Proof that an electromagnetic wave does not propagate faster than c in a dispersive medium

In a dispersive medium, the index of refraction $n(\omega)$ is a function of the angular frequency ω. In class, we saw that the group velocity of an electromagnetic wave is given by,

$$v_g = \frac{c}{n(\omega) + \omega \dfrac{dn}{d\omega}} \,. \tag{1}$$

In particular, in regions of anomalous dispersion, where $dn/d\omega < 0$, it is possible for the group velocity to be larger than c. Indeed, if we employ the model discussed in class based on an electron bound by a damped harmonic force, we obtained $n(\omega) = \sqrt{\epsilon(\omega)}$ [under the assumption that $\mu \simeq \mu_0$], where

$$\epsilon(\omega) = \epsilon_0 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\gamma\omega} \,, \tag{2}$$

where ω_p is the plasma frequency of the medium, γ is the damping coefficient (assumed to be positive) and ω_0 is the resonant frequency of the harmonic force. One can check that in the vicinity of the resonance, $dn/d\omega < 0$ and the group velocity can exceed c. As an example, Figure 1 on the next page exhibits an excerpt from a textbook by Panofsky and Phillips that illustrates the behavior of four different types of velocities associated with an electromagnetic wave moving through a dispersive medium. This figure shows that both the group velocity and phase velocity [$v_p = c/n(\omega)$] can exceed c.

Nevertheless, no "physical" velocity can exceed the c. In this note, I shall present a simple proof that the electric field of an electromagnetic wave does not propagate faster than c. Consider an electric field of an electromagnetic wave propagating in the z direction,

$$\vec{\boldsymbol{E}}(z,t) = \int_{-\infty}^{\infty} \vec{\boldsymbol{A}}(k) e^{i[kz - \omega(k)t]}\, dk \,, \tag{3}$$

where $\vec{\boldsymbol{A}}(k)$ is an amplitude of the wave packet that depends on the wave number k and the functional form for $\omega(k)$ is determined from the dispersion relation,

$$k^2 = n^2(\omega)\frac{\omega^2}{c^2} \,. \tag{4}$$

We shall prove that if $\vec{\boldsymbol{E}}(z,0) = 0$ for $z > 0$ then $\vec{\boldsymbol{E}}(z,t) = 0$ for $z > ct$. In particular, the wave front does not propagate faster than c. Our proof is based on two assumptions:

1. $\omega(k)$ is analytic in the upper half complex k-plane.

2. $\omega(k) \to ck$ as $k \to \infty$.

The practical difficulty of describing the transmission of a signal in the region of anomalous dispersion arises from the fact that the group, or packet, is highly distorted, so that the concept of group velocity as defined in Section 11–8 is no longer valid. The problem corresponding to a plane wave that starts through a medium at a particular time has been worked

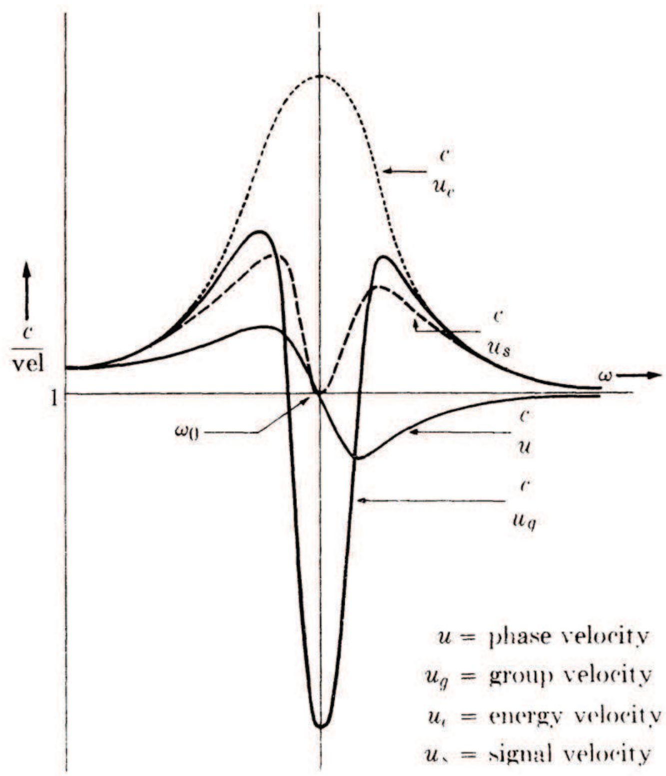


FIG. 22–4. Behavior of various velocities near a resonance frequency (after Brillouin).

out by Sommerfeld and by Brillouin, and details may be found in Volume II of *Congrès International d'Electricité*, Paris, 1932.* The definition of a signal velocity depends on both the nature of the pulse and the nature of the signal detector, but careful analysis shows that it is never greater than c. Figure 22–4 indicates the behavior of the four velocities in the neighborhood of an absorption frequency ω_0.

Figure 1: Excerpt from Wolfgang K.H. Panofsky and Melba Phillips, *Classical Electricity and Magnetism*, Second edition (Addison-Wesley Publishing Company, Inc., Reading, MA, 1962).

The two assumptions regarding the behavior of $\omega(k)$ are direct consequences of the behavior of $\epsilon(\omega)$ derived in class, where we showed that under the assumption of causality, $\epsilon(\omega)$ is analytic in the upper half complex ω-plane and $\epsilon \to \epsilon_0$ as $\omega \to \infty$.

Starting from eq. (3) evaluated at $z = 0$, we invert the Fourier transform to obtain,

$$\vec{\boldsymbol{A}}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \vec{\boldsymbol{E}}(z, 0) e^{-ikz} \, dz = \frac{1}{2\pi} \int_{-\infty}^{0} \vec{\boldsymbol{E}}(z, 0) e^{-ikz} \, dz \,, \tag{5}$$

under the assumption that $\vec{\boldsymbol{E}}(z, 0) = 0$ for $z > 0$.

We may now analytically continue $\vec{\boldsymbol{A}}(k)$ into the complex k-plane. In particular, for $k = k_R + ik_I$,

$$\frac{d^n \vec{\boldsymbol{A}}(k)}{dk^n} = \frac{(-i)^n}{2\pi} \int_{-\infty}^{0} \vec{\boldsymbol{E}}(z, 0) z^n e^{-ik_R z} e^{k_I z} \,, \tag{6}$$

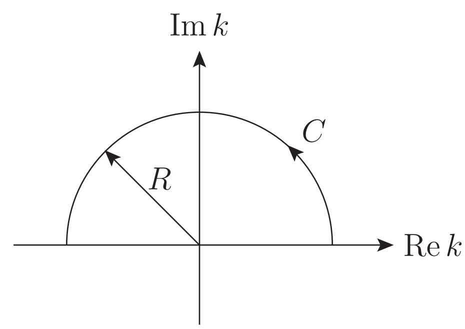
which converges for all $k_I > 0$, due to the exponential suppression factor $e^{k_I z}$ (note that $z < 0$ in the region of integration). That is, $\vec{\boldsymbol{A}}(k)$ is analytic in the upper half complex k-plane.

We wish to evaluate

$$\vec{\boldsymbol{E}}(z, t) = \int_{-\infty}^{\infty} \vec{\boldsymbol{A}}(k) e^{i[kz - \omega(k)t]} \, dk \,, \qquad \text{for } z > ct. \tag{7}$$

We can evaluate this integral by considering the integral along the closed contour C shown below in the limit of $R \to \infty$.

$$\vec{\boldsymbol{E}}(z, t) = \int_C \vec{\boldsymbol{A}}(k) e^{i[kz - \omega(k)t]} \, dk$$



The integral over the contour C is equal to the integral given by eq. (7) if we can show that the integrand vanishes for k lying on the semicircular arc in the limit of $R \to \infty$. Using $\omega(k) \to ck$ as $k \to \infty$, it follows that

$$kz - \omega(k)t \to k(z - ct) \,, \qquad \text{as } R \to \infty. \tag{8}$$

Thus, for $k = k_R + ik_I$,

$$\text{Re}\{i[kz - \omega(k)t]\} = \text{Re}[i(k_R + ik_I)(z - ct)] = -k_I(z - ct) \,, \tag{9}$$

which is negative on the semicircular arc when $z > ct$ (since $k_I > 0$ in the upper half complex k-plane). That is, the factor $e^{i[kz - \omega(k)t]}$ is exponentially suppressed in the upper half k-plane as $R \to \infty$, which means that the semicircular arc does not contribute to the integral.

Finally, we make use of the analyticity of $\vec{\boldsymbol{A}}(k)$ and $\omega(k)$ in the upper half complex k-plane. Consequently, there are no singularities (e.g., poles) contained inside the closed contour C. Hence, by Cauchy's theorem, it follows that

$$\vec{\boldsymbol{E}}(z, t) = 0 \,, \qquad \text{for } z > ct. \tag{10}$$

and our proof is complete.