromagnetic field at the surface Σ has status of *free field* – it corresponds with the solution of a homogeneous wave equation and it is equal to the prescribed field configuration at Σ.

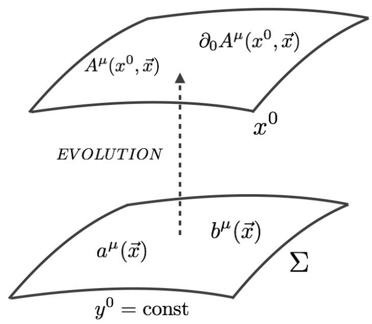


Figure 6.4: Cauchy problem for wave equation.

Cauchy problem for the homogeneous wave equation

In this section we study a Cauchy problem for *homogeneous* wave equation in 3+1 dimensions.[7] Let Σ be a three-surface at given instant of time $y^0 = const$ (surface of simultaneity in a certain reference frame). A *Cauchy problem* (initial problem) in electrodynamics consists on the *wave equation*

(1). Evolution equation

$$\partial^2 A^\mu(x) = 0 \tag{6.26}$$

and *initial conditions* specified at $x^0 = y^0$

(2). Initial conditions

$$A^\mu(x^0,\boldsymbol{x})\Big|_{x^0=y^0} = a^\mu(\boldsymbol{x}), \tag{6.27}$$

$$\frac{\partial}{\partial x^0} A^\mu(x^0,\boldsymbol{x})\Big|_{x^0=y^0} = b^\mu(\boldsymbol{x}). \tag{6.28}$$

The solution of the problem gives the electromagnetic four-potential $A^\mu(x^0,\boldsymbol{x})$ obtained as evolution of the initial field configuration described by $a^\mu(\boldsymbol{x})$ and $b^\mu(\boldsymbol{x})$, see Figure 6.4.

[7] In particular, it is the initial problem for hyperbolic differential equation.

By means of Fourier transform we map the Cauchy problem involving partial differential equation by an initial problem for ordinary differential equation. This requires transform acting on spatial variables $\boldsymbol{x}$. Plugging the four-potential represented by integrals

$$A^\mu(x^0,\boldsymbol{x}) = F^{-1}[\widetilde{A}^\mu(x^0,\boldsymbol{k})] = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} d^3k e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\widetilde{A}^\mu(x^0,\boldsymbol{k}) \tag{6.29}$$

into the wave equation (6.26) one gets

The harmonic oscillator equation

$$(\partial_0^2 + \boldsymbol{k}^2)\widetilde{A}^\mu(x^0,\boldsymbol{k}) = 0. \tag{6.30}$$

The *harmonic oscillator* equation (6.30) depends on a single variable x^0. The variable $\boldsymbol{k}$ has the status of free parameter. Applying the Fourier transform, the initial conditions yield

Fourier transform of the initial conditions

$$\widetilde{a}^\mu(\boldsymbol{k}) := F[a^\mu(\boldsymbol{x})] = \widetilde{A}^\mu(x^0,\boldsymbol{k})\Big|_{x^0=y^0}, \tag{6.31}$$

$$\widetilde{b}^\mu(\boldsymbol{k}) := F[b^\mu(\boldsymbol{x})] = \frac{\partial\widetilde{A}^\mu}{\partial x^0}(x^0,\boldsymbol{k})\Bigg|_{x^0=y^0}. \tag{6.32}$$

To solve (6.30) we choose two linearly independent solutions of the equation

$$(\partial_0^2 + k^2)\widetilde{D}(z^0,\boldsymbol{k}) = 0$$

where the variable x^0 has been substituted by $z^0 := x^0 - y^0$ and $k := |\boldsymbol{k}|$. They have the form

Linearly independent solutions of harmonic oscillator

$$\widetilde{D}(z^0,\boldsymbol{k}) = \frac{\sin(kz^0)}{k}, \qquad \widetilde{D}_1(z^0,\boldsymbol{k}) = \frac{\partial\widetilde{D}}{\partial x^0}(z^0,\boldsymbol{k}) = \cos(kz^0)$$

and satisfy conditions

$$\widetilde{D}(0,\boldsymbol{k}) = 0, \qquad \widetilde{D}_1(0,\boldsymbol{k}) = 1.$$

A general solution of (6.30) is given by linear combination of $\widetilde{D}$ and $\widetilde{D}_1$

$$\widetilde{A}^\mu(x^0,\boldsymbol{k}) = \alpha^\mu\widetilde{D}_1(x^0-y^0,\boldsymbol{k}) + \beta^\mu\widetilde{D}(x^0-y^0,\boldsymbol{k}), \tag{6.33}$$

where α^μ and β^μ are two constant four-vectors. Imposing the initial conditions (6.31) and (6.32) on (6.33) one gets

$$\alpha^\mu = \tilde{a}^\mu(\boldsymbol{k}), \qquad \beta^\mu = \tilde{b}^\mu(\boldsymbol{k}).$$

Thus, the solution of (6.30) reads

Solution of transformed equation

$$\widetilde{A}^\mu(x^0,\boldsymbol{k}) = \widetilde{a}^\mu(\boldsymbol{k})\widetilde{D}_1(x^0-y^0,\boldsymbol{k}) + \widetilde{b}^\mu(\boldsymbol{k})\widetilde{D}(x^0-y^0,\boldsymbol{k}). \tag{6.34}$$

The solution of the Cauchy problem is given by the inverse Fourier transform of $\widetilde{A}^\mu(x^0,\boldsymbol{k})$ which solves (6.34). Plugging (6.34) into (6.29) where coefficients are represented in the form

Solution of partial differential equation:

$$\widetilde{A}^\mu(x^0,\boldsymbol{k}) \to A^\mu(x^0,\boldsymbol{x})$$

$$\widetilde{a}^\mu(\boldsymbol{k}) = F[a^\mu(\boldsymbol{y})] = \int d^3y e^{-i\boldsymbol{k}\cdot\boldsymbol{y}}a^\mu(\boldsymbol{y}),$$

$$\widetilde{b}^\mu(\boldsymbol{k}) = F[b^\mu(\boldsymbol{y})] = \int d^3y e^{-i\boldsymbol{k}\cdot\boldsymbol{y}}b^\mu(\boldsymbol{y}),$$

and similarly

$$\widetilde{D}(x^0-y^0,\boldsymbol{k}) = F[D(x^0-y^0,\boldsymbol{z})] = \int d^3z e^{-i\boldsymbol{k}\cdot\boldsymbol{z}} D(x^0-y^0,\boldsymbol{z})$$

$$\widetilde{D}_1(x^0-y^0,\boldsymbol{k}) = F[D_1(x^0-y^0,\boldsymbol{z})] = \int d^3z e^{-i\boldsymbol{k}\cdot\boldsymbol{z}} D_1(x^0-y^0,\boldsymbol{z})$$

one gets

$$\begin{aligned} A^\mu(x^0,\boldsymbol{x}) =& \frac{1}{(2\pi)^3}\int d^3k e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \widetilde{A}^\mu(x^0,\boldsymbol{k}) = \\ =& \frac{1}{(2\pi)^3}\int d^3k e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \int d^3y e^{-i\boldsymbol{k}\cdot\boldsymbol{y}} \int d^3z e^{-i\boldsymbol{k}\cdot\boldsymbol{z}} \times \\ & \times \Big[a^\mu(\boldsymbol{y})D_1(x^0-y^0,\boldsymbol{z}) + b^\mu(\boldsymbol{y})D(x^0-y^0,\boldsymbol{z})\Big]. \end{aligned} \tag{6.35}$$

This expression can be simplified using integral representation for the Dirac delta distribution

$$\frac{1}{(2\pi)^3}\int d^3k e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{z})} = \delta^3(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{z}).$$

Integrating over variables z^1, z^2 and z^3 one gets

Solution of the wave equation that satisfy the initial conditions

$$\boxed{A^\mu(x^0,\boldsymbol{x}) = \int_\Sigma d^3y \Big[a^\mu(\boldsymbol{y})D_1(x^0-y^0,\boldsymbol{x}-\boldsymbol{y}) + b^\mu(\boldsymbol{y})D(x^0-y^0,\boldsymbol{x}-\boldsymbol{y})\Big]} \tag{6.36}$$

where the integral is taken over the hypersurface of initial conditions at $y^0 = const$.

We shall express this formula in the alternative form. First, one can see thet $D(z)$ is an anti-symmetric function of z^μ. This can be seen explicitly from

$$D(-z^0,-\boldsymbol{z}) = \frac{1}{(2\pi)^3}\int d^3k e^{-i\boldsymbol{k}\cdot\boldsymbol{z}} \frac{\sin(|\boldsymbol{k}|(-z^0))}{|\boldsymbol{k}|} = -D(z^0,\boldsymbol{z}), \tag{6.37}$$

where in the last step absorb the sign on definition of new variable of integration $\boldsymbol{k} \to -\boldsymbol{k}$

$$\int_{-\infty}^{\infty} dk^i \ldots \to \int_{+\infty}^{-\infty} (-dk^i) \ldots = \int_{-\infty}^{\infty} dk^i \ldots$$

Second, the function $D_1(x-x')$, defined as derivative of $D(x-x')$ with respect to x^0, can be represented alternatively as derivative with respect to x'^0. Thus we have

$$\begin{aligned} D_1(x-x') &= \frac{\partial}{\partial x^0} D(x^0-x'^0,\boldsymbol{x}-\boldsymbol{x}') = -\frac{\partial}{\partial x'^0} D(x^0-x'^0,\boldsymbol{x}-\boldsymbol{x}') \\ &= \frac{\partial}{\partial x'^0} D(x'^0-x^0,\boldsymbol{x}'-\boldsymbol{x}) = \frac{\partial}{\partial x'^0} D(x'-x). \end{aligned}$$

The function D_1 for $x' = y \equiv (y^0,\boldsymbol{y})$ *i.e.* $D_1(x^0-y^0,\boldsymbol{x}-\boldsymbol{y}) \equiv D_1(x-y)$ can be cast in the form

$$D_1(x-x')\Big|_{x'=y} = \frac{\partial}{\partial x'^0} D(x'-x)\Big|_{x'=y}. \tag{6.38}$$

Similarly, the function $D(x-y)$ reads

$$D(x-y) = D(x-x')\Big|_{x'=y} = -D(x'-x)\Big|_{x'=y}. \tag{6.39}$$

Finally, the functions $a^\mu(\boldsymbol{y})$ and $b^\mu(\boldsymbol{y})$ can be cast in the form

$$a^\mu(\boldsymbol{y}) = A^\mu(x')\Big|_{x'=y}, \qquad b^\mu(\boldsymbol{y}) = \frac{\partial A^\mu}{\partial x'^0}(x')\Big|_{x'=y}. \tag{6.40}$$

Plugging (6.38), (6.39) and (6.40) into (6.36) one gets[8]

$$\boxed{A^\mu(x) = \int_{\Sigma(y)} d^3x' \left[A^\mu(x')\frac{\partial D(x'-x)}{\partial x'^0} - \frac{\partial A^\mu(x')}{\partial x'^0}D(x'-x)\right]} \tag{6.41}$$

[8] A^μ and its derivative at $x' = y$ on the right hand side of expression (6.41) are prescribed (a priori given) functions.

where the four-vectors x' give events on $\Sigma(y)$.

Verification of formula (6.41):

It can be checked that (6.41) solves the Cauchy problem. Acting with ∂^2 on both sides of (6.41) one gets that $A^\mu(x)$ given by (6.41) satisfies the wave equation

$$\partial^2 A^\mu(x) = \int_{\Sigma(y)} d^3x' \Big[A^\mu(x')\frac{\partial}{\partial x'^0}\underbrace{\partial^2 D(x'-x)}_{0} - \frac{\partial A^\mu(x')}{\partial x'^0}\underbrace{\partial^2 D(x'-x)}_{0}\Big] = 0$$

(1). It satisfies the evolution equation

where $D(x'-x)$ is a solution of the homogeneous wave equation according to

$$\begin{aligned}\partial^2 D(x'-x) &= \frac{1}{(2\pi)^3}\int d^3k \left[\partial_0^2 - \nabla^2\right] e^{i\boldsymbol{k}\cdot(\boldsymbol{x}'-\boldsymbol{x})}\frac{\sin(|\boldsymbol{k}|(x'^0-x^0))}{|\boldsymbol{k}|}\\ &= \frac{1}{(2\pi)^3}\int d^3k\Big[-|\boldsymbol{k}|^2 + \boldsymbol{k}^2\Big]e^{i\boldsymbol{k}\cdot(\boldsymbol{x}'-\boldsymbol{x})}\frac{\sin(|\boldsymbol{k}|(x'^0-x^0))}{|\boldsymbol{k}|}\\ &= 0.\end{aligned}$$

The initial conditions at hypersurface $x \in \Sigma(y)$ *i.e.* $x^0 = x'^0 = y^0$ are satisfied. It can be seen from

$$\begin{aligned}D(x'-x)\Big|_{x\in\Sigma(y)} &= \frac{1}{(2\pi)^3}\int d^3k\, e^{i\boldsymbol{k}\cdot(\boldsymbol{x}'-\boldsymbol{x})}\frac{\sin(|\boldsymbol{k}|(y^0-x^0))}{|\boldsymbol{k}|}\Big|_{x^0=y^0} =\\ &= 0,\\ \frac{\partial D(x'-x)}{\partial x'^0}\Big|_{x\in\Sigma(y)} &= \frac{1}{(2\pi)^3}\int d^3k e^{i\boldsymbol{k}\cdot(\boldsymbol{x}'-\boldsymbol{x})}\cos(|\boldsymbol{k}|(y^0-x^0))\Big|_{x^0=y^0} =\\ &= \delta^3(\boldsymbol{x}'-\boldsymbol{x}),\\ \frac{\partial D(x'-x)}{\partial x^0}\Big|_{x\in\Sigma(y)} &= -\delta^3(\boldsymbol{x}'-\boldsymbol{x}),\\ \frac{\partial^2 D(x'-x)}{\partial x^0\partial x'^0}\Big|_{x\in\Sigma(y)} &= \frac{1}{(2\pi)^3}\int d^3k e^{i\boldsymbol{k}\cdot(\boldsymbol{x}'-\boldsymbol{x})}|\boldsymbol{k}|\sin(|\boldsymbol{k}|(y^0-x^0))\Big|_{x^0=y^0} =\\ &= 0.\end{aligned}$$

Thus (6.41) at $x^0 = y^0$ and with the above formulas yelds

(2). It satisfies the initial conditions

$$A^\mu(x)\Big|_{x^0=y^0} = \int_{\Sigma(y)} d^3x'\, A^\mu(y^0, \boldsymbol{x}')\, \delta^3(\boldsymbol{x}' - \boldsymbol{x}) = A^\mu(y^0, \boldsymbol{x}) = a^\mu(\boldsymbol{x}).$$

A derivative of (6.41) w.r.t. x^0 at $x^0 = y^0$ reads

$$\frac{\partial A^\mu(x)}{\partial x^0}\Big|_{x^0=y^0} = \int_{\Sigma(y)} d^3x'\, \frac{\partial A^\mu}{\partial x'^0}\, \delta^3(\boldsymbol{x}' - \boldsymbol{x}) = \frac{\partial A^\mu}{\partial x'^0}(x'^0, \boldsymbol{x})\Big|_{x'^0=y^0} = b^\mu(\boldsymbol{x}).$$

Generalization Cauchy problem to arbitrary Cauchy surface

The initial condition can be generalized by substitution of the surface of simultaneity a generic space-like surface.

A Cauchy surface $\Sigma(y)$ is a space-like three surface[9] which is intersected by every time-like curve (*e.g* worldline) exactly once. The interval associated with two events belonging to the Cauchy surface is negative, $\Delta s^2 < 0$.

[9] Locally it looks like a piece of the surface of simultaneity in a certain inertial reference frame.

When the hyperplane $x^0 = y^0$ is replaced by the Cauchy surface Σ then (6.41) must be modified in the wave that it contains Lorentz invariant (or covariant) expressions.

- The volume element $d^3x' = d^3\Sigma^0$ is replaced by

$$d^3\Sigma_0 \to d^3\Sigma_\nu = \frac{1}{3!}\sqrt{-g}\epsilon_{\nu\alpha\beta\gamma}\frac{\partial(x^\alpha x^\beta x^\gamma)}{\partial(t^1 t^2 t^3)} dt^1 dt^2 dt^3.$$

- The derivative of the electromagnetic four-potential with respect to x'^0 is replaced by

$$\frac{\partial A^\mu}{\partial x'^0} \to \frac{\partial A^\mu}{\partial x'^\nu}.$$

- The expression $\frac{\partial D(x'-x)}{\partial x'^0}\Big|_{x^0=x'^0} = \delta^3(\boldsymbol{x}' - \boldsymbol{x})$ requires substitution of the Dirac delta distribution by the following one

$$\delta^3(\boldsymbol{x}' - \boldsymbol{x}) \to n_\nu \delta^3(\boldsymbol{x}' - \boldsymbol{x}) \equiv \delta^3_\mu(\boldsymbol{x}' - \boldsymbol{x})$$

  where $n_\nu = n_\nu(x')$ is a time-like four-vector, $n^2 = 1$, orthogonal to the surface Σ *i.e* $n_\nu dx'^\nu = 0$. The spatial three-volume element $d^3\Sigma_\nu(x')n^\nu(x')$ is equal to $d^3\Sigma'_0(x')$ in a local rest frame S' at x' in which $n'^\nu = \delta^\nu_0$.

Thus

$$\frac{\partial D(x'-x)}{\partial x'^0}\Big|_{x^0=x'^0=y^0} \to \frac{\partial D(x'-x)}{\partial x'^\nu}\Big|_{x\in\Sigma} = n_\nu \delta^3(\boldsymbol{x}' - \boldsymbol{x}).$$

Finally, we get the generalization of (6.41) in the form

Solution of the Cauchy problem that satisfies initial conditions at the Cauchy surface Σ

$$\boxed{A^\mu(x) = \int_{\Sigma(y)} d^3\Sigma^\nu(x')\left[A^\mu(x')\frac{\partial}{\partial x'^\nu}D(x'-x) - D(x'-x)\frac{\partial A^\mu}{\partial x'^\nu}(x')\right]} \tag{6.42}$$

where $A^\mu(x')$ and $\frac{\partial A^\mu}{\partial x'^\nu}(x')$ are *prescribed functions*. They give the electromagnetic field at $\Sigma(y)$

$$A^\mu(x')\Big|_{x'\in\Sigma(y)} \equiv a^\mu(y), \qquad \frac{\partial A^\mu}{\partial x'^\nu}(x')\Bigg|_{x'\in\Sigma(y)} \equiv b^\mu(y).$$

Indeed, for $x \in \Sigma(y)$ the solution (6.42) reads

$$A^\mu(x)\Big|_{x\in\Sigma(y)} = \int_{\Sigma(y)} d^3\Sigma^\nu(x')\left[A^\mu(x')n_\nu\delta^3(\boldsymbol{x}'-\boldsymbol{x}) - 0\right] = a^\mu(y).$$

Similarly, taking derivative with respect to x^0 and evaluating the resulting expression at the Cauchy surface Σ one gets

$$\frac{\partial A^\mu(x)}{\partial x^0}\Big|_{x\in\Sigma(y)} = \int_\Sigma d^3\Sigma^\nu(x')\left[0 - \overbrace{\frac{\partial D}{\partial x^0}(x'-x)\Big|_{x\in\Sigma(y)}}^{-\delta^3(\boldsymbol{x}'-\boldsymbol{x})}\frac{\partial A^\mu}{\partial x'^\nu}(x')\right] = b^\mu(y).$$

Independence of the solution on the choice of the Cauchy surface

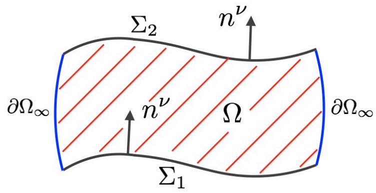


Figure 6.5: Region Ω and its border.

An important fact about (6.42) is that it does not depend on the choice of the Cauchy surface. To show this we take two different Cauchy surfaces Σ_1 and Σ_2 and the four-dimensional region Ω delimited by these surfaces and a time-like hypersurface $\partial\Omega_\infty$ at spatial infinity. This region is shown in Figure 6.5. We assume that A^μ vanishes sufficiently quickly for $|\boldsymbol{x}| \to \infty$.

We consider $A^\mu(x')$ and $D(x'-x)$ that satisfy equations

$$\partial'^2 A^\mu(x') = 0, \qquad \partial'^2 D(x'-x) = 0,$$

where $\partial'^2 = \frac{\partial}{\partial x'^\mu}\frac{\partial}{\partial x'_\mu}$. Integrating the expression

$$D(x'-x)\partial'^2 A^\mu(x') - A^\mu(x')\partial'^2 D(x'-x) \equiv 0$$

over the region Ω and making use of Gauss' integral theorem one gets

$$\begin{aligned}
0 &= \int_\Omega d^4x' \left[D(x'-x)\partial'^2 A^\mu(x') - A^\mu(x')\partial'^2 D(x'-x)\right] = \\
&= \int_\Omega d^4x'\, \partial'^\nu\left[D(x'-x)\partial'_\nu A^\mu(x') - A^\mu(x')\partial'_\nu D(x'-x)\right] - \\
&- \int_\Omega d^4x' \underbrace{\left[\partial'^\nu D(x'-x)\partial'_\nu A^\mu(x') - \partial'^\nu A^\mu(x')\partial'_\nu D(x'-x)\right]}_{0} = \\
&= \oint_{\partial\Omega} d^3\Sigma^\nu\left[D(x'-x)\partial'_\nu A^\mu(x') - A^\mu(x')\partial'_\nu D(x'-x)\right] = \\
&= \left[\int_{\Sigma_1} - \int_{\Sigma_2} + \int_{\partial\Omega_\infty}\right] d^3\Sigma^\nu\left[D(x'-x)\partial'_\nu A^\mu(x') - A^\mu(x')\partial'_\nu D(x'-x)\right]
\end{aligned}$$

where the integral vanishes at spatial infinity. It gives equality of integrals over surfaces Σ_1 and Σ_2. In particular, it means that any deformation of the surface Σ_k which does not change their character does not alternate the value of the integral.

If the the space-like surface Σ contains x then any event x' at Σ is connected with x by a space-like four-vector. It leads to vanishing of $D(x'-x)$ and gives

$$A^\mu(x) = \int_{\Sigma(y)} d^3\Sigma^\nu(x')\, A^\mu(x') \frac{\partial}{\partial x'^\nu} D(x'-x).$$

The solution is also unique. Let $A_1^\mu(x)$ and $A_2^\mu(x)$ be two solutions of the equation $\partial^2 A^\mu(x) = \frac{4\pi}{c} J^\mu(x)$ that satisfy simultaneously the initial conditions

The uniqueness of solution of the Cauchy problem

$$A_1^\mu(x)\Big|_{\Sigma(y)} = A_2^\mu(x)\Big|_{\Sigma(y)} = a^\mu(y) \tag{6.43}$$

and

$$\partial_0 A_1^\mu(x)\Big|_{\Sigma(y)} = \partial_0 A_2^\mu(x)\Big|_{\Sigma(y)} = b^\mu(y). \tag{6.44}$$

The function $A_0^\mu(x) := A_1^\mu(x) - A_2^\mu(x)$ is a solution of homogeneous d'Alembert equation $\partial^2 A_0^\mu(x) = 0$ and satisfies initial conditions

$$A_0^\mu(x)\Big|_{\Sigma(y)} = 0, \qquad \partial_0 A_0^\mu(x)\Big|_{\Sigma(y)} = 0. \tag{6.45}$$

It follows from (6.42) that the only solution of the homogeneous wave equation, which satisfies (6.42), is a constant solution $A_0^\mu(x) = 0$. We conclude that $A_1^\mu(x) = A_2^\mu(x)$ *i.e.* the solution of the inhomogeneous equation that satisfies initial conditions $A^\mu(x)\Big|_{\Sigma(y)} = a^\mu(y)$ and $\partial_0 A^\mu(x)\Big|_{\Sigma(y)} = b^\mu(y)$ is unique.

Retarded and advanced solutions

The Cauchy problem is well-possed if

1. a solution exists,
2. the solution is unique,
3. the solution's behaviour changes continuously with the initial conditions.

The initial field configuration is prescribed at the space-like surface $\Sigma(y)$. In particular, this configuration can be chosen at the simultaneity surface, $x^0 = y^0$. The field configurations at $\Sigma(y)$ are frequently determined in the remote past or in the remote future *i.e.* before (after) switching on (off) the external sources. Thus we shall consider

$$J^\mu(x)|_{x\in\Sigma(y)} \equiv 0.$$

It means that the initial (final) field configuration is a free electromagnetic field which solves the homogeneous wave equation. The solution defined in the remote past is denoted by $A^\mu_{in}(x)$ wheres the solution defined in the remote future is denoted by $A^\mu_{out}(x)$. The retarded and advanced potentials read

Retarded and advanced potentials

$$A^\mu_{ret}(x) := \frac{4\pi}{c}\int d^4x' D_{ret}(x-x')J^\mu(x'), \tag{6.46}$$

$$A^\mu_{adv}(x) := \frac{4\pi}{c}\int d^4x' D_{adv}(x-x')J^\mu(x'). \tag{6.47}$$

They solve the inhomogeneous wave equation, namely

$$\partial^2 A^\mu_{\substack{ret\\adv}}(x) = \frac{4\pi}{c}\int d^4x'\delta^4(x-x')J^\mu(x') = \frac{4\pi}{c}J^\mu(x).$$

One can construct general solutions by inclusion of incoming or outgoing free fields

$$A^\mu(x) = A^\mu_{in}(x) + A^\mu_{ret}(x), \tag{6.48}$$

$$A^\mu(x) = A^\mu_{out}(x) + A^\mu_{adv}(x). \tag{6.49}$$

Potentials (6.48) and (6.49) represent *the same field* at x, thus

$$A^\mu_{in}(x) + A^\mu_{ret}(x) = A^\mu_{out}(x) + A^\mu_{adv}(x). \tag{6.50}$$

The field $A^\mu_{ret}(x)$ describes *emission* of the field by sources J^μ whereas $A^\mu_{adv}(x)$ describes a field *absorption*.

The expression (6.48) is a solution of the inhomogeneous wave equation with the initial configuration which is a free electromagnetic field in the *remote past* $x^0 \to -\infty$. Similarly, expression (6.49) gives the solution of the inhomogeneous wave equation for the final configuration being free electromagnetic field in the *remote future* $x^0 \to +\infty$.

The Lorenz condition

Considering that the fields $A^\mu_{in}(x)$ and $A^\mu_{out}(x)$ satisfy the Lorenz condition

Free fields which (by assumption) satisfy the Lorenz condition

$$\partial_\mu A^\mu_{in}(x) = 0, \qquad \partial_\mu A^\mu_{out}(x) = 0, \tag{6.51}$$

one concludes that (6.48) and (6.49) satisfy this condition as well. To prove this statement we choose a region Ω. Its border consists of two space-like three-surfaces Σ_1 and Σ_2 and a time-like three-surface at spatial infinity Σ_∞,

$$\partial\Omega = \Sigma_1 \cup \Sigma_2 \cup \Sigma_\infty.$$

We assume that the four-currents vanish outside a certain compact region E^3. It means that

Four-currents (by assumption) vanish at Σ_∞

$$J^\mu\big|_{\Sigma_\infty} \equiv 0. \tag{6.52}$$

They are called *localized* currents. taking four-divergence of solutions

(6.48) and (6.49) one gets

$$\begin{aligned}
\frac{\partial A^\mu}{\partial x^\mu}(x) &= \overbrace{\frac{\partial}{\partial x^\mu} A^\mu_{\substack{in\\out}}(x)}^{0} + \frac{4\pi}{c}\int_\Omega d^4x' \frac{\partial}{\partial x^\mu} D_{\substack{ret\\adv}}(x-x')J^\mu(x') \\
&= -\frac{4\pi}{c}\int_\Omega d^4x' \frac{\partial}{\partial x'^\mu} D_{\substack{ret\\adv}}(x-x')J^\mu(x') \\
&= -\frac{4\pi}{c}\int_\Omega d^4x' \frac{\partial}{\partial x'^\mu}\Big(D_{\substack{ret\\adv}}(x-x')J^\mu(x')\Big) \\
&\quad + \frac{4\pi}{c}\int_\Omega d^4x' D_{\substack{ret\\adv}}(x-x')\overbrace{\frac{\partial J^\mu(x')}{\partial x'^\mu}}^{0} \\
&= -\frac{4\pi}{c}\oint_{\partial\Omega} d^3\Sigma_\mu D_{\substack{ret\\adv}}(x-x')J^\mu(x') \\
&= \frac{4\pi}{c}\Big(\int_{\Sigma_2} - \int_{\Sigma_1}\Big) d^3\Sigma_\mu D_{\substack{ret\\adv}}(x-x')J^\mu(x'). \qquad (6.53)
\end{aligned}$$

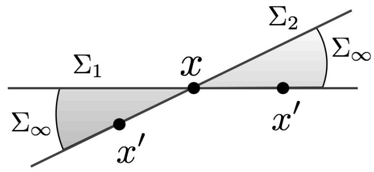


Figure 6.6: The choice of the Cauchy surfaces Σ_1 and Σ_2. The events x' at each surface and the event x are spatially separated.

We have made use of the assumption about vanishing of the four-currents at Σ_∞. Surfaces Σ_1 and Σ_2 can be chosen in a way that they pass through the point x. In such a case with two events x' and x that belong to Σ_k there is associated the interval $\Delta s^2 < 0$. Consequently, the retarded and advanced fundamental solutions vanish, $D_{\substack{ret\\adv}}(x-x') = 0$, since the four-vector $x - x'$ does not belong to the light-cone.[10] It leads to vanishing of (6.53) and results in vanishing of divergences $\partial_\mu A^\mu_{ret}(x) = 0$ and $\partial_\mu A^\mu_{adv}(x) = 0$.

[10] The fundamental solutions in 3+1 dimensions vanish identically outside the light-cone.

Kirchhoff's formula

The causal solution

$$D(x^0 - y^0, \boldsymbol{x} - \boldsymbol{y}) = \frac{1}{4\pi}\frac{\delta(x^0 - y^0 - |\boldsymbol{x}-\boldsymbol{y}|)}{|\boldsymbol{x}-\boldsymbol{y}|} \qquad (6.54)$$

where $x^0 > y^0$, substituted into the formula (6.36)

$$A^\mu(x^0, \boldsymbol{x}) = \int_{\Sigma_0} d^3y \left[a^\mu(\boldsymbol{y})\frac{\partial}{\partial x^0} D(x^0 - y^0, \boldsymbol{x}-\boldsymbol{y}) + b^\mu(\boldsymbol{y}) D(x^0 - y^0, \boldsymbol{x}-\boldsymbol{y})\right]$$

gives Kirchhoff's formula

$$\begin{aligned}
A^\mu(x^0, \boldsymbol{x}) = \frac{1}{4\pi c}\Bigg[&\int d^3y \frac{\delta(x^0 - y^0 - |\boldsymbol{x}-\boldsymbol{y}|)}{|\boldsymbol{x}-\boldsymbol{y}|}\, c\, b^\mu(\boldsymbol{y}) \\
&+ \frac{\partial}{\partial t}\int d^3y \frac{\delta(x^0 - y^0 - |\boldsymbol{x}-\boldsymbol{y}|)}{|\boldsymbol{x}-\boldsymbol{y}|}\, a^\mu(\boldsymbol{y})\Bigg], \qquad (6.55)
\end{aligned}$$

where $c\, b^\mu(\boldsymbol{y}) = \frac{\partial}{\partial t} A^\mu\Big|_{t=t_0}$.

Cauchy problem in 1+1 dimensions

The general solution of inhomogeneous wave equation in 1+1 dimensions can be obtained even though the inhomogeneity is present at the surface if initial data.

Since A^0 and A^1 obey the same equation, then it is sufficient to analyze the problem of a scalar field obeying the inhomogeneous wave equation and initial conditions

$$\begin{aligned} &u_{tt} - c^2 u_{xx} = f(t,x) \\ &u(0,x) = \varphi(x), \qquad u_t(0,x) = \psi(x). \end{aligned} \tag{6.56}$$

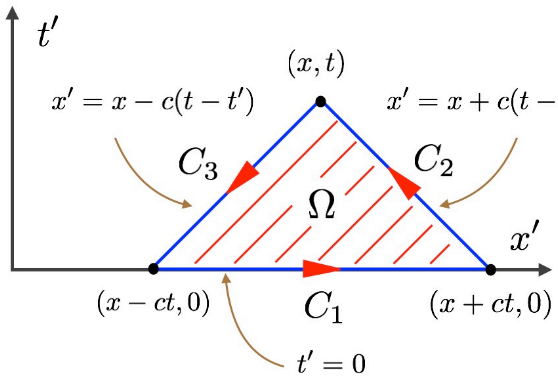


Figure 6.7: Dependence region of solution of wave equation in 1+1 dimension.

We shall solve the problem integrating the equation (6.56) over a region of dependence of the event (x,t). We order the coordinates in the way that x' is the *first* coordinate (horizontal axis) and t' as the *second* one (vertical axis). The region is a triangle delimited by segments C_1, C_2 and C_3, see Fig.6.2. The segment C_1 is located at the x' axis, $t' = 0$, and its endpoints correspond with events $(x - c\,t, 0)$ and $(x + c\,t, 0)$. The initial conditions for the wave equation are prescribed on this segment. The remaining two segments C_2 and C_3 represent light-cone lines. They are described by $x' = x + c(t - t')$ for C_2 and $x' = x - c(t - t')$ for C_3. We choose *anti-clockwise orientation* of the border $\partial\Omega$ in concordance with the axes ordering. Integrating the equation (6.56) one gets

$$\underbrace{\int_\Omega d\Omega\,(u_{t't'} - c^2 u_{x'x'})}_{I_{\text{left}}} = \underbrace{\int_\Omega d\Omega f(t',x')}_{I_{\text{right}}} \tag{6.57}$$

where I_{right} is given by the expression

$$I_{\text{right}} = \int_0^t dt' \int_{x-c(t-t')}^{x+c(t-t')} dx' f(t',x'). \tag{6.58}$$

On the other hand, the expression I_{left} can be evaluated applying Stokes theorem in two dimensions. Stokes theorem

$$\int_\Omega d\boldsymbol{a} \cdot (\nabla \times \boldsymbol{F}) = \oint_{\partial\Omega} \boldsymbol{F} \cdot d\boldsymbol{l}$$

simplifies to the form

$$\int_\Omega \underbrace{dy^1 dy^2}_{d\Omega} \left(\partial_1 F^2 - \partial_2 F^1\right) = \oint_{\partial\Omega} \left(F^1 dy^1 + F^2 dy^2\right) \tag{6.59}$$

where $dy^1 = dx'$ and $dy^2 = dt'$. The left hand side of the wave equation can be cast in the following form

$$\partial_{t'}^2 u - c^2 \partial_{x'}^2 u = \underbrace{\partial_{x'}}_{\partial_1} \underbrace{(-c^2 \partial_{x'} u)}_{F^2} - \underbrace{\partial_{t'}}_{\partial_2} \underbrace{(-\partial_{t'} u)}_{F^1} . \tag{6.60}$$

Thus the integral I_{left} can be replaced by the line integral taken along the border $\partial\Omega$

$$\begin{aligned} I_{\text{left}} &= \oint_{\partial\Omega} (F^1 dy^1 + F^2 dy^2) \\ &= \oint_{\partial\Omega} [-(\partial_{t'} u) dx' - c^2 (\partial_{x'} u) dt'] \\ &= I_1 + I_2 + I_3, \end{aligned}$$

where the integrals I_1, I_2, I_3 are evaluated on the segments C_1, C_2, C_3.

- Integral I_1:
  $dt' = 0$ along the line C_1 and thus

$$I_1 = \int_{x-ct}^{x+ct} dx' [-\partial_{t'} u(t', x')]_{t'=0} = - \int_{x-ct}^{x+ct} dx' \psi(x') \tag{6.61}$$

- Integral I_2:
  $dx' = -c\, dt'$ along the light-cone C_2. It gives

$$\begin{aligned} I_2 &= \int_{C_2} [-(\partial_{t'} u) dx' - c^2 (\partial_{x'} u) dt'] \\ &= \int_{C_2} \left[-(\partial_{t'} u)(-c\, dt') - c^2 (\partial_{x'} u) \left(-\frac{dx'}{c}\right)\right] \\ &= c \int_{C_2} [\partial_{t'} u\, dt' + \partial_{x'} u\, dx'] \\ &= c \int_{C_2} du \\ &= c[u(t,x) - u(0, x+ct)] = c[u(t,x) - \varphi(x+ct)]. \end{aligned} \tag{6.62}$$

- Integral I_3:

$dx' = +c\,dt'$ along the light-cone C_3. It gives

$$\begin{aligned}
I_3 &= \int_{C_3} [-(\partial_{t'} u) dx' - c^2 (\partial_{x'} u) dt'] \\
&= \int_{C_3} \left[-(\partial_{t'} u)(c\,dt') - c^2 (\partial_{x'} u) \left(\frac{dx'}{c} \right) \right] \\
&= -c \int_{C_3} [\partial_{t'} u\, dt' + \partial_{x'} u\, dx'] \\
&= -c \int_{C_3} du \\
&= -c[u(0, x - ct) - u(t, x)] = c[u(t, x) - \varphi(x - ct)]. \qquad (6.63)
\end{aligned}$$

Plugging the obtained result into $I_1 + I_2 + I_3 = I_{\text{right}}$ we get

$$\begin{aligned}
&2cu(t, x) - c[\varphi(x + ct) + \varphi(x - ct)] - \int_{x-ct}^{x+ct} dx' \psi(x') = \\
&= \int_0^t dt' \int_{x-c(t-t')}^{x+c(t-t')} dx' f(t', x')
\end{aligned}$$

which gives

$$\begin{aligned}
u(t, x) &= \frac{1}{2}[\varphi(x + ct) + \varphi(x - ct)] \\
&+ \frac{1}{2c} \int_{x-ct}^{x+ct} dx' \psi(x') + \frac{1}{2c} \int_0^t dt' \int_{x-c(t-t')}^{x+c(t-t')} dx' f(t', x'). \qquad (6.64)
\end{aligned}$$

Note that the last term can be also cast in the form

$$\begin{aligned}
&\frac{1}{2c} \int_0^t dt' \int_{x-c(t-t')}^{x+c(t-t')} dx' f(t', x') = \\
&= \int_{\mathbb{R}^2} dt' dx' \underbrace{\left[\frac{1}{2c} \theta\Big(c(t - t') - |x - x'| \Big) \right]}_{D^{(1)}_{ret}(t-t', x-x')} f(t', x'), \qquad (6.65)
\end{aligned}$$

where we define

$$\boxed{D^{(1)}_{ret}(t, x) := \frac{1}{2c} \theta(ct - |x|).}$$

6.3 *Radiation field from a moving particle*

In this section we pay attention to description of electromagnetic field from electrically charged particle in motion. First we derive the electromagnetic strength tensor and then obtain the associated energy momentum tensor. In further part we discuss the braking radiation. Finally we study radiation generated by large group of charges.

Radiation field

A radiation field is defined as the difference between outgoing and incoming fields. The radiation field in terms of four-potentials $A^{\mu}_{out}(x)$ and $A^{\mu}_{in}(x)$ reads

$$A^{\mu}_{rad}(x) := A^{\mu}_{out}(x) - A^{\mu}_{in}(x), \tag{6.66}$$

where $A^{\mu}_{out}(x)$ is the electromagnetic four-potential in *remote future* and $A^{\mu}_{in}(x)$ is the four-potential in *remote past*. The equality (6.50) allows us to put the radiation field in dependence on the retarded and advanced four-potentials

$$\begin{aligned} A^{\mu}_{rad}(x) &= A^{\mu}_{ret}(x) - A^{\mu}_{adv}(x) \\ &= \frac{4\pi}{c} \int d^4x' \Big[D_{ret}(x-x') - D_{adv}(x-x') \Big] J^{\mu}(x') \\ &= \frac{4\pi}{c} \int d^4x' \, D(x-x') J^{\mu}(x'), \end{aligned} \tag{6.67}$$

where $D(x-x')$ is defined in (6.22). Note that, (6.66) and (6.67) leads to relation

Scattering problem

$$A^{\mu}_{out}(x) = A^{\mu}_{in}(x) + \frac{4\pi}{c} \int d^4x' \, D(x-x') J^{\mu}(x') \tag{6.68}$$

which is a relation between asymptotic states of the electromagnetic field. Unlike the asymptotic fields, the disturbance represented by $\int d^4x' \, D(x-x') J^{\mu}(x')$ appears for finite times. Hence the expression (6.68) gives a relationship between two asymptotic states for the scattering process

$$A^{\mu}_{out}(x) = S A^{\mu}_{in}(x),$$

where S is the counterpart of the scattering matrix.

The electromagnetic field tensor $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$ takes the following form

$$F^{rad}_{\mu\nu}(x) = \frac{4\pi}{c} \int d^4x' \left[\frac{\partial D(x-x')}{\partial x^{\mu}} J_{\nu}(x') - \frac{\partial D(x-x')}{\partial x^{\nu}} J_{\mu}(x') \right] \tag{6.69}$$

where A^{μ} is given by (6.67).

Lienard-Wichert potentials

The simplest physical system containing electromagnetic radiation which can be described exactly is a point-like charged particle following some world-line. In the first stage we shall assume that the particle world line is given by the prescribed (explicitly given) curve. The particle in non-uniform motion is represents a forced physical system.[11] Thus we pay attention on kinematics but not dynamics of the particle. The particle dynamics in the presence of external electromagnetic field is studied in further part of this chapter. In such a case the particle world-line is determined by two factors: its interaction with the external field and the radiation process.

[11] It requires an external power source to maintain a prescribed motion of the particle.

At the present stage we shall assume that the world-line of the charged particle is explicitly given – described by the four vector $z^\mu(\tau)$. The electric four-current density of the particle reads

Four-current density of of a single point-like charged particle

$$J^\mu(x) = qc \int_{-\infty}^{\infty} d\tau \frac{dz^\mu}{d\tau} \delta^4(x - z(\tau)). \tag{6.70}$$

This four-current vanishes in everywhere except the world-line of the particle. With the help of (6.70) one gets the retarded (advanced) four-potentials

$$\begin{aligned} A^\mu_{\substack{ret\\adv}}(x) =& \frac{4\pi}{c} \int d^4x' D_{\substack{ret\\adv}}(x - x') J^\mu(x') \\ =& \frac{4\pi}{c} \frac{qc}{2\pi} \int d^4x' \int_{-\infty}^{\infty} d\tau\, \theta\Big(\pm (x^0 - x'^0) \Big) \delta\Big[(x - x')^2 \Big] \times \\ & \times \frac{dz^\mu}{d\tau} \delta^4(x' - z(\tau)) \\ =& 2q \int_{-\infty}^{\infty} d\tau\, \theta\Big(\pm (x^0 - z^0) \Big) \frac{dz^\mu}{d\tau} \delta\Big[(x - z(\tau))^2 \Big]. \end{aligned} \tag{6.71}$$

The last integral (6.71) is a relation between points (events) at the world-line of the particle and the point $(x^0, \boldsymbol{x})$ in spacetime where the field is evaluated. Equating to zero the argument of the Dirac delta we get the equation $(x - z(\tau))^2 = 0$. It has two solutions $\tau_{ret}(x)$ and $\tau_{adv}(x)$ which are proper times corresponding with the events at which the world-line cuts the past and future light-cone with vertex at x^μ. Thus, there are two lightlike four-vectors which connect the events $z(\tau_{\substack{ret\\adv}})$ at the world-line of the particle with the event x. We shall denote them

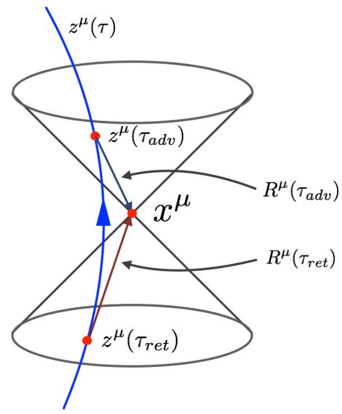


Figure 6.8: The world-line $z^\mu(\tau)$; the retarded and the advanced proper time.

$$R^\mu(\tau_{\substack{ret\\adv}}) := x^\mu - z^\mu(\tau_{\substack{ret\\adv}}).$$

The character of vectors means that they satisfy equations

$$R^\mu(\tau_{\substack{ret\\adv}}) R_\mu(\tau_{\substack{ret\\adv}}) = 0.$$

The Dirac delta of a function that has zeros is equal to the sum of deltas of binomials $\tau - \tau_{ret}$ and $\tau - \tau_{adv}$ which has the form

$$\delta\Big[(x - z(\tau))^2 \Big] = \frac{\delta(\tau - \tau_{ret})}{|-2\dot{z}^\mu R_\mu|_{\tau_{ret}}} + \frac{\delta(\tau - \tau_{adv})}{|-2\dot{z}^\mu R_\mu|_{\tau_{adv}}} \tag{6.72}$$

Plugging (6.72) into (6.71) one gets

$$
\begin{aligned}
A^{\mu}_{\substack{ret\\adv}}(x) &= q\int_{-\infty}^{\infty} \dot z^{\mu}\theta\Big(\pm(x^0 - z^0(\tau))\Big)\left[\frac{\delta(\tau-\tau_{ret})}{|\dot z^{\mu}R_{\mu}|_{\tau_{ret}}} + \frac{\delta(\tau-\tau_{adv})}{|\dot z^{\mu}R_{\mu}|_{\tau_{adv}}}\right]\\
&= q\,\theta\Big(\pm(x^0 - z^0(\tau_{ret}))\Big)\frac{\dot z^{\mu}(\tau_{ret})}{|\dot z^{\mu}R_{\mu}|_{\tau_{ret}}} + \\
&+ q\,\theta\Big(\pm(x^0 - z^0(\tau_{adv}))\Big)\frac{\dot z^{\mu}(\tau_{adv})}{|\dot z^{\mu}R_{\mu}|_{\tau_{adv}}}.
\end{aligned}
$$

Taking into account the identities

$$
\theta\Big(+(x^0 - z^0(\tau_{ret}))\Big) = 1, \qquad \theta\Big(-(x^0 - z^0(\tau_{adv}))\Big) = 1,
$$

$$
\theta\Big(+(x^0 - z^0(\tau_{adv}))\Big) = 0, \qquad \theta\Big(-(x^0 - z^0(\tau_{ret}))\Big) = 0,
$$

and the inequalities $\dot z^{\mu}R_{\mu}(\tau_{ret}) > 0$, $\dot z^{\mu}R_{\mu}(\tau_{adv}) < 0$ one gets *Lienart-Wichert potentials*

Lienard-Wichert potentials

$$
\boxed{A^{\mu}_{\substack{ret\\adv}}(x) = \pm q\left.\frac{\dot z^{\mu}}{R^{\alpha}\dot z_{\alpha}}\right|_{\tau_{\substack{ret\\adv}}(x)} = \frac{q}{c}\left.\frac{\dot z^{\mu}}{\rho}\right|_{\tau_{\substack{ret\\adv}}(x)}} \tag{6.73}
$$

where we have denoted

$$
\rho_{\substack{ret\\adv}} := \frac{1}{c}\Big|(x^{\alpha} - z^{\alpha}(\tau_{\substack{ret\\adv}}))\dot z_{\alpha}(\tau_{\substack{ret\\adv}})\Big| \equiv \pm\frac{1}{c}R^{\alpha}\dot z_{\alpha}\Big|_{\tau_{\substack{ret\\adv}}(x)}. \tag{6.74}
$$

The expression $\rho(x)$ has interpretation of *spatial distance* between the events x and $z(\tau_{\substack{ret\\adv}})$. In the instantaneous rest frame

$$
\rho(x)\Big|_{IR} = \pm\frac{1}{c}(x^0 - z^0)\Big|_{\substack{ret\\adv}} c = |\boldsymbol{x} - \boldsymbol{z}|.
$$

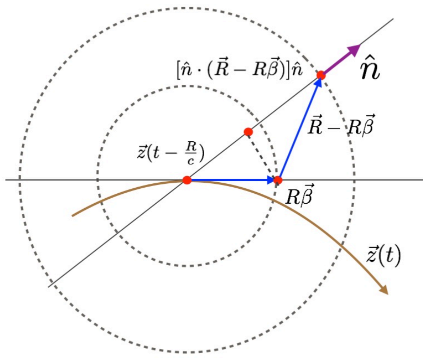


Figure 6.9: Relations between vectors $\boldsymbol{R}$ and $\boldsymbol{\beta}$.

This distance is evaluated in the *instantaneous rest frame* of a particle. Note that, $\tau_{\substack{ret\\adv}}$ is a function of x so the argument x enters to (6.73) explicitly and indirectly through the function $\tau_{\substack{ret\\adv}}(x)$.

In the last step we present the scalar and vector potentials. The four-vector $R^{\mu}(\tau_{ret})$ has components

$$
R^{\mu}(\tau_{ret}) \to (R, \boldsymbol{R}) = R\left(1, \frac{\boldsymbol{R}}{R}\right) = R(1, \hat{\boldsymbol{n}}), \tag{6.75}
$$

where $R^0 = R \equiv |\boldsymbol{R}|$.[12] Note, that orientation of the unit vector $\hat{\boldsymbol{n}}$ depends on time. We shall express the four-vector $A^{\mu}(x)$ in terms of the vectors $\hat{\boldsymbol{n}}$ and $\boldsymbol{\beta} = \frac{\boldsymbol{V}}{c}$. Contracting the four-vectors $\dot z^{\mu} = \gamma c(1, \boldsymbol{\beta})$ and $R^{\mu}(\tau_{ret}) = R(1, \hat{\boldsymbol{n}})$ one gets

[12] For the advanced solution $R^0 = -|\boldsymbol{R}|$.

$$
\dot z_{\mu}(\tau_{ret})R^{\mu}(\tau_{ret}) = R\gamma c\,(1 - \hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})|_{\tau_{ret}}.
$$

The relation between t' and t follows from the component $R^0(\tau_{ret}) = ct - ct'(\tau_{ret})$ and it reads

$$t'(\tau_{ret}) = t - \frac{R}{c}.$$

Thus, the retarded electromagnetic potentials can be expressed in the form

The retarded scalar potential and the vector potential

$$\begin{aligned} A^0(t, \boldsymbol{x}) &= \frac{q}{R} \frac{1}{1 - \hat{\boldsymbol{n}} \cdot \boldsymbol{\beta}}\bigg|_{t-\frac{R}{c}}, \\ \boldsymbol{A}(t, \boldsymbol{x}) &= \frac{q}{R} \frac{\boldsymbol{\beta}}{1 - \hat{\boldsymbol{n}} \cdot \boldsymbol{\beta}}\bigg|_{t-\frac{R}{c}}. \end{aligned} \tag{6.76}$$

The field tensor

The electromagnetic field tensor is given in terms of the retarded (advanced) potentials (6.73)

$$F_{\mu\nu}^{\substack{ret \\ adv}}(x) = \partial_\mu \left[\frac{q\dot{z}^\nu}{c\rho}\right]_{\tau(x)} - \partial_\nu \left[\frac{q\dot{z}^\mu}{c\rho}\right]_{\tau(x)}, \tag{6.77}$$

where $\tau(x) \equiv \tau_{\substack{ret \\ adv}}(x)$ is a function of x. For further convenience we define the symbol

$$(ab) := a^\alpha b_\alpha.$$

The first term of (6.77) reads

$z^\alpha = z^\alpha(\tau(x))$

$$\partial_\mu A_\nu^{\substack{ret \\ adv}}(x) = \frac{q}{c}\left[\frac{1}{\rho}\partial_\mu \dot{z}_\nu - \frac{1}{\rho^2}\partial_\mu \rho\, \dot{z}_\nu\right] \tag{6.78}$$

where partial derivatives of $z^\alpha = z^\alpha(\tau(x))$ contains derivatives of $\tau(x)$ according to $\frac{\partial z^\alpha}{\partial x^\mu} = \frac{dz^\alpha}{d\tau}\Big|_{\tau(x)} \frac{\partial \tau(x)}{\partial x^\mu}$. In particular, the derivative $\partial_\mu \rho$ reads

$$\begin{aligned} \partial_\mu \rho =& \partial_\mu \Big(\pm \frac{1}{c} R^\alpha \dot{z}_\alpha|_{\tau(x)} \Big) \\ =& \pm \frac{1}{c} \left[\partial_\mu (x^\alpha - z^\alpha)\dot{z}_\alpha + R^\alpha \partial_\mu \dot{z}_\alpha\right] \\ =& \pm \frac{1}{c} \Big[(\delta_\mu^\alpha - \dot{z}^\alpha \partial_\mu \tau(x))\dot{z}_\alpha + R^\alpha \ddot{z}_\alpha \partial_\mu \tau(x)\Big] \\ =& \pm \frac{1}{c} \Big[\dot{z}_\mu + \Big((R\ddot{z}) - c^2\Big)\partial_\mu \tau(x)\Big]. \end{aligned} \tag{6.79}$$

The expression $\partial_\mu \tau(x)$ follows from the equation $\partial_\mu (R^\alpha R_\alpha) = 0$ which gives

$$(\delta_\mu^\alpha - \dot{z}^\alpha \partial_\mu \tau(x))R_\alpha = 0.$$

Thus, $\partial_\mu \tau(x)$ reads

Expression $\partial_\mu \tau_{\substack{ret \\ adv}}(x)$

$$\partial_\mu \tau(x) = \frac{R_\mu}{(R\dot{z})} = \pm \frac{R_\mu}{c\,\rho} =: \pm k_\mu \qquad \Rightarrow \qquad R^\mu = c\rho k^\mu, \tag{6.80}$$

where $k^\mu \dot{z}_\mu = \pm 1$, $k^\mu k_\mu = 0$. The labels *ret*, *adv* have been omitted for simplicity. Hence

The lightlike four-vector k^μ:

$$k^\mu \dot{z}_\mu = \pm 1, \qquad k^\mu k_\mu = 0$$

$$R^\mu \equiv R^\mu_{\substack{ret\\adv}}, \qquad \rho \equiv \rho_{\substack{ret\\adv}}, \qquad k^\mu \equiv k^\mu_{\substack{ret\\adv}}.$$

Taking into account (6.80) one gets (6.79) in the form

$$\partial_\mu \rho = \pm \frac{1}{c} \dot{z}_\mu + \Big[\rho\,(k\ddot{z}) - c\Big] k_\mu. \tag{6.81}$$

Substituting this result into (6.78) one gets

$$\begin{aligned} \partial_\mu A_\nu^{\substack{ret\\adv}} &= \frac{q}{c}\left[\frac{1}{\rho}(\pm k_\mu \ddot{z}_\nu) - \frac{1}{c\,\rho^2}(\pm \dot{z}_\mu \dot{z}_\nu) - \frac{1}{\rho^2}\left(\rho\,(k\ddot{z}) - c\right) k_\mu \dot{z}_\nu\right] \\ &= q k_\mu \left[\frac{\dot{z}_\nu}{\rho^2} + \frac{\pm \ddot{z}_\nu - (k\ddot{z})\dot{z}_\nu}{c\,\rho}\right] \mp \frac{q}{c^2}\frac{\dot{z}_\mu \dot{z}_\nu}{\rho^2}. \end{aligned} \tag{6.82}$$

The term containing $\dot{z}_\mu \dot{z}_\nu$ does not contribute to the electromagnetic field tensor because it is symmetric in its indices μ and ν. This tensor takes the form

The electromagnetic field tensor

$$\boxed{F_{\mu\nu}^{\substack{ret\\adv}}(x) = k_\mu \xi_\nu - k_\nu \xi_\mu} \tag{6.83}$$

where

$$\xi_\nu := q\left[\frac{\dot{z}_\nu}{\rho^2} + \frac{\pm \ddot{z}_\nu - (k\ddot{z})\dot{z}_\nu}{c\,\rho}\right]. \tag{6.84}$$

The tensor (6.83) can be split into two terms

$$F_{\mu\nu}^{\substack{ret\\adv}}(x) = F^{(1)}_{\mu\nu}(x) + F^{(2)}_{\mu\nu}(x)$$

where

$$F^{(1)}_{\mu\nu}(x) = \frac{q}{\rho^2}(k_\mu \dot{z}_\nu - k_\nu \dot{z}_\mu), \tag{6.85}$$

$$F^{(2)}_{\mu\nu}(x) = \frac{q}{c\,\rho}\Big[\pm(k_\mu \ddot{z}_\nu - k_\nu \ddot{z}_\mu) - (k\ddot{z})(k_\mu \dot{z}_\nu - k_\nu \dot{z}_\mu)\Big]. \tag{6.86}$$

The Coulomb field

Note that the expression $F^{(1)}_{\mu\nu}(x)$ does not contain second derivatives with respect to proper time (independence on acceleration). To have insight into its physical meaning we assume $\dot{z}^\mu = const$. In such a case $\ddot{z}^\mu = 0$ and the electromagnetic field is given by the first term

$$F_{\mu\nu}^{\substack{ret\\adv}}(x) = F^{(1)}_{\mu\nu}(x).$$

We take a *space-like four-vector* w^μ such that

$w^2 < 0$

$$w^\mu \dot{z}_\mu = 0, \qquad w^\mu w_\mu = -c^2. \tag{6.87}$$

The light-like four-vector k^μ can be written in the from

$$k^\mu_{\substack{ret\\adv}} = \frac{1}{c^2}(\pm \dot{z}^\mu + w^\mu) \tag{6.88}$$

where the sign "+" stands for the retarded function and "−" for the advanced one. Substituting the expression (6.88) into (6.85) one gets

$$F^{(1)}_{\mu\nu}(x) = \frac{q}{c^2\rho^2}(\dot z_\mu w_\nu - \dot z_\nu w_\mu) \tag{6.89}$$

where the retarded and advanced distance have equal values, $\rho_{ret} = \rho_{adv} \equiv \rho$. The four-vectors $\dot z^\mu$ and w^μ in the *instantaneous rest frame* of a particle read

$$\dot z^\mu = (c, \mathbf{0}), \qquad w^\mu = (0, c\,\hat{\boldsymbol{n}}).$$

Thus the tensor $F^{(1)}_{\mu\nu}$ represents the *Coulomb field* associated with a charged particle

The Coulomb field

$$F^{(1)}_{0i} = \frac{q\,n^i}{\rho^2}, \qquad F^{(1)}_{ij} = 0.$$

The particle which moves without acceleration is surrounded by a field which is proportional to $\sim \rho^{-2}$. This is the proper field from a charged particle. Note that the electromagnetic field tensors (6.89) for the retarded and advanced solutions are equal

The proper field from a particle does not contribute to electromagnetic radiation

$$F^{(1)ret}_{\mu\nu} = F^{(1)adv}_{\mu\nu}.$$

This field does not contribute to electromagnetic radiation given by

$$F^{rad}_{\mu\nu} = F^{ret}_{\mu\nu} - F^{adv}_{\mu\nu} = F^{(1)ret}_{\mu\nu} - F^{(1)adv}_{\mu\nu} = 0.$$

The total field

In what follows, we look at total electromagnetic field which contains terms proportional to acceleration of the particle. We express all the formulas with the help of four-vector n^μ defined as follows

$$R^\mu \equiv R n^\mu \to R(\pm 1, \hat{\boldsymbol{n}})$$

where $R \equiv |\boldsymbol{R}|$ and $\hat{\boldsymbol{n}}^2 = 1$. It gives

$$k^\mu_{\substack{ret\\adv}} = \pm\frac{R^\mu}{(R\dot z)} = \pm\frac{n^\mu}{(n\dot z)} \qquad \text{and} \qquad \rho_{\substack{ret\\adv}} = \pm\frac{1}{c}(R\dot z) = \pm\frac{R}{c}(n\dot z).$$

Thus, the expression (6.84) can be cast in the form

$$\begin{aligned}
\xi_\nu &= q\left[\frac{\dot z_\nu}{\rho^2} + \frac{\pm\ddot z_\nu - (k\ddot z)\dot z_\nu}{c\,\rho}\right] \\
&= q\left[\frac{\dot z_\nu}{\frac{R^2}{c^2}(n\dot z)^2} + \frac{\pm\left(\ddot z_\nu - \frac{(n\ddot z)}{(n\dot z)}\dot z_\nu\right)}{\pm R(n\dot z)}\right] \\
&= \frac{qc^2}{R^2}\frac{\dot z_\nu}{(n\dot z)^2} + \frac{q}{R}\frac{(n\dot z)\ddot z_\nu - (n\ddot z)\dot z_\nu}{(n\dot z)^2}.
\end{aligned} \tag{6.90}$$

The electromagnetic field tensor $F^{\substack{ret\\adv}}_{\mu\nu} = F^{(1)}_{\mu\nu} + F^{(2)}_{\mu\nu}$ is given in terms of expressions

The components of the electromagnetic tensor of a charged particle in non-uniform motion

$$F^{(1)}_{\mu\nu} = \pm \frac{qc^2}{R^2} \left. \frac{n_\mu \dot{z}_\nu - n_\nu \dot{z}_\mu}{(n\dot{z})^3} \right|_{\substack{ret \\ adv}}, \tag{6.91}$$

$$F^{(2)}_{\mu\nu} = \pm \frac{q}{R} \left. \frac{(n\dot{z})(n_\mu \ddot{z}_\nu - n_\nu \ddot{z}_\mu) - (n\ddot{z})(n_\mu \dot{z}_\nu - n_\nu \dot{z}_\mu)}{(n\dot{z})^3} \right|_{\substack{ret \\ adv}}. \tag{6.92}$$

The part of the electromagnetic field tensor (6.91) proportional to R^{-2} has been already recognized as the Coulomb field from the charged particle. It is said that there is no external free field if the electromagnetic field surrounding the charged particle is equal to its Coulomb field.

On the other hand, the retarded and advanced tensors $F^{(2)}_{\mu\nu}$ differ from each other. It means that the radiation field $F^{rad}_{\mu\nu} = F^{ret}_{\mu\nu} - F^{adv}_{\mu\nu}$ is associates with the non-uniformly moving charged particle. In particular, in absence of the advanced field, the radiation field is determined by the retarded electromagnetic field.

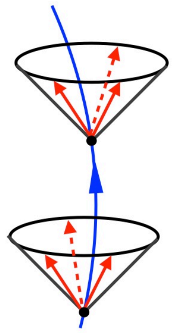


Figure 6.10: The retarded field sent from the world-line $z^\mu(\tau)$.

For instance, if the particle has constant velocity in the remote past, then assuming that there is no independent incoming free field, the electromagnetic field at any Cauchy surface[13] is the Coulomb field. The outgoing field must contain the Coulomb field. However, it could happen that the retarded field contains also some additional field. In such a case this extra field is interpreted as the radiation field – sent out by a charged particle. The radiation field is send out along the future light cone with the apex at the world-line of the particle, see Figure 6.10.

[13] In the past, i.e. before any perturbation occurs.

Electric and magnetic retarded fields

In this section we shall consider only *retarded fields* $\boldsymbol{E}$ and $\boldsymbol{B}$. The electric and magnetic field can be obtained directly from $F^{ret}_{\mu\nu}$. We shall denote by $\boldsymbol{E}_{(1)}$, $\boldsymbol{B}_{(1)}$ the Coulomb fields and by $\boldsymbol{E}_{(2)}$, $\boldsymbol{B}_{(2)}$ the radiation fields. The particle's four-velocity and four-acceleration read

$$\dot{z}^\mu \to \gamma c(1, \boldsymbol{\beta}),$$
$$\ddot{z}^\mu \to \gamma^4 (\boldsymbol{\beta} \cdot \boldsymbol{a})(1, \boldsymbol{\beta}) + \gamma^2 (0, \boldsymbol{a}),$$

where

$$\gamma = \frac{dt}{d\tau}, \qquad \boldsymbol{\beta} := \frac{1}{c}\frac{d\boldsymbol{z}}{dt}, \qquad \boldsymbol{a} := \frac{d^2\boldsymbol{z}}{dt^2}.$$

$\boldsymbol{a}$ is a three-dimensional acceleration in the laboratory reference frame. The expressions $(n\dot{z})$ and $(n\ddot{z})$ have the form

$$(n\dot{z}) = \gamma c(1 - \hat{\boldsymbol{n}} \cdot \boldsymbol{\beta}),$$
$$(n\ddot{z}) = \gamma^4 (\boldsymbol{\beta} \cdot \boldsymbol{a})(1 - \hat{\boldsymbol{n}} \cdot \boldsymbol{\beta}) - \gamma^2 \hat{\boldsymbol{n}} \cdot \boldsymbol{a}.$$

The electric field of the Coulomb part reads

The electric field associated with the Coulomb part

$$\begin{aligned}
E^i_{(1)} = F^{i0}_{(1)} &= \frac{qc^2}{R^2}\left.\frac{n^i \dot{z}^0 - n^0 \dot{z}^i}{(n\dot{z})^3}\right|_{t-\frac{R}{c}} \\
&= \frac{qc^2}{R^2}\left.\frac{\gamma c n^i - \gamma v^i}{\gamma^3 c^3 (1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})^3}\right|_{t-\frac{R}{c}} \\
&= \frac{q}{R^2}\,(1-\beta^2)\left.\frac{n^i - \beta^i}{(1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})^3}\right|_{t-\frac{R}{c}}, \qquad (6.93)
\end{aligned}$$

where the retarded time in the laboratory reference frame is given by the expression

$$t_{ret} = t - \frac{R}{c}.$$

The magnetic field has the form

The magnetic field associated with the Coulomb part

$$\begin{aligned}
B^i_{(1)} &= -\frac{1}{2}\epsilon_{ijk}F^{jk}_{(1)} = -\frac{1}{2}\epsilon_{ijk}\frac{qc^2}{R^2}\left.\frac{n^j\dot{z}^k - n^k\dot{z}^j}{(n\dot{z})^3}\right|_{t-\frac{R}{c}} \\
&= -\epsilon_{ijk}\frac{qc^2}{R^2}\left.\frac{n^j\dot{z}^k}{(n\dot{z})^3}\right|_{t-\frac{R}{c}} \\
&= \frac{q}{R^2}\,(1-\beta^2)\left.\frac{\epsilon_{ijk}n^j(-\beta^k)}{(1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})^3}\right|_{t-\frac{R}{c}} \\
&= \frac{q}{R^2}\,(1-\beta^2)\left.\frac{\epsilon_{ijk}n^j(n^k-\beta^k)}{(1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})^3}\right|_{t-\frac{R}{c}} \\
&= \left.(\hat{\boldsymbol{n}}\times\boldsymbol{E}_{(1)})^i\right|_{t-\frac{R}{c}}. \qquad (6.94)
\end{aligned}$$

The magnetic field (6.94) is perpendicular to the electric field (6.93) and its magnitude is proportional to magnitude of the electric field.

The electric field which depends on the acceleration is given by expression

The electric part of the radiation field

$$\begin{aligned}
E^i_{(2)} = F^{i0}_{(2)} &= \frac{q}{R}\left.\frac{(n\dot{z})(n^i\ddot{z}^0 - n^0\ddot{z}^i) - (n\ddot{z})(n^i\dot{z}^0 - n^0\dot{z}^i)}{(n\dot{z})^3}\right|_{t-\frac{R}{c}} \\
&= \frac{q}{R}\left.\frac{[(n\dot{z})\ddot{z}^0 - (n\ddot{z})\dot{z}^0]n^i - [(n\dot{z})\ddot{z}^i - (n\ddot{z})\dot{z}^i]}{(n\dot{z})^3}\right|_{t-\frac{R}{c}}. \qquad (6.95)
\end{aligned}$$

Substituting expressions

$$\begin{aligned}
(n\dot{z})\ddot{z}^0 - (n\ddot{z})\dot{z}^0 &= \gamma c(1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})\,[\gamma^4(\boldsymbol{\beta}\cdot\boldsymbol{a})] \\
&\quad - [\gamma^4(\boldsymbol{\beta}\cdot\boldsymbol{a})(1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta}) - \gamma^2\hat{\boldsymbol{n}}\cdot\boldsymbol{a}]\gamma c \\
&= \gamma^3 c\,\hat{\boldsymbol{n}}\cdot\boldsymbol{a} \qquad (6.96)
\end{aligned}$$

and

$$\begin{aligned}
(n\dot{z})\ddot{z}^i - (n\ddot{z})\dot{z}^i &= \gamma c(1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})[\cancel{\gamma^4(\boldsymbol{\beta}\cdot\boldsymbol{a})\beta^i} + \gamma^2 a^i] \\
&\quad - [\cancel{\gamma^4(\boldsymbol{\beta}\cdot\boldsymbol{a})(1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})} - \gamma^2(\hat{\boldsymbol{n}}\cdot\boldsymbol{a})]\gamma c\beta^i \\
&= \gamma^3 c\,[(\hat{\boldsymbol{n}}\cdot\boldsymbol{a})\beta^i + (1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})a^i] \qquad (6.97)
\end{aligned}$$

into the formula (6.95) one gets

$$\begin{aligned} E^i_{(2)} &= \frac{q}{R}\left.\frac{\gamma^3 c\,(\hat{\boldsymbol{n}}\cdot\boldsymbol{a})n^i - \gamma^3 c\,[(\hat{\boldsymbol{n}}\cdot\boldsymbol{a})\beta^i + (1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})a^i]}{\gamma^3 c^3(1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})^3}\right|_{t-\frac{R}{c}} \\ &= \frac{q}{Rc^2}\left.\frac{(\hat{\boldsymbol{n}}\cdot\boldsymbol{a})(n^i-\beta^i) - \hat{\boldsymbol{n}}\cdot(\hat{\boldsymbol{n}}-\boldsymbol{\beta})a^i}{(1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})^3}\right|_{t-\frac{R}{c}} \\ &= \frac{q}{Rc^2}\left.\frac{(\hat{\boldsymbol{n}}\times[(\hat{\boldsymbol{n}}-\boldsymbol{\beta})\times\boldsymbol{a}])^i}{(1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})^3}\right|_{t-\frac{R}{c}}. \end{aligned} \tag{6.98}$$

The electric field (6.98) has origin in accelerated motion of the charged particle, $\boldsymbol{a} \neq 0$.

Finally, the magnetic field takes the form

The magnetic field associated with the radiation part

$$\begin{aligned} B^i_{(2)} &= -\frac{1}{2}\epsilon_{ijk}F^{jk}_{(2)} \\ &= -\frac{1}{2}\frac{q}{R}\epsilon_{ijk}\left.\frac{(n\dot{z})[n^j\ddot{z}^k - n^k\ddot{z}^j] - (n\ddot{z})[n^j\dot{z}^k - n^k\dot{z}^j]}{(n\dot{z})^3}\right|_{t-\frac{R}{c}} \\ &= -\frac{q}{R}\epsilon_{ijk}\left.\frac{(n\dot{z})[n^j\ddot{z}^k] - (n\ddot{z})[n^j\dot{z}^k]}{(n\dot{z})^3}\right|_{t-\frac{R}{c}} \\ &= \frac{q}{R}\left.\frac{\epsilon_{ijk}n^j[(n\ddot{z})\dot{z}^k - (n\dot{z})\ddot{z}^k]}{(n\dot{z})^3}\right|_{t-\frac{R}{c}} \\ &\overset{(6.97)}{=} \frac{q}{R}\epsilon_{ijk}n^j\left.\frac{\gamma^3 c\,[(\hat{\boldsymbol{n}}\cdot\boldsymbol{a})(-\beta^k) - (1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})a^k]}{\gamma^3c^3(1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})^3}\right|_{t-\frac{R}{c}} \\ &= \frac{q}{Rc^2}\epsilon_{ijk}n^j\left.\frac{(\hat{\boldsymbol{n}}\cdot\boldsymbol{a})(n^k-\beta^k) - \hat{\boldsymbol{n}}\cdot(\hat{\boldsymbol{n}}-\boldsymbol{\beta})a^k}{(1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})^3}\right|_{t-\frac{R}{c}} \\ &= \frac{q}{Rc^2}\left.\left[\hat{\boldsymbol{n}}\times\frac{\hat{\boldsymbol{n}}\times[(\hat{\boldsymbol{n}}-\boldsymbol{\beta})\times\boldsymbol{a}]}{(1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})^3}\right]^i\right|_{t-\frac{R}{c}} \\ &= [\hat{\boldsymbol{n}}\times\boldsymbol{E}_{(2)}]^i\Big|_{t-\frac{R}{c}}. \end{aligned} \tag{6.99}$$

To summarize our results we define the vector

Summary

$$\boldsymbol{K} := \frac{\hat{\boldsymbol{n}}-\boldsymbol{\beta}}{(1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})^3} \tag{6.100}$$

which allows us to put (6.93), (6.94), (6.98) and (6.99) in the following form

$$\boldsymbol{E}_{(1)} = \frac{q}{R^2}\,(1-\beta^2)\boldsymbol{K}\Big|_{t-\frac{R}{c}}, \qquad \boldsymbol{B}_{(1)} = \hat{\boldsymbol{n}}\times\boldsymbol{E}_{(1)}, \tag{6.101}$$

$$\boldsymbol{E}_{(2)} = \frac{q}{Rc^2}\,\hat{\boldsymbol{n}}\times(\boldsymbol{K}\times\boldsymbol{a})\Big|_{t-\frac{R}{c}}, \qquad \boldsymbol{B}_{(2)} = \hat{\boldsymbol{n}}\times\boldsymbol{E}_{(2)}. \tag{6.102}$$

It follows from these expressions that the total magnetic field is a cross product of $\hat{\boldsymbol{n}}$ and the total electric field $\boldsymbol{E} = \boldsymbol{E}_{(1)} + \boldsymbol{E}_{(2)}$.

The Coulomb field

In this section we show the equivalence of two expressions for the Coulomb field from the charged particle. This expressions are obtained with the help of retarded potentials and, alternatively, as the Lorentz boost of the electrostatic field surrounding a charged particle.

According to our previous considerations, the Lorentz transformation of the electrostatic field from a point charge reads

$$\boldsymbol{E}(t,\boldsymbol{x}) = q\frac{\boldsymbol{R}'}{R'^3}\frac{1-\beta^2}{(1-\beta^2\sin^2\vartheta')^{3/2}}, \qquad \boldsymbol{B} = \boldsymbol{\beta}\times\boldsymbol{E} \tag{6.103}$$

where, according to Figure 6.11,

$$\boldsymbol{R}' = \boldsymbol{R} - R\boldsymbol{\beta}, \qquad R' = R|\hat{\boldsymbol{n}} - \boldsymbol{\beta}|.$$

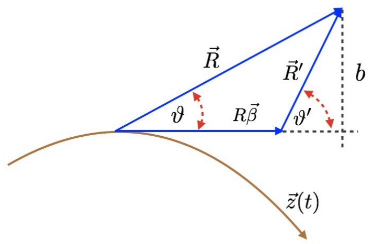


Figure 6.11: Relation between $\boldsymbol{R}$ and $\boldsymbol{R}'$.

The electric $\boldsymbol{E}(t,\boldsymbol{x})$ and magnetic $\boldsymbol{B}(t,\boldsymbol{x})$ Coulomb fields are given in terms of particle's position and velocity taken *at the retarded time* $t' = t - \frac{R}{c}$. The accelerated and non-accelerated motions differ by actual position of the particle at t. If the acceleration is zero, this position is given by the vector $R\boldsymbol{\beta}$, otherwise the particle's position is given by the corresponding trajectory $\boldsymbol{z}(t)$. The Coulomb electric field points out in direction of the $\boldsymbol{R}'$ vector. Clearly, this direction has nothing to do with the actual position of the particle. In particular, for $\boldsymbol{a} \neq 0$ the position of the particle *does not coincide* with $R\boldsymbol{\beta}$ *i.e.* with a tail of $\boldsymbol{R}'$.

In what follows, we show that (6.103) and (6.101) are equivalent. According to Figure 6.11 the angle ϑ and ϑ' satisfy $\sin\vartheta = \frac{b}{R}$ and $\sin\vartheta' = \frac{b}{R'}$. It gives

$$\sin\vartheta' = \frac{R}{R'}\sin\vartheta = \frac{\sin\vartheta}{|\hat{\boldsymbol{n}}-\boldsymbol{\beta}|}. \tag{6.104}$$

Thus, the electric field can be cast in the form

$$\begin{aligned}
\boldsymbol{E}(t,\boldsymbol{x}) &= q\frac{\boldsymbol{R}'}{R'^3}\frac{1-\beta^2}{\left(1-\beta^2\frac{\sin^2\vartheta}{|\hat{\boldsymbol{n}}-\boldsymbol{\beta}|^2}\right)^{3/2}}\Bigg|_{t-\frac{R}{c}} \\
&= q\underbrace{\boldsymbol{R}'}_{R(\hat{\boldsymbol{n}}-\boldsymbol{\beta})}\underbrace{\frac{|\hat{\boldsymbol{n}}-\boldsymbol{\beta}|^3}{R'^3}}_{\frac{1}{R^3}}\frac{1-\beta^2}{(|\hat{\boldsymbol{n}}-\boldsymbol{\beta}|^2-\beta^2\sin^2\vartheta)^{3/2}}\Bigg|_{t-\frac{R}{c}} \\
&= q\frac{\hat{\boldsymbol{n}}-\boldsymbol{\beta}}{R^2}\frac{1-\beta^2}{(1-2\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta}+\underbrace{\beta^2(1-\sin^2\vartheta)}_{(\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})^2})^{3/2}}\Bigg|_{t-\frac{R}{c}} \\
&= \frac{q}{R^2}\frac{(1-\beta^2)(\hat{\boldsymbol{n}}-\boldsymbol{\beta})}{[(1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})^2]^{3/2}}\Bigg|_{t-\frac{R}{c}} = \frac{q}{R^2}\frac{(1-\beta^2)(\hat{\boldsymbol{n}}-\boldsymbol{\beta})}{(1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})^3}\Bigg|_{t-\frac{R}{c}}
\end{aligned} \tag{6.105}$$

Thus the electric field has the form (6.101). The magnetic field is a consequence of (6.105) and the equality

$$\boldsymbol{\beta} \times (\hat{\boldsymbol{n}} - \boldsymbol{\beta}) = \boldsymbol{\beta} \times \hat{\boldsymbol{n}} = -\hat{\boldsymbol{n}} \times \boldsymbol{\beta} = \hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} - \boldsymbol{\beta}). \tag{6.106}$$

It reads

$$\boldsymbol{B}(t, \boldsymbol{x}) = \hat{\boldsymbol{n}} \times \boldsymbol{E}\Big|_{t-\frac{R}{c}}. \tag{6.107}$$

Thus, in the case of uniform motion both formulas are equivalent. The retarded potentials, however, give deeper insight into the problem.

Radiated power

Emission of radiation is a process in which an accelerated particle *looses* its energy. This emission is described by *energy flux* which represents the amount of energy per unit of time emitted in the solid angle $d\Omega$ in the direction of $\hat{\boldsymbol{n}}$. The energy emitted during infinitesimal time interval dt' is given by $dP(t')dt'$, where $dP(t')$ stands for *radiated power*.[14] This power is a function of angles parametrising vector $\hat{\boldsymbol{n}}$. We denote by dt' the interval in which certain infinitesimal amount of energy is emitted and by dt the interval in which this amount of energy is registered –pass through the infinitesimal area $d\boldsymbol{a} = \hat{\boldsymbol{n}} R^2 d\Omega$. Since both energies are equal, then

[14] More precisely, it is a power distribution.

Radiated power

$$dP(t')dt' = dP(t)dt. \tag{6.108}$$

Equality of energies emitted by the particle and registered in the laboratory reference frame

Instants of time t' and t are related by the expression

$$t = t' + \frac{1}{c} R(t') \tag{6.109}$$

where t' and t stand, respectively, for emission and registration instants of time ($\boldsymbol{E}(t)$ and $\boldsymbol{B}(t)$ are evaluated at t). The expression (6.109) leads to the relation

$$\frac{dt}{dt'} = 1 + \frac{1}{c} \frac{dR(t')}{dt'}.$$

Taking the derivative of $R(t')^2 = \boldsymbol{R}(t') \cdot \boldsymbol{R}(t')$ with respect to t' one gets

$$\cancel{2R(t')} \frac{dR(t')}{dt'} = \cancel{2R(t')} \hat{\boldsymbol{n}}(t') \cdot \frac{d\boldsymbol{R}(t')}{dt'}$$

where $\boldsymbol{R}(t') = \boldsymbol{x} - \boldsymbol{z}(t') = R(t')\hat{\boldsymbol{n}}(t')$. It leads to the expression

$$\frac{dR(t')}{dt'} = \hat{\boldsymbol{n}}(t') \cdot \frac{d}{dt'}(\boldsymbol{x} - \boldsymbol{z}(t')) = -\hat{\boldsymbol{n}}(t') \cdot \boldsymbol{v}(t') = -c\, \hat{\boldsymbol{n}}(t') \cdot \boldsymbol{\beta}(t').$$

The relation between the time intervals dt' and dt reads

Relation between the intervals dt' and dt

$$\boxed{\frac{dt}{dt'} = 1 - \hat{\boldsymbol{n}} \cdot \boldsymbol{\beta}} \tag{6.110}$$

where $\hat{\boldsymbol{n}} \cdot \boldsymbol{\beta}$ is taken at the retarded time t'. This expression allows us to get the radiated power $P(t')$ in terms of the power measured at the sphere with radius R.[15] Comparing the expression

$$dP(t)dt = \left[dP(t)\frac{dt}{dt'}\right]dt' = dP(t')dt' \tag{6.111}$$

with (6.108) one gets the radiated power

The radiated power in terms of the registered power

$$dP(t') = (1 - \hat{\boldsymbol{n}} \cdot \boldsymbol{\beta})dP(t). \tag{6.112}$$

The registered power distribution $dP(t)$ can be expressed in terms of the Poynting vector $\boldsymbol{S}(t)$ and it has the form

$$dP(t) = \boldsymbol{S}(t) \cdot d\boldsymbol{a} = \boldsymbol{S}(t) \cdot \hat{\boldsymbol{n}}\, R^2 d\Omega \tag{6.113}$$

where

The Poynting vector

$$\boldsymbol{S}(t) = \frac{c}{4\pi}\boldsymbol{E} \times \boldsymbol{B} = \frac{c}{4\pi}\Big[\boldsymbol{E} \times (\hat{\boldsymbol{n}} \times \boldsymbol{E})\Big] = \frac{c}{4\pi}\Big[\hat{\boldsymbol{n}}\,\boldsymbol{E}^2 - (\hat{\boldsymbol{n}} \cdot \boldsymbol{E})^2\Big]. \tag{6.114}$$

Since the electric field $\boldsymbol{E}$ satisfies[16] $\boldsymbol{E} \cdot \hat{\boldsymbol{n}} = 0$, then

$$\begin{aligned} dP(t) &= \frac{c}{4\pi}\boldsymbol{E}^2 R^2 d\Omega = \frac{c}{4\pi} R^2 \frac{q^2}{R^2 c^4}\left[\hat{\boldsymbol{n}} \times (\boldsymbol{K} \times \boldsymbol{a})\right]^2\Big|_{t-\frac{R}{c}} d\Omega \\ &= \frac{q^2}{4\pi c^3}\Big[(\hat{\boldsymbol{n}} \cdot \boldsymbol{a})\boldsymbol{K} - (\hat{\boldsymbol{n}} \cdot \boldsymbol{K})\boldsymbol{a}\Big]^2\Big|_{t-\frac{R}{c}} d\Omega \\ &= \frac{q^2}{4\pi c^3}\Big[(\hat{\boldsymbol{n}} \cdot \boldsymbol{a})^2\boldsymbol{K}^2 + (\hat{\boldsymbol{n}} \cdot \boldsymbol{K})^2\boldsymbol{a}^2 - 2(\hat{\boldsymbol{n}} \cdot \boldsymbol{a})(\hat{\boldsymbol{n}} \cdot \boldsymbol{K})(\boldsymbol{a} \cdot \boldsymbol{K})\Big]_{t-\frac{R}{c}} d\Omega. \end{aligned} \tag{6.115}$$

Substituting $\boldsymbol{K}$ given by (6.100) into (6.115) one gets

$$\begin{aligned} dP(t) &= \frac{q^2}{4\pi c^3}\frac{1}{(1 - \hat{\boldsymbol{n}} \cdot \boldsymbol{\beta})^6}[(\hat{\boldsymbol{n}} \cdot \boldsymbol{a})^2(\hat{\boldsymbol{n}} - \boldsymbol{\beta})^2 + (1 - \hat{\boldsymbol{n}} \cdot \boldsymbol{\beta})^2\boldsymbol{a}^2 \\ &\quad - 2(\hat{\boldsymbol{n}} \cdot \boldsymbol{a})(1 - \hat{\boldsymbol{n}} \cdot \boldsymbol{\beta})(\hat{\boldsymbol{n}} \cdot \boldsymbol{a} - \boldsymbol{\beta} \cdot \boldsymbol{a})]_{t-\frac{R}{c}} d\Omega \\ &= \frac{q^2}{4\pi c^3}\frac{1}{(1 - \hat{\boldsymbol{n}} \cdot \boldsymbol{\beta})^6}\Bigg[(\hat{\boldsymbol{n}} \cdot \boldsymbol{a})^2 \underbrace{(1 + \beta^2 - 2\hat{\boldsymbol{n}} \cdot \boldsymbol{\beta})}_{-(1-\beta^2)+\cancel{2(1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})}} + (1 - \hat{\boldsymbol{n}} \cdot \boldsymbol{\beta})^2\boldsymbol{a}^2 \\ &\quad - \cancel{2(\hat{\boldsymbol{n}} \cdot \boldsymbol{a})^2(1 - \hat{\boldsymbol{n}} \cdot \boldsymbol{\beta})} + 2(\hat{\boldsymbol{n}} \cdot \boldsymbol{a})(1 - \hat{\boldsymbol{n}} \cdot \boldsymbol{\beta})(\hat{\boldsymbol{n}} \cdot \boldsymbol{\beta})\Bigg]_{t-\frac{R}{c}} d\Omega. \end{aligned} \tag{6.116}$$

Finally, substituting (6.116) into (6.112) we get angular distribution of the radiated power

Angular distribution of the radiated power

$$dP(t') = \frac{q^2}{4\pi c^3}W(t', \hat{\boldsymbol{n}})d\Omega \tag{6.117}$$

where

$$W(t', \hat{\boldsymbol{n}}) := \left[\frac{\boldsymbol{a}^2}{(1 - \hat{\boldsymbol{n}} \cdot \boldsymbol{\beta})^3} + 2\frac{(\hat{\boldsymbol{n}} \cdot \boldsymbol{a})(\boldsymbol{\beta} \cdot \boldsymbol{a})}{(1 - \hat{\boldsymbol{n}} \cdot \boldsymbol{\beta})^4} - (1 - \beta^2)\frac{(\hat{\boldsymbol{n}} \cdot \boldsymbol{a})^2}{(1 - \hat{\boldsymbol{n}} \cdot \boldsymbol{\beta})^5}\right]_{t-\frac{R}{c}}. \tag{6.118}$$

[15] Note, that the fields are singular at the position of a charged particle, hence the Poynting vector is also singular at the particle.

[16] Here $\boldsymbol{E} \equiv \boldsymbol{E}_{(2)}$. We omit the Coulomb component because it behaves as R^{-2} so it became irrelevand far from the radiation source.

Acceleration parallel to velocity

In this section we look at the case of mutually parallel velocity and acceleration of the particle *i.e.* $\boldsymbol{\beta} \times \boldsymbol{a} = 0$. The electric field (6.102) is proportional to

$$\hat{\boldsymbol{n}} \times (\boldsymbol{K} \times \boldsymbol{a}) = \frac{\hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \boldsymbol{a})}{(1 - \hat{\boldsymbol{n}} \cdot \boldsymbol{\beta})^3} = \frac{(\hat{\boldsymbol{n}} \cdot \boldsymbol{a})\hat{\boldsymbol{n}} - \boldsymbol{a}}{(1 - \hat{\boldsymbol{n}} \cdot \boldsymbol{\beta})^3}.$$

The power distribution (6.117) reads

$$dP(t') = \frac{q^2}{4\pi c^3} \frac{\boldsymbol{a}^2 - (\hat{\boldsymbol{n}} \cdot \boldsymbol{a})^2}{(1 - \hat{\boldsymbol{n}} \cdot \boldsymbol{\beta})^5}\bigg|_{t-\frac{R}{c}} d\Omega. \tag{6.119}$$

Let $\hat{\boldsymbol{z}}$ be a versor parallel to $\boldsymbol{\beta}$, defined as $\hat{\boldsymbol{z}} := \boldsymbol{\beta}/\beta$. The acceleration reads $\boldsymbol{a} = a\hat{\boldsymbol{z}}$. We introduce spherical coordinates where ϑ in an angle between $\hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{z}}$, hence $\hat{\boldsymbol{n}} \cdot \boldsymbol{a} = a \cos\vartheta$. The power radiated into the infinitesimal solid angle $d\Omega = \sin\vartheta \, d\vartheta \, d\varphi$ reads

$$\boxed{dP(t') = \frac{q^2}{4\pi c^3} \frac{a^2 \sin^2\vartheta}{(1 - \beta\cos\vartheta)^5}\bigg|_{t-\frac{R}{c}} d\Omega.} \tag{6.120}$$

Note, that this formula does not depend on the fact if velocity and acceleration are parallel $\vartheta = 0$ or antiparallel $\vartheta = \pi$. The function $W(t, \hat{\boldsymbol{n}})$ is shown in Figure 6.12, Figure 6.13 and Figure 6.14 for three different velocities β.

The total value of radiated power is obtained by integration of (6.120) over angles

$$P(t') = \frac{q^2 a^2}{4\pi c^3}(2\pi) \int_0^\pi d\vartheta \frac{\sin^3\vartheta}{(1 - \beta\cos\vartheta)^5}\bigg|_{t-\frac{R}{c}}. \tag{6.121}$$

Defining new variable $u := \cos\vartheta$ one gets

$$P(t') = \frac{q^2 a^2}{2c^3} \int_{-1}^{1} du \frac{(1 - u^2)}{(1 - \beta u)^5}.$$

Changing once again a variable of integration $y := 1 - \beta u$ we get

$$\begin{aligned}
\int_{-1}^{1} du \frac{(1 - u^2)}{(1 - \beta u)^5} &= \int_{1+\beta}^{1-\beta} \left(-\frac{dy}{\beta}\right) \frac{1 - \frac{(1-y)^2}{\beta^2}}{y^5} \\
&= -\frac{1}{\beta^3} \int_{1+\beta}^{1-\beta} dy \frac{\beta^2 - (1-y)^2}{y^5} \\
&= \frac{2y(4 - 3y) - 3(1 - \beta^2)}{12\beta^3 y^4}\bigg|_{1+\beta}^{1-\beta} \\
&= \frac{4}{3} \frac{1}{(1 - \beta^2)^3}.
\end{aligned}$$

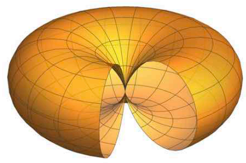


Figure 6.12: The angular distribution of the radiated power for $\beta = 0.1$.

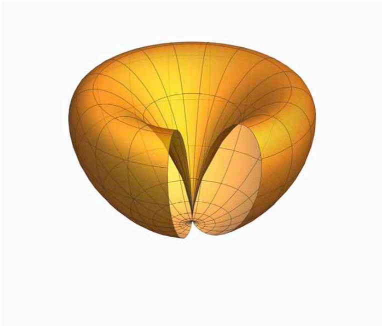


Figure 6.13: The angular distribution of the radiated power for $\beta = 0.5$.

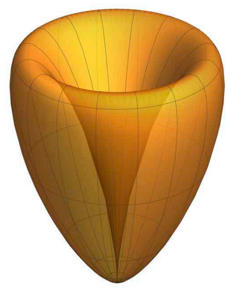


Figure 6.14: The angular distribution of the radiated power for $\beta = 0.8$.

Thus, the total radiated power reads

$$\boxed{P(t') = \frac{2q^2}{3c^3}\frac{a^2}{(1-\beta^2)^3}\Big|_{t-\frac{R}{c}} \sim \gamma^6.} \tag{6.122}$$

For small velocities the last formula simplifies to the following one

$$P(t') = \frac{2}{3}\frac{q^2a^2}{c^3}. \tag{6.123}$$

Acceleration perpendicular to velocity

Now we look at the case of acceleration perpendicular to velocity *i.e.* $\boldsymbol{\beta}\cdot\boldsymbol{a} = 0$. The radiated power distribution can be obtained directly from (6.117) and (6.118). It reads

$$dP(t') = \frac{q^2}{4\pi c^3}\left[\frac{\boldsymbol{a}^2}{(1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})^3} - (1-\beta^2)\frac{(\hat{\boldsymbol{n}}\cdot\boldsymbol{a})^2}{(1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})^5}\right]_{t-\frac{R}{c}} d\Omega. \tag{6.124}$$

We consider an instantaneous Cartesian frame of reference defined as $\hat{\boldsymbol{z}} := \boldsymbol{\beta}/\beta$ and $\hat{\boldsymbol{x}} := \boldsymbol{a}/a$, see Figure 6.15. Since

$$\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta} = \beta\cos\vartheta, \qquad \hat{\boldsymbol{n}}\cdot\boldsymbol{a} = a\sin\vartheta\cos\varphi$$

then the angular distribution of radiated power reads

$$\boxed{dP(t') = \frac{q^2}{4\pi c^3}\left[\frac{a^2}{(1-\beta\cos\vartheta)^3} - (1-\beta^2)\frac{a^2\sin^2\vartheta\cos^2\varphi}{(1-\beta\cos\vartheta)^5}\right]_{t-\frac{R}{c}} d\Omega.} \tag{6.125}$$

The distribution (6.125) depends explicitly on the angle φ. It vanishes for $\varphi = \{0, \pi\}$ and $\cos\vartheta = \beta$. The total radiated power is given by the integral

$$P(t') = \frac{q^2}{4\pi c^3}\int_0^{\pi} d\vartheta \int_0^{2\pi} d\varphi \left[\frac{a^2\sin\vartheta}{(1-\beta\cos\vartheta)^3} - (1-\beta^2)\frac{a^2\sin^3\vartheta\cos^2\varphi}{(1-\beta\cos\vartheta)^5}\right]_{t-\frac{R}{c}} \tag{6.126}$$

where $\int_0^{2\pi} d\varphi\cos^2\varphi = \pi$. Defining new variable $u := \cos\vartheta$ we get the expression (6.126) in the form

$$P(t') = \frac{q^2a^2}{4c^3}\left[2\int_{-1}^{1}\frac{du}{(1-\beta u)^3} - (1-\beta^2)\int_{-1}^{1} du\frac{1-u^2}{(1-\beta u)^5}\right]_{t-\frac{R}{c}} \tag{6.127}$$

where the integrals read

$$\int_{-1}^{1}\frac{du}{(1-\beta u)^3} = \frac{2}{(1-\beta^2)^2}, \qquad \int_{-1}^{1} du\frac{1-u^2}{(1-\beta u)^5} = \frac{4}{3}\frac{1}{(1-\beta^2)^3}.$$

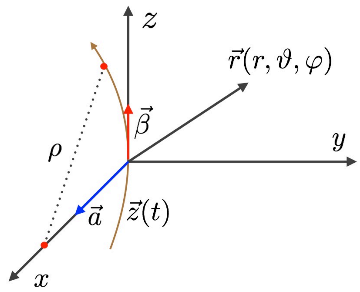


Figure 6.15: The charged particle in a circular motion.

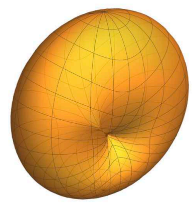


Figure 6.16: The angular distribution of radiated power for $\beta = 0.1$.

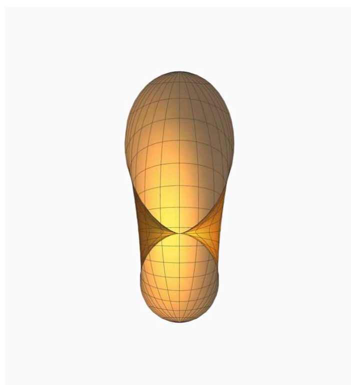


Figure 6.17: The angular distribution of radiated power (section $\varphi = 0$) for $\beta = 0.1$.

The total radiated power has the form

$$\boxed{P(t') = \frac{2q^2}{3c^3}\frac{a^2}{(1-\beta^2)^2}\bigg|_{t-\frac{R}{c}} \sim \gamma^4.} \tag{6.128}$$

The looses of energy in circular motion are proportional to γ^4 whereas they are proportional to γ^6 in rectilinear motion (6.122). The acceleration expressed by particle's velocity and the radius of its trajectory ρ has the form $a = \frac{v^2}{\rho} = \frac{c^2\beta^2}{\rho}$. The radiated power for ultrarelativistic case $a \approx \frac{c^2}{\rho}$ reads

$$P(t') = \frac{2}{3}\frac{q^2 c}{(1-\beta^2)^2}\frac{1}{\rho^2}.$$

It means that radiative losses of energy can be reduced for accelerators with big radius.

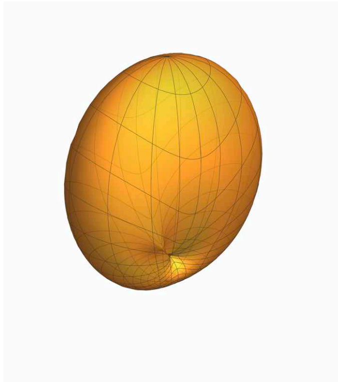


Figure 6.18: The angular distribution of radiated power for $\beta = 0.3$.

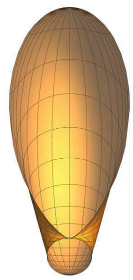


Figure 6.19: The angular distribution of radiated power (section $\varphi = 0$) for $\beta = 0.3$.

The retarded energy-momentum tensor

Substituting the field-strength tensor

$$F^{\mu\nu}_{ret} = k^\mu \xi^\nu - k^\nu \xi^\mu \qquad \text{where} \qquad k^\mu k_\mu = 0$$

into the energy momentum tensor

$$T^{\mu\nu} = \frac{1}{4\pi}\left[-F^{\mu\alpha}F^{\nu}{}_{\alpha} + \frac{1}{4}\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right]$$

one gets

$$\boxed{T^{\mu\nu}_{ret} = \frac{1}{4\pi}\left[-\xi^2 k^\mu k^\nu + (k\xi)(k^\mu\xi^\nu + k^\nu\xi^\mu) - \frac{1}{2}(k\xi)^2\eta^{\mu\nu}\right].} \tag{6.129}$$

The coefficients ξ^2 and $(k\xi) \equiv k_\mu \xi^\mu$ in (6.129) are given in terms of the expression

$$\xi^\mu = q\left[\frac{\dot{z}^\mu}{\rho^2} + \frac{\ddot{z}^\mu - (k\ddot{z})\dot{z}^\mu}{c\,\rho}\right].$$

Taking into account the relation (6.87) and (6.88) one gets

$$k^\mu_{ret}\dot{z}_\mu = \frac{1}{c^2}(\dot{z}^\mu\dot{z}_\mu + w^\mu\dot{z}_\mu) = 1,$$

where $w^\mu \dot{z}_\mu = 0$. It gives

$$(k\xi) = \frac{q}{\rho^2} \tag{6.130}$$

and

$$\begin{aligned}\xi^2 &= q^2\left[\frac{\dot{z}_\mu}{\rho^2} + \frac{\ddot{z}_\mu - (k\ddot{z})\dot{z}_\mu}{c\,\rho}\right]\left[\frac{\dot{z}^\mu}{\rho^2} + \frac{\ddot{z}^\mu - (k\ddot{z})\dot{z}^\mu}{c\,\rho}\right] \\ &= q^2\left[\frac{c^2}{\rho^4} - 2c\frac{(k\ddot{z})}{\rho^3} + \frac{(\ddot{z}_\mu - (k\ddot{z})\dot{z}_\mu)^2}{c^2\rho^2}\right] \\ &= q^2\left[\frac{c^2}{\rho^4} - 2c\frac{(k\ddot{z})}{\rho^3} + \frac{\ddot{z}^2 + c^2(k\ddot{z})^2}{c^2\rho^2}\right].\end{aligned} \tag{6.131}$$

The energy-momentum tensor (6.129) takes the form

$$T^{\mu\nu}_{ret} = -\frac{q^2}{4\pi}\left[\left(\frac{\ddot{z}^2 + c^2(k\ddot{z})^2}{c^2\rho^2} - 2c\frac{(k\ddot{z})}{\rho^3} + \frac{c^2}{\rho^4}\right)k^\mu k^\nu + \frac{\eta^{\mu\nu}}{2\rho^4}\right] + \frac{q^2}{4\pi}\left(\frac{k^\mu \dot{z}^\nu + k^\nu \dot{z}^\mu}{\rho^4} + \frac{k^\mu(\ddot{z}^\nu - (k\ddot{z})\dot{z}^\nu) + k^\nu(\ddot{z}^\mu - (k\ddot{z})\dot{z}^\mu)}{c\rho^3}\right). \tag{6.132}$$

In the limit $\rho \to \infty$ the terms proportional to ρ^{-2} are dominant, hence we shall consider only these terms in further analysis, namely

The leading term of the energy-momentum tensor at spatial infinity

$$\boxed{T^{\mu\nu}_{ret} = -\frac{q^2}{4\pi}\left[\frac{\ddot{z}^2 + c^2(k\ddot{z})^2}{c^2\rho^2}\right]k^\mu k^\nu + \mathcal{O}(\rho^{-3}).} \tag{6.133}$$

Note that (6.132) satisfyies

$$T^{\mu\nu}_{ret}k_\nu = \frac{1}{8\pi}\frac{q^2}{\rho^4}k^\mu, \qquad T^{\mu\nu}_{ret}k_\mu k_\nu = 0.$$

The retarded angular momentum tensor

The retarded angular momentum tensor density is defined as follows

$$M^{\mu\nu\alpha}_{ret} := x^\mu T^{\nu\alpha}_{ret} - x^\nu T^{\mu\alpha}_{ret}. \tag{6.134}$$

We assume that $T^{\nu\alpha}_{ret}$ is the energy-momentum tensor for electromagnetic field od a point-like charged particle at the world-line $z^\mu(\tau)$. We express the four-vector x^μ by two four-vectors $z^\mu(\tau_{ret})$ and $R^\mu_{ret} = c\rho k^\mu$, where $R^\mu_{ret} := x^\mu - z^\mu(\tau_{ret})$. Substituting $x^\mu = z^\mu + c\rho\, k^\mu$ into (6.134) one gets

$$M^{\mu\nu\alpha}_{ret} = z^\mu T^{\nu\alpha}_{ret} - z^\nu T^{\mu\alpha}_{ret} + c\rho(k^\mu T^{\nu\alpha}_{ret} - k^\nu T^{\mu\alpha}_{ret}). \tag{6.135}$$

The expression $k^\mu T^{\nu\alpha}_{ret} - k^\nu T^{\mu\alpha}_{ret}$ does not contain contributions from the dominating term (6.133) due to symmetric character of the expression $k^\mu k^\nu$. Hence, the dominating term in (6.135) would be proportional to ρ^{-2} and not to ρ^{-1} in the limit $\rho \to \infty$. Thus

$$\begin{aligned} k^\mu T^{\nu\alpha}_{ret} - k^\nu T^{\mu\alpha}_{ret} =& \frac{1}{4\pi}k^\mu\left[-\xi^2 k^\nu k^\alpha + (k\xi)(k^\nu\xi^\alpha + k^\alpha\xi^\nu) - \frac{1}{2}(k\xi)^2\eta^{\nu\alpha}\right] \\ &- \frac{1}{4\pi}k^\nu\left[-\xi^2 k^\mu k^\alpha + (k\xi)(k^\mu\xi^\alpha + k^\alpha\xi^\mu) - \frac{1}{2}(k\xi)^2\eta^{\mu\alpha}\right] \\ =& \frac{1}{4\pi}\left[(k\xi)(k^\mu\xi^\nu - k^\nu\xi^\mu)k^\alpha - \frac{1}{2}(k\xi)^2(k^\mu\eta^{\nu\alpha} - k^\nu\eta^{\mu\alpha})\right]. \end{aligned}$$

Note that, $(k^\mu T^{\nu\alpha}_{ret} - k^\nu T^{\mu\alpha}_{ret})k_\alpha = 0$. The angular momentum density tensor (6.134) reads

The four-tensor of angular momentum density for a radiating particle

$$M^{\mu\nu\alpha}_{ret} = z^\mu T^{\nu\alpha}_{ret} - z^\nu T^{\mu\alpha}_{ret} + \frac{c\rho}{4\pi}\left[(k\xi)(k^\mu\xi^\nu - k^\nu\xi^\mu)k^\alpha - \frac{1}{2}(k\xi)^2(k^\mu\eta^{\nu\alpha} - k^\nu\eta^{\mu\alpha})\right] \tag{6.136}$$

The expression (6.136) can be put in the form with explicit dependence on four-velocity and four-acceleration of the particle. It takes the form

$$\begin{aligned} M^{\mu\nu\alpha}_{ret} &= z^\mu T^{\nu\alpha}_{ret} - z^\nu T^{\mu\alpha}_{ret} + \\ &+ \frac{q^2 c}{4\pi}\left[\frac{k^\mu \dot z^\nu - k^\nu \dot z^\mu}{\rho^3} + \frac{k^\mu(\ddot z^\nu - (k\ddot z)\dot z^\nu) - k^\nu(\ddot z^\mu - (k\ddot z)\dot z^\mu)}{c\rho^2}\right]k^\alpha \\ &- \frac{q^2 c}{8\pi}\frac{k^\mu \eta^{\nu\alpha} - k^\nu \eta^{\mu\alpha}}{\rho^3}. \end{aligned} \tag{6.137}$$

Thus, the leading term of the angular-momentum tensor behaves as ρ^{-2} at spatial infinity

The leading term of the angular momentum density tensor at spatial infinity

$$\begin{aligned} M^{\mu\nu\alpha}_{ret} &= z^\mu T^{\nu\alpha}_{ret} - z^\nu T^{\mu\alpha}_{ret} + \\ &+ \frac{q^2}{4\pi}\frac{k^\mu(\ddot z^\nu - (k\ddot z)\dot z^\nu) - k^\nu(\ddot z^\mu - (k\ddot z)\dot z^\mu)}{\rho^2}k^\alpha + \mathcal{O}(\rho^{-3}) \end{aligned} \tag{6.138}$$

where

$$z^\mu T^{\nu\alpha}_{ret} - z^\nu T^{\mu\alpha}_{ret} = -\frac{q^2}{4\pi}\left[\frac{\ddot z^2 + c^2(k\ddot z)^2}{c^2\rho^2}\right](z^\mu k^\nu - z^\nu k^\mu)k^\alpha + \mathcal{O}(\rho^{-3}).$$

6.4 *Braking radiation (Bremsstrahlung)*

In this section we study a charged particle in external electromagnetic field. We take into account the radiative loses of the four-momentum and the angular momentum. The solution of this problem requires knowledge of the form of the energy-momentum tensor and the angular-momentum density tensor. It turns out that only the dominating terms far from the particle are relevant. Finally, we derive relativistic equation of motion for the particle.

Four-momentum and angular momentum carried by electromagnetic radiation

In what follows, we evaluate infinitesimal four-momentum dP^μ and angular momentum $dM^{\mu\nu}$ carried by electromagnetic radiation. Integrating these expressions over angles we are able to get total four-momentum and total angular momentum which carried by radiation.

These quantities are evaluated in an inertial reference frame S' that moves along the world-line

Asymptotic relations

$$y^\mu(\tau') = b^\mu + u^\mu \tau' \qquad b^\mu = const, \qquad u^\mu = const \tag{6.139}$$

where τ' is a proper time in S'.

The light-like four-vector $R^\mu = x^\mu - z^\mu(\tau_{ret})$ satisfies the equation $R^\mu R_\mu = 0$. Its counterpart $\bar R^\mu$, that connects x^μ and $y^\mu(\tau')$ at the world-line of the observer, has the form

$$\bar R^\mu := x^\mu - y^\mu(\tau'_{ret}), \qquad \bar R^\mu \bar R_\mu = 0. \tag{6.140}$$

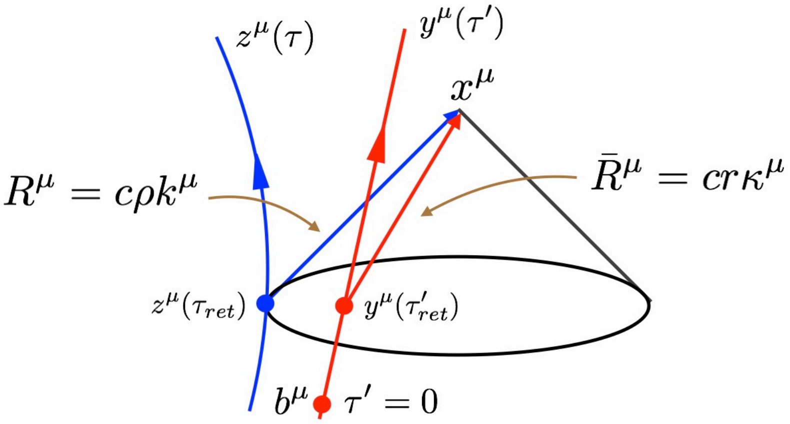


Figure 6.20: The accelerated charged particle that moves along z^μ. The inertial observer has the world-line given by y^μ.

The counterparts of expressions ρ and k^μ are denoted by r and κ^μ and they are defined in the following way

$$r := \frac{1}{c}(\bar{R}u),$$
$$\kappa_\mu := \partial_\mu \tau'(x) = \frac{\bar{R}_\mu}{(\bar{R}u)}. \tag{6.141}$$

OBSERVER		PARTICLE
τ'	$\longleftrightarrow$	τ
y^μ	$\longleftrightarrow$	z^μ
u^μ	$\longleftrightarrow$	$\dot{z}^\mu$
$\bar{R}^\mu$	$\longleftrightarrow$	R^μ
κ^μ	$\longleftrightarrow$	k^μ
r	$\longleftrightarrow$	ρ

which gives

$$\bar{R}^\mu = c\, r\kappa^\mu, \qquad \kappa^\mu \kappa_\mu = 0, \qquad \kappa^\mu u_\mu = 1. \tag{6.142}$$

The coordinate r represents spatial distance between the inertial observer and the event x measured in the observer's rest frame at τ'_{ret}.

The energy-momentum tensor and the angular momentum density tensor can be expressed using quantities that characterize *the observer*. It requires knowledge of asymptotic relations between the expressions

$$d\tau \leftrightarrow d\tau', \qquad \rho \leftrightarrow r, \qquad k^\mu \leftrightarrow \kappa^\mu.$$

The relation $d\tau \leftrightarrow d\tau'$

To get relation between the parameters τ' and τ we consider the equation

$$(x - z(\tau))^2 = 0. \tag{6.143}$$

Its solution is a function $\tau_{ret} = \tau(x)$. Substituting the expression

$$x^\mu = y^\mu(\tau') + c\, r\kappa^\mu \tag{6.144}$$

into (6.143) one gets

$$\left[y^\mu(\tau') + c\, r\kappa^\mu - z^\mu(\tau)\right]^2 = 0. \tag{6.145}$$

Using relations (6.142) equation (6.145) one gets (6.145) in the form

$$\kappa_\mu[y^\mu(\tau') - z^\mu(\tau)] = \frac{1}{2c}\frac{(y^\mu(\tau') - z^\mu(\tau))^2}{r}. \tag{6.146}$$

The right hand side of (6.146) vanishes at spatial infinity, $r \to \infty$, hence

$$\kappa_\mu \Big(y^\mu(\tau') - z^\mu(\tau) \Big) = \mathcal{O}(r^{-1}). \tag{6.147}$$

which can be solved with respect to τ giving some function

Solution: $\tau = \bar{\tau}(\tau', \kappa^\mu)$

$$\tau = \bar{\tau}(\tau', \kappa^\mu) \tag{6.148}$$

Let us consider the limit case for which the right hand side of (6.147) vanishes. Substituting (6.139) into (6.147) and using $\kappa_\mu u^\mu = 1$ one gets

$$\tau' + \kappa_\mu b^\mu - \kappa_\mu z^\mu(\tau) = 0. \tag{6.149}$$

The equation (6.149) implies the relation between differentials $d\tau$ and $d\tau'$, namely

$$\boxed{d\tau' = (\kappa \dot{z}) d\tau.} \tag{6.150}$$

The relation $\rho \leftrightarrow r$

The relation between ρ and r can be deduced from the expression $x^\mu \dot{z}_\mu(\bar{\tau})$, where x^μ is expressed in two ways – involving world-line of the observer and world-line of the particle

$$[z^\mu(\bar{\tau}) + c\,\rho k^\mu]\dot{z}_\mu(\bar{\tau}) = [y^\mu(\tau') + c\,r\kappa^\mu]\dot{z}_\mu(\bar{\tau})$$

where $k^\mu \dot{z}_\mu(\bar{\tau}) = 1$. Solving with respect to ρ we get

$$\rho = \kappa_\mu \dot{z}^\mu(\bar{\tau})\,r + \frac{1}{c}\Big(y^\mu(\tau') - z^\mu(\bar{\tau})\Big)\dot{z}^\mu.$$

Hence, $\rho(r) = (\kappa \dot{z})\,r + \rho_0$, where ρ_0 is a term which does not depend on r. At spatial infinity dominates the first what gives asymptotic relation

$$\boxed{\rho = (\kappa \dot{z}) r.} \tag{6.151}$$

The relation $k^\mu \leftrightarrow \kappa^\mu$

Let us consider the expression x^μ written in two different ways

$$z^\mu(\bar{\tau}) + c\,\rho k^\mu = y^\mu(\tau') + c\,r\kappa^\mu. \tag{6.152}$$

where τ and τ' are related by $\tau = \bar{\tau}(\tau', \kappa^\mu)$. The inverse of $\bar{\tau}$ for the asymptotic case is given by (6.149). Contracting both sides of (6.152) with the light-like four-vector κ^μ one gets

$$k^\mu \kappa_\mu = \frac{\Big(y^\mu(\tau') - z^\mu(\bar{\tau})\Big)\kappa_\mu}{c\rho}.$$

According to (6.146) and (6.151) the expression $k^\mu \kappa_\mu$ behaves at spatial infinity as

$$k^\mu \kappa_\mu = \mathcal{O}(r^{-2}). \tag{6.153}$$

The four-vector k^μ derived from the equality (6.152) reads

$$k^\mu = \frac{r}{\rho}\kappa^\mu + \frac{y^\mu(\tau') - z^\mu(\bar{\tau})}{c\rho}.$$

It has the following dominant behaviour at spatial infinity

$$\boxed{k^\mu = \frac{\kappa^\mu}{(\kappa\dot{z})} + \mathcal{O}(r^{-1}).} \tag{6.154}$$

Using the relations (6.150), (6.151) and (6.154) we get asymptotic form of the tensors $T_{ret}^{\mu\nu}$ and $M_{ret}^{\mu\nu\alpha}$ expressed in terms of quantities associated withe the observer's reference frame S'. Thus the leading term of the energy-momentum tensor reads

Asymptotic form of tensors $T_{ret}^{\mu\nu}$ and $M_{ret}^{\mu\nu\alpha}$ at spatial infinity

$$T_{ret}^{\mu\nu} = -\frac{q^2}{4\pi c^2}\frac{1}{r^2(\kappa\dot{z})^4}\left[\ddot{z}^2 + c^2\frac{(\kappa\ddot{z})^2}{(\kappa\dot{z})^4}\right]\kappa^\mu\kappa^\nu + \mathcal{O}(r^{-3}). \tag{6.155}$$

For the angular momentum tensor we get

$$\begin{aligned} M_{ret}^{\mu\nu\alpha} &= z^\mu T_{ret}^{\nu\alpha} - z^\nu T_{ret}^{\mu\alpha} \\ &+ \frac{q^2}{4\pi}\frac{1}{r^2(\kappa\dot{z})^4}\left[\kappa^\mu\left(\ddot{z}^\nu - \frac{(\kappa\ddot{z})}{(\kappa\dot{z})}\dot{z}^\nu\right) - \kappa^\nu\left(\ddot{z}^\mu - \frac{(\kappa\ddot{z})}{(\kappa\dot{z})}\dot{z}^\mu\right)\right]\kappa^\alpha + \\ &+ \mathcal{O}(r^{-3}) \end{aligned} \tag{6.156}$$

where

$$\dot{z} \equiv \left.\frac{dz}{d\tau}\right|_{\tau=\bar{\tau}(\tau',\kappa)} \qquad \text{and} \qquad \ddot{z} \equiv \left.\frac{d^2z}{d\tau^2}\right|_{\tau=\bar{\tau}(\tau',\kappa)}.$$

The amount of four-momentum and angular momentum emitted by the charged particle into the solid angle $d\Omega$ in the interval of time $d\tau'$ at the reference frame S' read

Tensors dP^μ and $dM^{\mu\nu}$ are taken at spatial infinity

$$dP^\mu = \lim_{r\to\infty}\frac{1}{c}T_{ret}^{\mu\nu}d^3\Sigma_\nu, \tag{6.157}$$

$$dM^{\mu\nu} = \lim_{r\to\infty}\frac{1}{c}M_{ret}^{\mu\nu\alpha}d^3\Sigma_\alpha \tag{6.158}$$

where $d^3\Sigma_\mu$ is the spherical area element, $r = const$.

We shall integrate expressions (6.157) and (6.158) choosing $d^3\Sigma_\mu$ as the area element on three-surface $r = const$. It is parametrized by coordinates $(c\tau', \vartheta, \varphi)$. The three-area element is obtained from equation defining the three-surface $r = r(x^\mu)$.

In what follows we consider a generic case of three-area element $d^3\Sigma_\mu$ associated with the surface obtained by fixing the value of one of four coordinates. Thus, we consider the diffeomorphism

The three-area element $d^3\Sigma_\mu$

$$\zeta^\mu = \zeta^\mu(x^\nu), \qquad \mu,\nu = 0,1,2,3$$

where $\{x^\nu\}$ are Cartesian coordinates and $\{\zeta^\mu\}$ are curvilinear coordinates. The volume element $d^4\Omega$ is given by

$d\zeta^4 \equiv d\zeta^0 d\zeta^1 d\zeta^2 d\zeta^3$

$$
\begin{aligned}
d^4\Omega &= \frac{1}{4!}\epsilon_{\mu\nu\alpha\beta}\frac{\partial(x^\mu x^\nu x^\alpha x^\beta)}{\partial(\zeta^0\zeta^1\zeta^2\zeta^3)}d^4\zeta \\
&= \epsilon_{\mu\nu\alpha\beta}\frac{\partial x^\mu}{\partial\zeta^0}\frac{\partial x^\nu}{\partial\zeta^1}\frac{\partial x^\alpha}{\partial\zeta^2}\frac{\partial x^\beta}{\partial\zeta^3}d^4\zeta \\
&= \det\left(\frac{\partial x^\rho}{\partial\zeta^\sigma}\right)d^4\zeta = \sqrt{-g(\zeta)}\,d^4\zeta \qquad (6.159)
\end{aligned}
$$

where $d^4\zeta \equiv d\zeta^0 d\zeta^1 d\zeta^2 d\zeta^3$ and $g(\zeta) \equiv \det\big(\hat{g}(\zeta)\big)$. On the other hand, the four-volume element can be cast in the form

$d\zeta^3 \equiv d\zeta^1 d\zeta^2 d\zeta^3$

$$
\begin{aligned}
d^4\Omega &= \left(\frac{\partial x^\mu}{\partial\zeta^0}d\zeta^0\right)\left(\epsilon_{\mu\nu\alpha\beta}\frac{\partial x^\nu}{\partial\zeta^1}\frac{\partial x^\alpha}{\partial\zeta^2}\frac{\partial x^\beta}{\partial\zeta^3}d^3\zeta\right) \\
&= \left(\frac{\partial x^\mu}{\partial\zeta^0}d\zeta^0\right)\left(\frac{1}{3!}\epsilon_{\mu\nu\alpha\beta}\frac{\partial(x^\nu x^\alpha x^\beta)}{\partial(\zeta^1\zeta^2\zeta^3)}d^3\zeta\right) \\
&= \left(\frac{\partial x^\mu}{\partial\zeta^0}d\zeta^0\right)d^3\Sigma_\mu. \qquad (6.160)
\end{aligned}
$$

Comparing (6.159) and (6.160) one gets

$$
\frac{\partial x^\mu}{\partial\zeta^0}d^3\Sigma_\mu = \sqrt{-g(\zeta)}\,d^3\zeta
$$

which can be cast in the form[17]

$$
\frac{\partial x^\mu}{\partial\zeta^0}\left[d^3\Sigma_\mu - \frac{\partial\zeta^0}{\partial x^\mu}\sqrt{-g(\zeta)}\,d^3\zeta\right] = 0.
$$

[17] We apply the formula

$$
\frac{\partial x^\mu}{\partial\zeta^\beta}\frac{\partial\zeta^\alpha}{\partial x^\mu} = \delta^\alpha_\beta
$$

for $\alpha = \beta = 0$.

It gives

$$
\boxed{d^3\Sigma_\mu = \frac{\partial\zeta^0}{\partial x^\mu}\sqrt{-g(\zeta)}\,d^3\zeta.} \qquad (6.161)
$$

Now we go back to the particular case of spherical coordinates. We choose

The case of the surface $r = const$ in spherical coordinates

$$
\zeta_0 \equiv r, \qquad \sqrt{-g(\zeta)}d^3\zeta = r^2\sin\vartheta\,d\vartheta d\varphi(cd\tau').
$$

The expression (6.161) gives

$d\Omega \equiv \sin\vartheta\,d\vartheta d\varphi$

$$
d^3\Sigma_\mu = (\partial_\mu r)\,r^2(c\,d\tau')d\Omega. \qquad (6.162)
$$

Taking derivative of (6.144) with respect to x^μ and contraction of the obtained result with u^μ we get

$$
\partial_\mu x^\alpha = u^\alpha\partial_\mu\tau'(x) + c(\partial_\mu r)\,\kappa^\alpha.
$$

Hence

$$
\delta^\alpha_\mu = u^\alpha\kappa_\mu + c\kappa^\alpha\partial_\mu r.
$$

Then, contracting this expression with u_α, where $\kappa_\alpha u^\alpha = 1$, we get $u_\mu = c^2\kappa^\mu + c\partial_\mu r$. It gives

$$
\partial_\mu r = \frac{u_\mu}{c} - c\,\kappa_\mu. \qquad (6.163)
$$

Note that $\partial_\mu r$ is a space-like vector that satisfies

$$(\partial_\mu r)(\partial^\mu r) = \frac{u^2}{c^2} - 2(\kappa u) + c^2\kappa^2 = -1.$$

The element $d^3\Sigma_\mu$ associated wit the surface $r = const$ takes the form

$$\boxed{d^3\Sigma_\mu = \left(u_\mu - c^2\kappa_\mu\right) r^2 d\tau' d\Omega} \tag{6.164}$$

The four-momentum dP^μ that crosses the area element $r^2 d\Omega$ of the surface $r = const$ during the time interval $d\tau'$ has asymptotic value

Asymptotic form of quantities dP^μ and $dM^{\mu\nu}$

$$\begin{aligned} dP^\mu &= \lim_{r\to\infty} \frac{1}{c} T^{\mu\nu}_{ret} d^3\Sigma_\nu = \lim_{r\to\infty} \frac{1}{c} T^{\mu\nu}_{ret} (u_\nu - c^2\kappa_\nu)\, d\tau' r^2 d\Omega \\ &= -\frac{q^2}{4\pi c^3} \frac{1}{r^2(\kappa\dot{z})^4} \left[\ddot{z}^2 + c^2 \frac{(\kappa\ddot{z})^2}{(\kappa\dot{z})^4}\right] \kappa^\mu\kappa^\nu (u_\nu - c^2\kappa_\nu)\, d\tau' r^2 d\Omega \\ &= -\frac{q^2}{4\pi c^3} \frac{1}{(\kappa\dot{z})^4} \left[\ddot{z}^2 + c^2 \frac{(\kappa\ddot{z})^2}{(\kappa\dot{z})^4}\right] \kappa^\mu d\tau' d\Omega. \end{aligned} \tag{6.165}$$

The term $T^{\mu\nu}_{ret}\kappa_\nu$ vanishes in the lowest orders, $T^{\mu\nu}_{ret}\kappa_\nu \propto r^{-4}$, because $T^{\mu\nu}_{ret}$ given by (6.155) is only asymptotic form of the energy momentum tensor. Indeed, according to (6.133) the retarded tensor $T^{\mu\nu}_{ret}$ behaves at spatial infinity as $\sim \rho^{-2} k^\mu k^\nu$, where $\rho = (\kappa\dot{z})r$. Since $k^\nu\kappa_\nu \propto r^{-2}$ then $T^{\mu\nu}_{ret}\kappa_\nu \sim r^{-4}$. Its contribution can be discarded because the integral of this quantity involving the element $r^2 d\Omega$ vanishes for $r \to \infty$.

Similarly, the asymptotic expression for the angular momentum is given by (6.156) and (6.164)

$$\begin{aligned} dM^{\mu\nu} &= \lim_{r\to\infty} \frac{1}{c} M^{\mu\nu\alpha}_{ret} d^3\Sigma_\alpha = \lim_{r\to\infty} \frac{1}{c} M^{\mu\nu\alpha}_{ret} (u_\alpha - c^2\kappa_\alpha)\, r^2 d\tau' d\Omega \\ &= z^\mu dP^\nu - z^\nu dP^\mu \\ &+ \frac{q^2}{4\pi c} \frac{1}{r^2(\kappa\dot{z})^4} \left[\kappa^\mu \left(\ddot{z}^\nu - \frac{(\kappa\ddot{z})}{(\kappa\dot{z})}\dot{z}^\nu\right) - \right. \\ &- \left.\kappa^\nu \left(\ddot{z}^\mu - \frac{(\kappa\ddot{z})}{(\kappa\dot{z})}\dot{z}^\mu\right)\right] \kappa^\alpha (u_\alpha - c^2\kappa_\alpha) d\tau' r^2 d\Omega \\ &= z^\mu dP^\nu - z^\nu dP^\mu \\ &+ \frac{q^2}{4\pi c} \frac{1}{(\kappa\dot{z})^4} \left[\kappa^\mu \left(\ddot{z}^\nu - \frac{(\kappa\ddot{z})}{(\kappa\dot{z})}\dot{z}^\nu\right) - \kappa^\nu \left(\ddot{z}^\mu - \frac{(\kappa\ddot{z})}{(\kappa\dot{z})}\dot{z}^\mu\right)\right] d\tau' d\Omega. \end{aligned} \tag{6.166}$$

Integrating the expressions (6.165) and (6.166) over angles ϑ and φ we get total amount of four-momentum and angular momentum carried by radiation. These integrals are evaluated in *instantaneous rest frame of the particle*. In this reference frame, the observer's four-velocity coincide with particle's four-velocity. The same is true for $\dot{z}^\alpha\kappa_\alpha$ and $u^\alpha\kappa_\alpha = 1$. Thus

Integration over angles

$$(u\kappa) = 1 = (\dot{z}\kappa), \qquad d\tau' = d\tau. \tag{6.167}$$

There are three different types of integrals over angles

$$\begin{aligned} I_\mu &:= \frac{1}{4\pi}\oint \kappa_\mu \, d\Omega, \\ I_{\mu\nu} &:= \frac{1}{4\pi}\oint \kappa_\mu \kappa_\nu \, d\Omega, \\ I_{\mu\nu\alpha} &:= \frac{1}{4\pi}\oint \kappa_\mu \kappa_\nu \kappa_\alpha \, d\Omega. \end{aligned} \tag{6.168}$$

The formula (6.167) implies that

$$I_\mu u^\mu = 1, \qquad I_{\mu\nu} u^\nu = I_\mu, \qquad I_{\mu\nu\alpha} u^\alpha = I_{\mu\nu}. \tag{6.169}$$

The condition $\kappa^\mu \kappa_\mu = 0$ implies

$$I_{\mu\nu}\eta^{\mu\nu} = 0, \qquad I_{\mu\nu\alpha}\eta^{\mu\nu} = I_{\mu\nu\alpha}\eta^{\mu\alpha} = I_{\mu\nu\alpha}\eta^{\nu\alpha} = 0. \tag{6.170}$$

The integrals (6.168) can be represented by combinations of symmetric expressions build of the four-vector u_μ and the metric tensor $\eta_{\mu\nu}$. The most general such combinations read

$$I_\mu = \alpha u_\mu, \tag{6.171}$$

$$I_{\mu\nu} = \beta_1 u_\mu u_\nu + \beta_2 \eta_{\mu\nu} \tag{6.172}$$

$$I_{\mu\nu\alpha} = \gamma_1 u_\mu u_\nu u_\alpha + \gamma_2 \left[\eta_{\mu\nu} u_\alpha + \eta_{\alpha\mu} u_\nu + \eta_{\nu\alpha} u_\mu\right] \tag{6.173}$$

where free constants can be determined from identities (6.169).

The first integral (6.171) contracted with u^μ gives $I_\mu u^\mu = 1 = \alpha c^2$. Thus $\alpha = \frac{1}{c^2}$. The expression $I_{\mu\nu}u^\nu = I_\mu$ reduces to the equation $c^2\beta_1 + \beta_2 = \frac{1}{c^2}$. The second and independent equation follows from the first relation (6.170), namely $\eta^{\mu\nu} I_{\mu\nu} = 0 = c^2\beta^1 + 4\beta_2$. These equation have solution

$$\beta_1 = \frac{4}{3c^4}, \qquad \beta_2 = -\frac{1}{3c^2}.$$

Finally, the expression (6.173) takes the form

$$\begin{aligned} I_{\mu\nu\alpha} u^\alpha &= c^2\gamma_1 u_\mu u_\nu + \gamma_2 [c^2 \eta_{\mu\nu} + 2 u_\mu u_\nu] \\ &= \underbrace{\left[c^2\gamma_1 + 2\gamma_2\right]}_{\beta_1} u_\mu u_\nu + \underbrace{c^2\gamma_2}_{\beta_2} \eta_{\mu\nu}. \end{aligned}$$

It gives

$$\gamma_1 = \frac{2}{c^6}, \qquad \gamma_2 = -\frac{1}{3c^4}.$$

Substituting these results into (6.171) and (6.173) one gets

Integrals (6.171) and (6.173)

$$I_\mu = \frac{1}{c^2} u_\mu, \qquad I_{\mu\nu} = \frac{1}{3c^4}\left[4u_\mu u_\nu - c^2\eta_{\mu\nu}\right], \tag{6.174}$$

$$I_{\mu\nu\alpha} = \frac{1}{c^6}\left[2u_\mu u_\nu u_\alpha - \frac{c^2}{3}(\eta_{\mu\nu}u_\alpha + \eta_{\nu\alpha}u_\mu + \eta_{\alpha\mu}u_\nu)\right]. \tag{6.175}$$

The result of integration of the expression (6.165) *over angles* has the form

$$\oint dP^\mu = \Big(\frac{dP^\mu}{d\tau}\Big)d\tau$$

where[18]

[18] Note that $u^\mu \to \dot{z}^\mu$.

$$\begin{aligned}\frac{dP^\mu}{d\tau} &= -\frac{q^2}{4\pi c^3}\oint\Big[\ddot{z}^2 + c^2(\kappa\ddot{z})^2\Big]\kappa^\mu d\Omega\\ &= -\frac{q^2}{c^3}\Big[\ddot{z}^2 I^\mu + c^2 I^{\mu\nu\alpha}\ddot{z}_\nu\ddot{z}_\alpha\Big].\end{aligned} \tag{6.176}$$

Substituting into (6.176) the integral $I^\mu = \frac{1}{c^2}\dot{z}^\mu$ and the expression

$$\begin{aligned}I^{\mu\nu\alpha}\ddot{z}_\nu\ddot{z}_\alpha &= \frac{1}{c^6}\Big[2\overbrace{\dot{z}^\mu(\dot{z}^\nu\ddot{z}_\nu)(\dot{z}^\alpha\ddot{z}_\alpha)}^{0} - \frac{c^2}{3}(\eta^{\mu\nu}\dot{z}^\alpha + \eta^{\alpha\mu}\dot{z}^\nu + \eta^{\nu\alpha}\dot{z}^\mu)\ddot{z}_\nu\ddot{z}_\alpha\Big]\\ &= -\frac{1}{3c^4}\ddot{z}^2\dot{z}^\mu,\end{aligned}$$

where $\dot{z}^\alpha\ddot{z}_\alpha = 0$, one gets

The total four-momentum per unit of time carried by the radiation (it has dimension of force)

$$\boxed{\frac{dP^\mu}{d\tau} = -\frac{2q^2}{3c^5}\ddot{z}^2\dot{z}^\mu.} \tag{6.177}$$

Similarly, integrating the expression (6.166) over angles one gets

$$\oint dM^{\mu\nu} = \Big(\frac{dM^{\mu\nu}}{d\tau}\Big)d\tau$$

where

$$\begin{aligned}\frac{dM^{\mu\nu}}{d\tau} &= z^\mu\frac{dP^\nu}{d\tau} - z^\nu\frac{dP^\mu}{d\tau} +\\ &+ \frac{q^2}{4\pi c}\oint\Big[\kappa^\mu\big(\ddot{z}^\nu - (\kappa\ddot{z})\dot{z}^\nu\big) - \kappa^\nu\big(\ddot{z}^\mu - (\kappa\ddot{z})\dot{z}^\mu\big)\Big]d\Omega\\ &= z^\mu\frac{dP^\nu}{d\tau} - z^\nu\frac{dP^\mu}{d\tau} +\\ &+ \frac{q^2}{c}\underbrace{\Big[I^\mu\ddot{z}^\nu - I^\nu\ddot{z}^\mu - \big(I^{\mu\alpha}\dot{z}^\nu\ddot{z}_\alpha - I^{\nu\alpha}\dot{z}^\mu\ddot{z}_\alpha\big)\Big]}_{W^{\mu\nu}}.\end{aligned} \tag{6.178}$$

The expression $W^{\mu\nu}$ reads

$$\begin{aligned}W^{\mu\nu} &= \frac{1}{c^2}\big(\dot{z}^\mu\ddot{z}^\nu - \dot{z}^\nu\ddot{z}^\mu\big) -\\ &- \frac{1}{3c^4}\Big[4\overbrace{\big(\dot{z}^\mu\dot{z}^\alpha\dot{z}^\nu\ddot{z}_\alpha - \dot{z}^\nu\dot{z}^\alpha\dot{z}^\mu\ddot{z}_\alpha\big)}^{0} - c^2\big(\eta^{\mu\alpha}\dot{z}^\nu\ddot{z}_\alpha - \eta^{\nu\alpha}\dot{z}^\mu\ddot{z}_\alpha\big)\Big]\\ &= \frac{1}{c^2}\big(\dot{z}^\mu\ddot{z}^\nu - \dot{z}^\nu\ddot{z}^\mu\big) - \frac{1}{3c^2}\big(\dot{z}^\mu\ddot{z}^\nu - \dot{z}^\nu\ddot{z}^\mu\big)\\ &= \frac{2}{3c^2}\big(\dot{z}^\mu\ddot{z}^\nu - \dot{z}^\nu\ddot{z}^\mu\big).\end{aligned} \tag{6.179}$$

The angular four-momentum per unit of time carried out by radiation reads

$$\boxed{\frac{dM^{\mu\nu}}{d\tau} = z^{\mu}\frac{dP^{\nu}}{d\tau} - z^{\nu}\frac{dP^{\mu}}{d\tau} + \frac{2q^2}{3c^3}\left(\dot{z}^{\mu}\ddot{z}^{\nu} - \dot{z}^{\nu}\ddot{z}^{\mu}\right)} \tag{6.180}$$

The total angular four-momentum per unit of time carried by radiation

This quantity has physical dimension of torque.

The Lorentz-Dirac equation

A charged particle in the presence of an external electromagnetic field experiences the four-force

$$f^{\mu} = \frac{q}{c}F^{\mu}{}_{\nu}\dot{z}^{\nu}. \tag{6.181}$$

The particle's four-momentum id denoted by p^{μ} and their angular four-momentum reads

$$m^{\mu\nu} := z^{\mu}p^{\nu} - z^{\nu}p^{\mu}. \tag{6.182}$$

Neglecting electromagnetic radiation, one gets that the rate of change of particle's four-momentum is equal to the four-force (6.181). Similarly, the rate of change change of its angular four-momentum is equal to a torque associated with the external electromagnetic field

$$\frac{dp^{\mu}}{d\tau} = f^{\mu}, \tag{6.183}$$

$$\frac{dm^{\mu\nu}}{d\tau} = z^{\mu}f^{\nu} - z^{\nu}f^{\mu}, \tag{6.184}$$

where

$$\frac{dm^{\mu\nu}}{d\tau} = z^{\mu}\frac{dp^{\nu}}{d\tau} - z^{\nu}\frac{dp^{\nu}}{d\tau}, \tag{6.185}$$

$$\frac{dz^{\mu}}{d\tau}p^{\nu} - \frac{dz^{\nu}}{d\tau}p^{\mu} = 0. \tag{6.186}$$

According to (6.185) and (6.186), the rate of change of angular four-momentum of the particle depends only on the rate of change of particle's four-momentum if the braking radiation is not taken into account.

The particle's equation of motion must be modified when braking radiation (Bremsstrahlung) is taken into account. To do it we postulate that the particle experiences additional *reaction four-force* "$-\frac{dP^{\mu}}{d\tau}$" and additional *reaction four-torque* "$-\frac{dM^{\mu\nu}}{d\tau}$". The reaction force and torque are expressed in terms of total four-momentum and four-angular momentum per unit of time carried out by radiation. Equations of motion (6.183) and (6.184) must be substituted by the following ones

$$\frac{dp^{\mu}}{d\tau} = f^{\mu} - \frac{dP^{\mu}}{d\tau}, \tag{6.187}$$

$$\frac{dm^{\mu\nu}}{d\tau} = z^{\mu}f^{\nu} - z^{\nu}f^{\mu} - \frac{dM^{\mu\nu}}{d\tau}. \tag{6.188}$$

Substituting (6.177) into the right hand side of (6.187) one gets

$$\frac{dp^\mu}{d\tau} = f^\mu + \frac{2q^2}{3c^5}\ddot{z}^2\dot{z}^\mu. \tag{6.189}$$

The relation between four-momentum of the particle and its four-velocity is obtained from (6.187) and (6.188). The expression $\frac{dm^{\mu\nu}}{d\tau}$ on the left hand side of (6.188) cannot simplify to (6.186) (otherwise, there would be $p^\mu \propto \dot{z}^\mu$). Indeed, the equation (6.188) can be cast in the form

$$\frac{d}{d\tau}(z^\mu p^\nu - z^\nu p^\mu) = z^\nu f^\mu - z^\mu f^\nu - \frac{2q^2}{3c^3}(\dot{z}^\mu\ddot{z}^\nu - \dot{z}^\nu\ddot{z}^\mu)$$

which gives

$$\left(\frac{dz^\mu}{d\tau}p^\nu - \frac{dz^\nu}{d\tau}p^\mu\right) = \overbrace{-\left(z^\mu\frac{dp^\nu}{d\tau} - z^\nu\frac{dp^\mu}{d\tau}\right) + z^\mu\left(f^\nu - \frac{dP^\nu}{d\tau}\right) - z^\nu\left(f^\mu - \frac{dP^\mu}{d\tau}\right)}^{=0 \text{ from } (6.187)}$$
$$- \frac{2q^2}{3c^3}(\dot{z}^\mu\ddot{z}^\nu - \dot{z}^\nu\ddot{z}^\mu).$$

The last equation reads

$$\dot{z}^\mu\left(p^\nu + \frac{2q^2}{3c^3}\ddot{z}^\nu\right) - \dot{z}^\nu\left(p^\mu + \frac{2q^2}{3c^3}\ddot{z}^\mu\right) = 0. \tag{6.190}$$

It has the solution

The relation between four-momentum four-velocity and four-acceleration of the particle

$$p^\mu = m\dot{z}^\mu - \frac{2q^2}{3c^3}\ddot{z}^\mu \tag{6.191}$$

where the term $m\dot{z}^\mu$ is must be included because it is a solution in the case of vanishing acceleration. Note that four momentum of a charged particle (6.191) depends on particle's four-velocity *and four-acceleration*. Taking derivative with respect to τ of both sides of (6.191) and substituting $\frac{dp^\mu}{d\tau}$ from (6.187) one gets the *Lorentz-Dirac equation*

The Lorentz-Dirac equation

$$\boxed{m\ddot{z}^\mu - \frac{2q^2}{3c^3}\left(\dddot{z}^\mu + \frac{1}{c^2}\ddot{z}^2\dot{z}^\mu\right) = \frac{q}{c}F^\mu{}_\nu\dot{z}^\nu.} \tag{6.192}$$

This equation describes a process of braking the particle due to emission of radiation. Its solutions concord with experimental data to some extend. There are some problems associated with this equation. One of them is the existence of third order derivative. It requires an additional initial condition (initial value of the acceleration). Another problem is the existence of non-physical solutions (self-accelerating). For instance, in absence of external electromagnetic field $f^\mu = 0$ the ansatz $\ddot{z}^2 = -a^2$ gives the linear equation

$$m\ddot{z}^\mu - \frac{2q^2}{3c^3}\left(\dddot{z}^\mu - \frac{a^2}{c^2}\dot{z}^\mu\right) = 0.$$

Among solutions of this equations there exists also a pathological solution. Such a solution can be eliminated requiring $\ddot{z}^\mu \to 0$ for $\tau \to \infty$.

6.5 *Radiation generated by continuous distributions of charges*

The retarded potential

In this section we study the problem of generation of electromagnetic radiation by physical systems which contains a macroscopic number of charged particles described by four-current $J^\mu(x)$. When distribution of charges is approximated by some continuous function, the associated four-current is also a smooth function of coordinates. The retarded potential takes the form

$$\begin{aligned} A^\mu(x) &= \frac{4\pi}{c}\int d^4x'\, D_{ret}(x-x')J^\mu(x') \\ &= \frac{1}{c}\int d^3x' \int dx'^0 \theta(x^0 - x'^0)\frac{\delta(x^0 - x'^0 - |\boldsymbol{x}-\boldsymbol{x}'|)}{|\boldsymbol{x}-\boldsymbol{x}'|} J^\mu(x'^0, \boldsymbol{x}') \\ &= \frac{1}{c}\int d^3x' \frac{J^\mu(x^0 - |\boldsymbol{x}-\boldsymbol{x}'|, \boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}, \end{aligned} \tag{6.193}$$

where $\theta(|\boldsymbol{x}-\boldsymbol{x}'|) = 1$. The retarded time t' is given by solution of the equation $x^0 - x'^0 - |\boldsymbol{x}-\boldsymbol{x}'|$ and it reads

The retarded time

$$\boxed{t_{ret} \equiv t' = t - \frac{1}{c}|\boldsymbol{x}-\boldsymbol{x}'|.} \tag{6.194}$$

Note that, with each $\boldsymbol{x}'$ at t. there is associated different value of retarded time t'. In what follows we work with the variable t instead of x^0. The retarded four-potential takes the following form

The retarded four-potential

$$\boxed{A^\mu(t, \boldsymbol{x}) = \frac{1}{c}\int d^3x' \frac{J^\mu(t - \frac{1}{c}|\boldsymbol{x}-\boldsymbol{x}'|, \boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}.} \tag{6.195}$$

The expression (6.195) provides the relation between currents and potentials for antennas. Unlike for the case of radiating particle, the radiation from antennas is obtained using approximated expressions based on series expansions of (6.195).

Retarded potentials in the radiation zone

The radiation zone is a region far from the source. Strictly speaking, the distance from the source must be much larger than the characteristic size of the source *i.e.* $|\boldsymbol{x}| \gg |\boldsymbol{x}'|$, see Figure 6.21.

The approximated solution is restricted to leading terms obtained expanding retarder potential in the radiation zone. Expanding $|\boldsymbol{x}-\boldsymbol{x}'|$ one gets

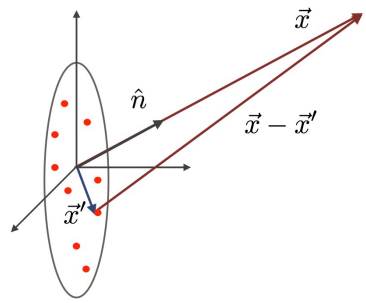


Figure 6.21: The source of an electromagnetic radiation (antenna).

$$\begin{aligned} |\boldsymbol{x}-\boldsymbol{x}'| &= \left(|\boldsymbol{x}|^2 - 2\boldsymbol{x}\cdot\boldsymbol{x}' + |\boldsymbol{x}'|^2\right)^{1/2} = |\boldsymbol{x}|\left(1 - 2\frac{\boldsymbol{x}\cdot\boldsymbol{x}'}{|\boldsymbol{x}|^2} + \frac{|\boldsymbol{x}'|^2}{|\boldsymbol{x}|^2}\right)^{1/2} \\ &= |\boldsymbol{x}| - \frac{\boldsymbol{x}\cdot\boldsymbol{x}'}{|\boldsymbol{x}|} + \dots \end{aligned}$$

We denote $r := |\boldsymbol{x}|$ and $\hat{\boldsymbol{n}} := \boldsymbol{x}/|\boldsymbol{x}|$. It gives

$$|\boldsymbol{x} - \boldsymbol{x}'| = r - \hat{\boldsymbol{n}} \cdot \boldsymbol{x}' + \ldots \tag{6.196}$$

We shall restrict our consideration to first two terms of the expansion

The retarded time

$$\boxed{t_{ret} = t - \frac{r}{c} + \frac{1}{c}\hat{\boldsymbol{n}} \cdot \boldsymbol{x}' + \ldots} \quad \text{where} \quad \boxed{t_e := t - \frac{r}{c}.} \tag{6.197}$$

We define *emission time* t_e as the retarded time of a source at the origin of the reference frame. By assumption, the origin of this frame is localized inside the radiating region V. The expression t_e is the most rough approximation for the retarded time. Thus, the term $\frac{1}{c}\hat{\boldsymbol{n}} \cdot \boldsymbol{x}'$ represents *linear corrections* to the retarded time associated with the finite size of the radiated system.

The partial derivatives of the retarded time with respect to spatial coordinates x^i read

$$\begin{aligned}
\partial_i t_{ret} &= \partial_i \left(t - \frac{r}{c} + \frac{1}{c}\hat{\boldsymbol{n}} \cdot \boldsymbol{x}' + \ldots \right) \\
&= \frac{1}{c}\left(-\partial_i r + \partial_i n^j x'^j \right) + \ldots \\
&= \frac{1}{c}\left(-n^i + \frac{\delta_{ij} - n^i n^j}{r} x'^j \right) + \ldots \\
&= \frac{1}{c}\left(-n^i + \frac{1}{r}\left(x'^i - n^i (\hat{\boldsymbol{n}} \cdot \boldsymbol{x}') \right) \right) + \ldots \\
&= \frac{1}{c}\left(-n^i - \frac{1}{r}\left[\hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \boldsymbol{x}') \right]^i \right) + \ldots
\end{aligned}$$

which gives the gradient of t_{ret}

The gradient of t_{ret}

$$\boxed{\nabla t_{ret} = -\frac{1}{c}\left(\hat{\boldsymbol{n}} + \frac{1}{r}\hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \boldsymbol{x}') \right).} \tag{6.198}$$

The four-potential A^μ in the radiation zone is obtained expanding the expression (6.195) and it reads

$$\begin{aligned}
A^\mu(t, \boldsymbol{x}) &= \frac{1}{c}\int_V d^3x' \frac{J^\mu(t - \frac{1}{c}(r - \hat{\boldsymbol{n}} \cdot \boldsymbol{x}' + \ldots), \boldsymbol{x}')}{r - \hat{\boldsymbol{n}} \cdot \boldsymbol{x}' + \ldots} \\
&= \frac{1}{c\,r}\int_V d^3x' (1 + \frac{\hat{\boldsymbol{n}} \cdot \boldsymbol{x}'}{r} + \ldots) J^\mu(t_e + \frac{\hat{\boldsymbol{n}} \cdot \boldsymbol{x}'}{c} + \ldots, \boldsymbol{x}') \\
&= \frac{1}{c\,r}\int_V d^3x' \left(1 + \frac{\hat{\boldsymbol{n}} \cdot \boldsymbol{x}'}{r} + \ldots \right) \times \\
&\times \left(J^\mu(t_e, \boldsymbol{x}') + \frac{\hat{\boldsymbol{n}} \cdot \boldsymbol{x}'}{c}\partial_t J^\mu(t_e, \boldsymbol{x}') + \ldots \right)
\end{aligned}$$

The region of integration is fixed (does not depend on time). Thus the integral of partial derivative is equal to total derivative of the integral

$$\int_V d^3x\, \partial_t f = \frac{d}{dt}\int_V d^3x f.$$

It gives

The leading terms of the expansion

$$A^{\mu}(t,\boldsymbol{x}) = \frac{1}{c\,r}\int_V d^3x' J^{\mu}(t_e,\boldsymbol{x}')+ \\ + \frac{1}{c^2\,r}\frac{d}{dt}\int_V d^3x'(\hat{\boldsymbol{n}}\cdot\boldsymbol{x}')J^{\mu}(t_e,\boldsymbol{x}') + \ldots \tag{6.199}$$

We shall express the formula (6.199) in terms of dipole moments

$$\boldsymbol{p}(t) := \frac{1}{c}\int d^3x'\boldsymbol{x}'J^0(t,\boldsymbol{x}'), \tag{6.200}$$

$$\boldsymbol{m}(t) := \frac{1}{2c}\int d^3x'\boldsymbol{x}'\times\boldsymbol{J}(t,\boldsymbol{x}'). \tag{6.201}$$

The scalar potential is of the form

$$\boxed{A^0(t,\boldsymbol{x}) = \frac{Q}{r} + \frac{1}{c\,r}\hat{\boldsymbol{n}}\cdot\dot{\boldsymbol{p}}(t_e) + \ldots} \tag{6.202}$$

where $Q = \text{const}$ is the total electric charge of the configuration.

The vector potential is obtained from (6.199), however, it requires some work. First, we observe that the first integral (6.199) with $\mu = \{1,2,3\}$ is equal to the time derivative of the electric dipole moment. It can be seen as follows. We take a closed surface $S = \partial\Omega$ which contains all the sources and has no common points or intersections with the region V containing four-currents, $V \subset \Omega$. It means that there are no electromagnetic sources at the surface S *i.e.* $J^i \equiv 0$ at S. Hence

$$\oint_S (x'^i J^j)da^j = 0. \tag{6.203}$$

The left hand side of this integral can be put in the form

$$\begin{aligned}
\oint_S (x'^i J^j)da^j &= \int_{\Omega} d^3x'\partial'_j(x'^i J^j) \\
&= \int_V d^3x'\partial'_j(x'^i J^j) \\
&= \int_V d^3x'\left[\delta_{ij}J^j + x'^i(\partial'_j J^j)\right] \\
&= \int_V d^3x' J^i - \int_V d^3x'\, x'^i\partial_t\rho \\
&= \int_V d^3x' J^i - \frac{d}{dt}\int_V d^3x'\, x'^i\rho \\
&= \int_V d^3x' J^i - \frac{dp^i}{dt}
\end{aligned}$$

The integral (6.203) gives

$$\int_V d^3x'\boldsymbol{J}(t_e,\boldsymbol{x}') = \dot{\boldsymbol{p}}(t_e). \tag{6.204}$$

The second integral (6.199) can be transformed as follows

$$
\begin{aligned}
n^i \int d^3x'\, x'^i J^j &= n^i \frac{1}{2} \int d^3x \left[x'^i J^j - x'^j J^i\right] + n^i \underbrace{\frac{1}{2} \int d^3x \left[x'^i J^j + x'^j J^i\right]}_{\text{a quadrupole moment}} \\
&= n^i c \underbrace{\frac{1}{2c} \int d^3x \left[x'^i J^j - x'^j J^i\right]}_{\epsilon_{ijk} m^k} + \cdots \\
&= -c[\hat{\boldsymbol{n}} \times \boldsymbol{m}]^j + \cdots
\end{aligned}
\tag{6.205}
$$

The vector potential reads

The vector potential

$$
\boxed{\boldsymbol{A}(t, \boldsymbol{x}) = \frac{1}{cr} \left[\dot{\boldsymbol{p}}(t_e) - \hat{\boldsymbol{n}} \times \dot{\boldsymbol{m}}(t_e)\right] + \cdots} \tag{6.206}
$$

where $\dot{\boldsymbol{p}}(t_e)$ and $\dot{\boldsymbol{m}}(t_e)$ are time derivatives of electric and magnetic dipole moment.

Electric and magnetic field

Electric and magnetic fields are given by the four-potential components

$$
\boldsymbol{E} = -\nabla A^0 - \frac{1}{c}\partial_t \boldsymbol{A} \qquad\qquad \boldsymbol{B} = \nabla \times \boldsymbol{A} \tag{6.207}
$$

We shall skip all the terms which decrease faster than $1/r$. Thus, expressions

$$
\nabla \frac{1}{r} \sim \frac{1}{r^2} \qquad\qquad \frac{1}{r} \nabla n^i \sim \frac{1}{r^2}.
$$

are not relevant. The electric field takes the form

$$
\begin{aligned}
\boldsymbol{E} &= -\nabla \left[\frac{Q}{r} + \frac{1}{c\,r} \hat{\boldsymbol{n}} \cdot \dot{\boldsymbol{p}}(t_e)\right] - \frac{1}{c^2 r} \left[\ddot{\boldsymbol{p}}(t_e) - \hat{\boldsymbol{n}} \times \ddot{\boldsymbol{m}}(t_e)\right] \\
&= -\frac{1}{c\,r} \underbrace{(\nabla t_e)}_{-\frac{\hat{\boldsymbol{n}}}{c}+\cdots} \hat{\boldsymbol{n}} \cdot \ddot{\boldsymbol{p}}(t_e) - \frac{1}{c^2 r} \left[\ddot{\boldsymbol{p}}(t_e) - \hat{\boldsymbol{n}} \times \ddot{\boldsymbol{m}}(t_e)\right] + \cdots \\
&= \frac{1}{c^2\, r} \Big[\underbrace{(\hat{\boldsymbol{n}} \cdot \ddot{\boldsymbol{p}})\hat{\boldsymbol{n}} - \ddot{\boldsymbol{p}}}_{\hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \ddot{\boldsymbol{p}})} + \hat{\boldsymbol{n}} \times \ddot{\boldsymbol{m}} \Big] + \cdots
\end{aligned}
$$

and thus

$$
\boxed{\boldsymbol{E} = \frac{1}{c^2\, r} \hat{\boldsymbol{n}} \times \left[\hat{\boldsymbol{n}} \times \ddot{\boldsymbol{p}} + \ddot{\boldsymbol{m}}\right]\Big|_{t_e}} \tag{6.208}
$$

The magnetic field reads

$$
\begin{aligned}
\boldsymbol{B} &= \nabla \times \left[\frac{1}{cr} \left[\dot{\boldsymbol{p}}(t_e) - \hat{\boldsymbol{n}} \times \dot{\boldsymbol{m}}(t_e)\right]\right] \\
&= \frac{1}{c\,r} \underbrace{(\nabla t_e)}_{-\frac{\hat{\boldsymbol{n}}}{c}+\ldots} \times \left[\ddot{\boldsymbol{p}}(t_e) - \hat{\boldsymbol{n}} \times \ddot{\boldsymbol{m}}(t_e)\right] + \cdots
\end{aligned}
$$

which gives

$$\boxed{\boldsymbol{B} = \frac{1}{c^2 r}\hat{\boldsymbol{n}} \times [\hat{\boldsymbol{n}} \times \ddot{\boldsymbol{m}} - \ddot{\boldsymbol{p}}]\Big|_{t_e}} \tag{6.209}$$

The expressions (6.208) and (6.209) imply that $\boldsymbol{B} = \hat{\boldsymbol{n}} \times \boldsymbol{E}$.

Angular distribution of radiation

The radiated power is a function of angles. It can be cast in the form

$$\begin{aligned} dP(t) &= (\hat{\boldsymbol{n}} \cdot \boldsymbol{S})r^2 d\Omega = \frac{c}{4\pi}|\boldsymbol{E}|^2 r^2 d\Omega = \frac{c}{4\pi}|\boldsymbol{B}|^2 r^2 d\Omega \\ &= \frac{1}{4\pi c^3}\,[\hat{\boldsymbol{n}} \times [\hat{\boldsymbol{n}} \times \ddot{\boldsymbol{p}} + \ddot{\boldsymbol{m}}]]^2 d\Omega \\ &= \frac{1}{4\pi c^3}[[\hat{\boldsymbol{n}} \times \ddot{\boldsymbol{p}} + \ddot{\boldsymbol{m}}]^2 - [\underbrace{\hat{\boldsymbol{n}} \cdot (\hat{\boldsymbol{n}} \times \ddot{\boldsymbol{p}})}_{0} + \hat{\boldsymbol{n}} \cdot \ddot{\boldsymbol{m}}]^2]d\Omega \\ &= \frac{1}{4\pi c^3}[(\hat{\boldsymbol{n}} \times \ddot{\boldsymbol{p}})^2 + 2\underbrace{(\hat{\boldsymbol{n}} \times \ddot{\boldsymbol{p}}) \cdot \ddot{\boldsymbol{m}}}_{\hat{\boldsymbol{n}}\cdot(\ddot{\boldsymbol{p}}\times\ddot{\boldsymbol{m}})} + \underbrace{\ddot{\boldsymbol{m}}^2 - (\hat{\boldsymbol{n}} \cdot \ddot{\boldsymbol{m}})^2}_{(\hat{\boldsymbol{n}}\times\ddot{\boldsymbol{m}})^2}]d\Omega \end{aligned}$$

which gives

$$\boxed{dP(t) = \frac{1}{4\pi c^3}[(\hat{\boldsymbol{n}} \times \ddot{\boldsymbol{p}})^2 + (\hat{\boldsymbol{n}} \times \ddot{\boldsymbol{m}})^2 + 2\hat{\boldsymbol{n}} \cdot (\ddot{\boldsymbol{p}} \times \ddot{\boldsymbol{m}})]\Big|_{t_e}.} \tag{6.210}$$

Angular distribution of the power associated with the radiation field

Integrating (6.210) over the angles one gets

$$\begin{aligned} P(t) = \frac{1}{4\pi c^3}\Bigg[&\int_V (\hat{\boldsymbol{n}} \times \ddot{\boldsymbol{p}})^2 d\Omega + \int_V (\hat{\boldsymbol{n}} \times \ddot{\boldsymbol{m}})^2 d\Omega + \\ &+ 2\int_V \hat{\boldsymbol{n}} \cdot (\ddot{\boldsymbol{p}} \times \ddot{\boldsymbol{m}})d\Omega\Bigg]. \end{aligned} \tag{6.211}$$

It is convenient to choose spherical coordinates independently for each integral (6.211). We choose these coordiantes in such a way that the z axis is aligned with $\ddot{\boldsymbol{p}}$ in the first integral, with $\ddot{\boldsymbol{m}}$ in the second one and with $\ddot{\boldsymbol{p}} \times \ddot{\boldsymbol{m}}$ for third integral. It leads to expressions

$$\int (\hat{\boldsymbol{n}} \times \ddot{\boldsymbol{p}})^2 d\Omega = 2\pi|\ddot{\boldsymbol{p}}|^2 \int_0^\pi d\vartheta \sin^3\vartheta = \frac{8\pi}{3}|\ddot{\boldsymbol{p}}|^2, \tag{6.212}$$

$$\int (\hat{\boldsymbol{n}} \times \ddot{\boldsymbol{m}})^2 d\Omega' = 2\pi|\ddot{\boldsymbol{m}}|^2 \int_0^\pi d\vartheta' \sin^3\vartheta' = \frac{8\pi}{3}|\ddot{\boldsymbol{m}}|^2, \tag{6.213}$$

$$\int \hat{\boldsymbol{n}} \cdot (\ddot{\boldsymbol{p}} \times \ddot{\boldsymbol{m}})d\Omega'' = 2\pi|\ddot{\boldsymbol{p}} \times \ddot{\boldsymbol{m}}| \int_0^\pi d\vartheta'' \sin\vartheta'' \cos\vartheta'' = 0. \tag{6.214}$$

The total radiated power calculated in the radiation zone reads

The total pawer carried by radiation

$$\boxed{P(t) = \frac{2}{3c^2}\left(|\ddot{\boldsymbol{p}}|^2 + |\ddot{\boldsymbol{m}}|^2\right).} \tag{6.215}$$

Note, that $|\ddot{\boldsymbol{m}}|^2 \sim c^{-2}$. It means that variation of magnetic dipole moment contributes less than variation of electric dipole moment.

Taking for instance the electric charge q oscillating along the z axis one gets $\boldsymbol{p} = q\boldsymbol{z}$. Hence $\ddot{\boldsymbol{p}} = q\ddot{\boldsymbol{z}} = q\boldsymbol{a}$.The total radiated power is given by the *Larmor formula*

$$\boxed{P(t) = \frac{2q^2}{3c^3}a^2.} \tag{6.216}$$