Evaluation of some integrals over solid angles—Part 2

The angular distribution of the power liberated by an accelerating point charge e moving with velocity $\vec{\boldsymbol{v}} \equiv c\,\vec{\boldsymbol{\beta}}$ and acceleration $\vec{\boldsymbol{a}} \equiv c\,\vec{\boldsymbol{\alpha}} = d\vec{\boldsymbol{v}}/dt$ is given by eq. (14.38) of Jackson,

$$\frac{dP(t')}{d\Omega} = \frac{e^2}{4\pi c}\,\frac{\left|\hat{\boldsymbol{n}} \times [(\hat{\boldsymbol{n}} - \vec{\boldsymbol{\beta}}) \times \vec{\boldsymbol{\alpha}}]\right|^2}{(1 - \hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{\beta}})^5}\,, \tag{1}$$

where t' is the retarded time. One can simplify the numerator of eq. (1) by employing a number of vector identities (such as the BAC–CAB rule). We then obtain:

$$\hat{\boldsymbol{n}} \times [(\hat{\boldsymbol{n}} - \vec{\boldsymbol{\beta}}) \times \vec{\boldsymbol{\alpha}}] = \hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{\alpha}}(\hat{\boldsymbol{n}} - \vec{\boldsymbol{\beta}}) - \vec{\boldsymbol{\alpha}}(1 - \hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{\beta}})\,, \tag{2}$$

and

$$\begin{aligned}
\left|\hat{\boldsymbol{n}} \times [(\hat{\boldsymbol{n}} - \vec{\boldsymbol{\beta}}) \times \vec{\boldsymbol{\alpha}}]\right|^2 &= (\hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{\alpha}})^2(\hat{\boldsymbol{n}} - \vec{\boldsymbol{\beta}}) \cdot (\hat{\boldsymbol{n}} - \vec{\boldsymbol{\beta}}) + \alpha^2(1 - \hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{\beta}})^2 \\
&\qquad -2\,\vec{\boldsymbol{\alpha}} \cdot (\hat{\boldsymbol{n}} - \vec{\boldsymbol{\beta}})\,\hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{\alpha}}\,(1 - \hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{\beta}}) \\
&= \left[1 + \beta^2 - 2\hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{\beta}}\right](\hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{\alpha}})^2 + \alpha^2(1 - \hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{\beta}})^2 \\
&\qquad -2\,(\hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{\alpha}})^2(1 - \hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{\beta}}) + 2\,\hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{\alpha}}(1 - \hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{\beta}})\vec{\boldsymbol{\alpha}} \cdot \vec{\boldsymbol{\beta}} \\
&= (\beta^2 - 1)(\hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{\alpha}})^2 + \alpha^2(1 - \hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{\beta}})^2 + 2\,\hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{\alpha}}(1 - \hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{\beta}})\vec{\boldsymbol{\alpha}} \cdot \vec{\boldsymbol{\beta}}\,,
\end{aligned} \tag{3}$$

where $\alpha^2 \equiv |\vec{\boldsymbol{\alpha}}|^2$ and $\beta^2 \equiv |\vec{\boldsymbol{\beta}}|^2$.

In order to compute total power liberated by an accelerating charge, we must integrate eq. (1) over all solid angles. In particular, we need to compute the following three integrals over the solid angle Ω,

$$I_1 = \int \frac{(\hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{\alpha}})^2}{(1 - \vec{\boldsymbol{\beta}} \cdot \hat{\boldsymbol{n}})^5}\,d\Omega\,, \tag{4}$$

$$I_2 = \int \frac{d\Omega}{(1 - \vec{\boldsymbol{\beta}} \cdot \hat{\boldsymbol{n}})^3}\,, \tag{5}$$

$$I_3 = \int \frac{\hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{\alpha}}}{(1 - \vec{\boldsymbol{\beta}} \cdot \hat{\boldsymbol{n}})^4}\,d\Omega\,. \tag{6}$$

Then, the total power is given by

$$P(t') = \frac{e^2}{4\pi c}\left[(\beta^2 - 1)I_1 + \alpha^2 I_2 + 2\,\vec{\boldsymbol{\alpha}} \cdot \vec{\boldsymbol{\beta}}\,I_3\right]. \tag{7}$$

Let's begin with I_2. Choose the z axis to lie along the direction of $\vec{\boldsymbol{\beta}}$. Then, it follow that $\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\hat{\boldsymbol{n}} = 1-\beta\cos\theta$, where $\beta \equiv |\vec{\boldsymbol{\beta}}|$. Writing $d\Omega = d\cos\theta\, d\phi$ and introducing $w \equiv \cos\theta$, it follows that

$$I_2 = 2\pi \int_{-1}^{1} \frac{dw}{(1-\beta w)^3} = \frac{2\pi}{\beta}\int_{1-\beta}^{1+\beta}\frac{dy}{y^3} = -\frac{\pi}{\beta}\left[\frac{1}{(1+\beta)^2} - \frac{1}{(1-\beta)^2}\right], \tag{8}$$

after changing the integration variable to $y = 1-\beta w$. Hence,

$$\boxed{I_2 = \frac{4\pi}{(1-\beta^2)^2}} \tag{9}$$

Next, we can take the derivative of I_2 with respect to $\vec{\boldsymbol{\beta}}$ by making use of eq. (5),

$$\frac{\partial I_2}{\partial\vec{\boldsymbol{\beta}}} = 3\int \frac{\hat{\boldsymbol{n}}\, d\Omega}{(1-\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\hat{\boldsymbol{n}})^4}\,. \tag{10}$$

Thus, we can identify

$$I_3 = \tfrac{1}{3}\vec{\boldsymbol{\alpha}}\boldsymbol{\cdot}\frac{\partial I_2}{\partial\vec{\boldsymbol{\beta}}}\,. \tag{11}$$

We can evaluate the right hand side of eq. (11) by using the result obtained in eq. (8). Since eq. (8) is a function of $\beta = |\vec{\boldsymbol{\beta}}|$, we can use the chain rule to write

$$\frac{\partial I_2}{\partial\vec{\boldsymbol{\beta}}} = \frac{\partial\beta}{\partial\vec{\boldsymbol{\beta}}}\frac{\partial I_2}{\partial\beta} = \frac{\vec{\boldsymbol{\beta}}}{\beta}\frac{\partial I_2}{\partial\beta}\,. \tag{12}$$

To obtain the last step above, we noted that $\beta = (\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{\beta}})^{1/2}$. Hence, it follows that

$$\frac{\partial\beta}{\partial\vec{\boldsymbol{\beta}}} = \frac{\partial}{\partial\vec{\boldsymbol{\beta}}}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{\beta}})^{1/2} = \tfrac{1}{2}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{\beta}})^{-1/2}\frac{\partial}{\partial\vec{\boldsymbol{\beta}}}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{\beta}}) = (\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{\beta}})^{-1/2}\vec{\boldsymbol{\beta}} = \frac{\vec{\boldsymbol{\beta}}}{\beta}\,. \tag{13}$$

Finally, we can use eq. (8) to evaluate $\partial I_2/\partial\beta$,

$$\frac{\partial I_2}{\partial\beta} = \frac{16\pi\beta}{(1-\beta^2)^3}\,. \tag{14}$$

Hence, we end up with

$$\boxed{I_3 = \frac{16\pi}{3}\,\frac{\vec{\boldsymbol{\alpha}}\boldsymbol{\cdot}\vec{\boldsymbol{\beta}}}{(1-\beta^2)^3}\,.} \tag{15}$$

Finally, we can use eq. (5) to obtain

$$\frac{\partial I_2}{\partial\beta_i} = 3\int\frac{\hat{\boldsymbol{n}}_i\, d\Omega}{(1-\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\hat{\boldsymbol{n}})^4}\,, \tag{16}$$

$$\frac{\partial^2 I_2}{\partial\beta_i\partial\beta_j} = 12\int\frac{\hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j\, d\Omega}{(1-\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\hat{\boldsymbol{n}})^5}\,, \tag{17}$$

after two successive differentiations. Hence, we can identify,

$$I_1 = \frac{1}{12}\sum_{i,j}\alpha_i\alpha_j\,\frac{\partial^2 I_2}{\partial\beta_i\partial\beta_j}\,. \tag{18}$$

Note that eqs. (13) and (14) are equivalent to

$$\frac{dI_2}{d\beta_i} = \frac{16\pi\beta_i}{(1-\beta^2)^3}\,. \tag{19}$$

The second derivative can now be easily evaluated with the help of eq. (13),

$$\begin{aligned}\frac{\partial^2 I_2}{\partial\beta_i\partial\beta_j} &= \frac{16\pi\delta_{ij}}{(1-\beta^2)^3} + 16\pi\beta_i\frac{\beta_j}{\beta}\frac{\partial}{\partial\beta}\left(\frac{1}{(1-\beta^2)^3}\right) \\ &= \frac{16\pi\delta_{ij}}{(1-\beta^2)^3} + \frac{96\pi\beta_i\beta_j}{(1-\beta^2)^4}\,. \end{aligned}\tag{20}$$

Consequently, eq. (18) yields,

$$\boxed{I_1 = \frac{4\pi}{3}\,\frac{\alpha^2}{(1-\beta^2)^3} + \frac{8\pi(\vec{\boldsymbol{\alpha}}\cdot\vec{\boldsymbol{\beta}})^2}{(1-\beta^2)^4}\,.} \tag{21}$$

An alternative technique for evaluating the integrals I_1 and I_3 that does not rely on a separate computation of I_2 is presented in Appendix A.

Inserting the results of eqs. (9), (15) and (21) into eq. (7), we obtain

$$\begin{aligned} P(t') &= \frac{e^2}{4\pi c}\left[-\frac{4\pi}{3}\,\frac{\alpha^2}{(1-\beta^2)^2} - \frac{8\pi(\vec{\boldsymbol{\alpha}}\cdot\vec{\boldsymbol{\beta}})^2}{(1-\beta^2)^3} + \frac{4\pi\alpha^2}{(1-\beta^2)^2} + \frac{32\pi}{3}\,\frac{(\vec{\boldsymbol{\alpha}}\cdot\vec{\boldsymbol{\beta}})^2}{(1-\beta^2)}\right] \\ &= \frac{e^2}{4\pi c}\left(\frac{8\pi}{3}\right)\left[\frac{\alpha^2}{(1-\beta^2)^2} + \frac{(\vec{\boldsymbol{\alpha}}\cdot\vec{\boldsymbol{\beta}})^2}{(1-\beta^2)^3}\right] \\ &= \frac{2e^2\left[\alpha^2(1-\beta^2) + (\vec{\boldsymbol{\alpha}}\cdot\vec{\boldsymbol{\beta}})^2\right]}{3c(1-\beta^3)}\,. \end{aligned}\tag{22}$$

Finally, we employ the vector identity

$$\alpha^2 - |\vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{\alpha}}|^2 = (1-\beta^2)\alpha^2 + (\vec{\boldsymbol{\alpha}}\cdot\vec{\boldsymbol{\beta}})^2\,, \tag{23}$$

Introducing $\gamma \equiv (1-\beta^2)^{-1/2}$, we arrive at our final result:

$$P(t') = \frac{2e^2\gamma^6}{3c}\left[\alpha^2 - |\vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{\alpha}}|^2\right]\,, \tag{24}$$

which is the relativistic generalization of Larmor's formula.

APPENDIX: An alternative technique for evaluating $\boldsymbol{I_1}$ and $\boldsymbol{I_3}$

We can define the following two integrals,

$$J_{ij} = \int \frac{\hat{\boldsymbol{n}}_i \hat{\boldsymbol{n}}_j \, d\Omega}{(1 - \vec{\boldsymbol{\beta}} \cdot \hat{\boldsymbol{n}})^5} \,, \tag{A.1}$$

$$K_i = \int \frac{\hat{\boldsymbol{n}}_i \, d\Omega}{(1 - \vec{\boldsymbol{\beta}} \cdot \hat{\boldsymbol{n}})^4} \,. \tag{A.2}$$

By the covariance properties of Euclidean tensors, it follows that

$$J_{ij} = c_1 \delta_{ij} + c_2 \beta_i \beta_j \,, \tag{A.3}$$

$$K_i = \kappa \beta_i \,. \tag{A.4}$$

Consider first the evaluation of K_i. Multiplying by β_i and summing over i yields

$$\kappa \beta^2 = \int \frac{\vec{\boldsymbol{\beta}} \cdot \hat{\boldsymbol{n}} \, d\Omega}{(1 - \vec{\boldsymbol{\beta}} \cdot \hat{\boldsymbol{n}})^4} \,. \tag{A.5}$$

The integral above is now easily evaluated by employing the same method used to obtain eq. (8). Thus, we can obtain an explicit expression for κ. I will leave it as an exercise for the reader to show that

$$\kappa = \frac{16\pi}{3} \frac{1}{(1 - \beta^2)^3} \,. \tag{A.6}$$

Likewise, to evaluate J_{ij}, we first multiply by δ_{ij} and sum over i and j to get one equation. A second equation is obtained by multiplying by $\beta_i \beta_j$ and summing over i and j, Thus, we get two equations for the two unknowns c_1 and c_2,

$$3c_1 + c_2 \beta^2 = \int \frac{d\Omega}{(1 - \vec{\boldsymbol{\beta}} \cdot \hat{\boldsymbol{n}})^5} \,, \tag{A.7}$$

$$c_1 \beta^2 + c_2 \beta^4 = \int \frac{(\vec{\boldsymbol{\beta}} \cdot \hat{\boldsymbol{n}})^2 \, d\Omega}{(1 - \vec{\boldsymbol{\beta}} \cdot \hat{\boldsymbol{n}})^5} \,. \tag{A.8}$$

Again, the two integrals above are easily evaluated by employing the same method used to obtain eq. (8). One can then solve for c_1 and c_2. I will leave it as an exercise for the reader to carry out the remaining computations to obtain,

$$c_1 = \frac{4\pi}{3} \frac{1}{(1 - \beta^2)^3} \,, \qquad c_2 = \frac{8\pi}{(1 - \beta^2)^4} \,. \tag{A.9}$$

Finally, we obtain

$$I_1 = \sum_{i,j} \alpha_i \alpha_j J_{ij} \,, \qquad I_3 = \sum_i \alpha_i K_i \,. \tag{A.10}$$

Using eqs. (A.3), (A.4), (A.6) and (A.9), we recover the results obtained in eqs. (21) and (15), respectively.